Symmetry, Integrability and Geometry: Methods and Applications SIGMA **17** (2021), 014, 9 pages

The Subelliptic Heat Kernel of the Octonionic Anti-De Sitter Fibration

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Received July 31, 2020, in final form January 29, 2021; Published online February 10, 2021
https://doi.org/10.3842/SIGMA.2021.014

Abstract. In this note, we study the sub-Laplacian of the 15-dimensional octonionic anti-de Sitter space which is obtained by lifting with respect to the anti-de Sitter fibration the Laplacian of the octonionic hyperbolic space $\mathbb{O}H^1$. We also obtain two integral representations for the corresponding subelliptic heat kernel.

Key words: sub-Laplacian; 15-dimensional octonionic anti-de Sitter space; the anti-de Sitter fibration

2020 Mathematics Subject Classification: 58J35; 53C17

1 Introduction and results

In this note we study the sub-Laplacian and the corresponding sub-Riemannian heat kernel of the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \mathrm{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1.$$

This paper follows the previous works [2, 3, 10] which respectively concerned:

1. *The complex anti-de Sitter fibrations*:

$$\mathbb{S}^1 \hookrightarrow \mathrm{AdS}^{2n+1}(\mathbb{C}) \to \mathbb{C}H^n.$$

2. *The quaternionic anti-de Sitter fibrations*:

$$\mathbb{S}^3 \hookrightarrow \mathrm{AdS}^{4n+3}(\mathbb{H}) \to \mathbb{H}H^n.$$

The 15-dimensional anti-de Sitter fibration is the last model space that remained to be studied of a sub-Riemannian manifold arising from a H-type semi-Riemannian submersion over a rank-one symmetric space, see the Table 3 in [4].

Similarly to the complex and quaternionic case, the sub-Laplacian is defined as the lift on $\mathrm{AdS}^{15}(\mathbb{O})$ of the Laplace–Beltrami operator of the octonionic hyperbolic space $\mathbb{O}H^1$. However, in the complex and quaternionic case the Lie group structure of the fiber played an important role that we can not use here, since the fiber $\mathbb{S}^7$ is not a group. Instead, we make use of some algebraic properties of $\mathbb{S}^7$ that were already pointed out and used by the authors in [1] for the study of the octonionic Hopf fibration:

$$\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{O}P^1.$$

Let us briefly describe our main results. Due to the cylindrical symmetries of the fibration, the heat kernel of the sub-Laplacian only depends on two variables: the variable r which is the

Riemannian distance on $\mathbb{O}H^1$ (the starting point is specified with inhomogeneous coordinate in Section 3) and the variable η which is the Riemannian distance starting at a pole on the fiber $\mathbb{S}^7$. We prove in Proposition 3.1 that in these coordinates, the radial part of the sub-Laplacian $\tilde{L}$ writes

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r} + \tanh^2 r\left(\frac{\partial^2}{\partial \eta^2} + 6\cot\eta\frac{\partial}{\partial \eta}\right).$$

As a consequence of this expression for the sub-Laplacian, we are able to derive two equivalent formulas for the heat kernel. The first formula, see Proposition 4.1, reads as follows: for $r \geq 0$, $\eta \in [0, \pi)$, $t > 0$

$$p_t(r, \eta) = \int_0^\infty s_t(\eta, \mathrm{i}u) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, \mathrm{d}u,$$

where s_t is the heat kernel of the Jacobi operator

$$\tilde{\triangle}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6\cot\eta\frac{\partial}{\partial \eta}$$

with respect to the measure $\sin^6 \eta \, \mathrm{d}\eta$, and where $q_{t,15}$ is the Riemannian heat kernel on the 15-dimensional real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). The second formula, see Proposition 4.2, writes as follows:

$$p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, \mathrm{d}u \, \mathrm{d}\varphi,$$

where $q_{t,9}$ is Riemannian heat kernel on the 9-dimensional hyperbolic space $\mathbb{H}^9$ and $G_t(\eta, \varphi, u)$ is given in (4.3).

Similarly to [2, 3, 10], it might be expected that explicit integral representations of the heat kernel might be used to study small-time asymptotics, inside and outside of the cut-locus. Integral representations of heat kernels can also be used to obtain sharp heat kernel estimates, see [7]. Those applications of the heat kernel representations we obtain will possibly be addressed in a future research project.

2 The octonionic anti-de Sitter fibration

Let

$$\mathbb{O} = \left\{ x = \sum_{j=0}^{7} x_j e_j, \, x_j \in \mathbb{R} \right\},$$

be the division algebra of octonions (see [9] for explicit representations of this algebra). We recall that the multiplication rules are given by

$$\begin{aligned} e_i e_j &= e_j \quad \text{if} \quad i = 0, \\ e_i e_j &= e_i \quad \text{if} \quad j = 0, \\ e_i e_j &= -\delta_{ij} e_0 + \epsilon_{ijk} e_k \quad \text{otherwise}, \end{aligned}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the completely antisymmetric tensor with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$ (also see [1]). The octonionic norm is defined for $x \in \mathbb{O}$ by

$$||x||^2 = \sum_{j=0}^{7} x_j^2.$$

The octonionic anti-de Sitter space $\mathrm{AdS}^{15}(\mathbb{O})$ is the quadric defined as the pseudo-hyperbolic space by:

$$\mathrm{AdS}^{15}(\mathbb{O}) = \left\{(x, y) \in \mathbb{O}^2,\ ||(x, y)||^2_{\mathbb{O}} = -1\right\},$$

where

$$||(x, y)||^2_{\mathbb{O}} := ||x||^2 - ||y||^2.$$

In real coordinates we have $x = \sum_{j=0}^{7} x_j e_j$, $y = \sum_{j=0}^{7} y_j e_j$, and the pseudo-norm can be written as

$$x_0^2 + \cdots + x_7^2 - y_0^2 - \cdots - y_7^2.$$

As such, $\mathrm{AdS}^{15}(\mathbb{O})$ is embedded in the flat 16-dimensional space $\mathbb{R}^{8,8}$ endowed with the Lorentzian real signature $(8, 8)$ metric

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 + \cdots + \mathrm{d}x_7^2 - \mathrm{d}y_0^2 - \cdots - \mathrm{d}y_7^2.$$

Consequently, $\mathrm{AdS}^{15}(\mathbb{O})$ is naturally endowed with a pseudo-Riemannian structure of signature $(8, 7)$.

Let $\mathbb{O}H^1$ denote the octonionic hyperbolic space. The map $\pi\colon \mathrm{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$, given by $(x, y) \mapsto [x : y] = y^{-1}x$ is a pseudo-Riemannian submersion with totally geodesic fibers isometric to the seven-dimensional sphere $\mathbb{S}^7$. Notice that, as a topological manifold, $\mathbb{O}H^1$ can therefore be identified with the unit open ball in $\mathbb{O}$. The pseudo-Riemannian submersion π yields the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \mathrm{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1.$$

For further information on semi-Riemannian submersions over rank-one symmetric spaces, we refer to [6].

3 Cylindrical coordinates and radial part of the sub-Laplacian

The sub-Laplacian L on $\mathrm{AdS}^{15}(\mathbb{O})$ we are interested in is the horizontal Laplacian of the Riemannian submersion $\pi\colon \mathrm{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$, i.e., the horizontal lift of the Laplace–Beltrami operator of $\mathbb{O}H^1$. It can be written as

$$L = \Box_{\mathrm{AdS}^{15}(\mathbb{O})} + \triangle_{\mathcal{V}}, \tag{3.1}$$

where $\Box_{\mathrm{AdS}^{15}(\mathbb{O})}$ is the d'Alembertian, i.e., the Laplace–Beltrami operator of the pseudo-Riemannian metric and $\triangle_{\mathcal{V}}$ is the vertical Laplacian. Since the fibers of π are totally geodesic and isometric to $\mathbb{S}^7 \subset \mathrm{AdS}^{15}(\mathbb{O})$, we note that $\Box_{\mathrm{AdS}^{15}(\mathbb{O})}$ and $\triangle_{\mathcal{V}}$ are commuting operators, and we can identify

$$\triangle_{\mathcal{V}} = \triangle_{\mathbb{S}^7}. \tag{3.2}$$

The sub-Laplacian L is associated with a canonical sub-Riemannian structure on $\mathrm{AdS}^{15}(\mathbb{O})$ which is of H-type, see [4].

To study L, we introduce a set of coordinates that reflect the cylindrical symmetries of the octonionic unit sphere which provides an explicit local trivialization of the octonionic anti-de Sitter fibration. Consider the coordinates $w \in \mathbb{O}H^1$, where w is the inhomogeneous coordinate on $\mathbb{O}H^1$ given by $w = y^{-1}x$, with $x, y \in \mathrm{AdS}^{15}(\mathbb{O})$. Consider the north pole $p \in \mathbb{S}^7$ and take

$Y_1, \dots, Y_7$ to be an orthonormal frame of $T_p\mathbb{S}^7$. Let us denote $\exp_p$ the Riemannian exponential map at p on $\mathbb{S}^7$. Then the cylindrical coordinates we work with are given by

$$(w, \theta_1, \dots, \theta_7) \mapsto \left(\frac{\exp_p\left(\sum_{i=1}^7 \theta_i Y_i\right) w}{\sqrt{1-\rho^2}}, \frac{\exp_p\left(\sum_{i=1}^7 \theta_i Y_i\right)}{\sqrt{1-\rho^2}} \right) \in \mathrm{AdS}^{15}(\mathbb{O}),$$

where $\rho = \|w\|$ and $\|\theta\| = \sqrt{\theta_1^2 + \cdots + \theta_7^2} < \pi$.

A function f on $\mathrm{AdS}^{15}(\mathbb{O})$ is called radial cylindrical if it only depends on the two coordinates $(\rho, \eta) \in [0,1) \times [0,\pi]$ where $\eta = \sqrt{\sum_{i=1}^7 \theta_i^2}$. More precisely f is radial cylindrical if there exists a function g so that

$$f\left(\frac{\exp_p\left(\sum_{i=1}^7 \theta_i Y_i\right) w}{\sqrt{1-\rho^2}}, \frac{\exp_p\left(\sum_{i=1}^7 \theta_i Y_i\right)}{\sqrt{1-\rho^2}} \right) = g(\rho, \eta).$$

We denote by $\mathcal{D}$ the space of smooth and compactly supported functions on $[0,1) \times [0,\pi)$. Then the radial part of L is defined as the operator $\widetilde{L}$ such that for any $f \in \mathcal{D}$, we have

$$L(f \circ \psi) = \big(\widetilde{L} f\big) \circ \psi. \tag{3.3}$$

We now compute $\widetilde{L}$ in cylindrical coordinates.

Proposition 3.1. *The radial part of the sub-Laplacian on* $\mathrm{AdS}^{15}(\mathbb{O})$ *is given in the coordinates* (r, η) *by the operator*

$$\widetilde{L} = \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r} + \tanh^2 r \left(\frac{\partial^2}{\partial \eta^2} + 6\cot\eta \frac{\partial}{\partial \eta} \right),$$

where $r = \tanh^{-1}\rho$ *is the Riemannian distance on* $\mathbb{O}H^1$ *from the origin.*

Proof. Note that the radial part of the Laplace–Beltrami operator on the octonionic hyperbolic space $\mathbb{O}H^1$ is

$$\widetilde{\triangle}_{\mathbb{O}H^1} = \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r},$$

and the radial part of the Laplace–Beltrami operator on $\mathbb{S}^7$ is

$$\widetilde{\triangle}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6\cot\eta \frac{\partial}{\partial \eta}. \tag{3.4}$$

Since the octonionic anti-de Sitter fibration defines a totally geodesic submersion with base space $\mathbb{O}H^1$ and fiber $\mathbb{S}^7$, the semi-Riemannian metric on $\mathrm{AdS}^{15}(\mathbb{O})$ is locally given by a warped product between the Riemannian metric of $\mathbb{O}H^1$ and the Riemannian metric on $\mathbb{S}^7$. Hence the radial part of the d'Alembertian becomes

$$\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r} + g(r)\left(\frac{\partial^2}{\partial \eta^2} + 6\cot\eta \frac{\partial}{\partial \eta} \right), \tag{3.5}$$

for some smooth function g to be computed.

On the other hand, from the isometric embedding $\mathrm{AdS}^{15}(\mathbb{O}) \subset \mathbb{O} \times \mathbb{O}$, the d'Alembertian on $\mathrm{AdS}^{15}(\mathbb{O})$ is a restriction of the d'Alembertian on $\mathbb{O} \times \mathbb{O} \simeq \mathbb{R}^{8,8}$ in the sense that for a smooth $f\colon \mathrm{AdS}^{15}(\mathbb{O}) \to \mathbb{R}$

$$\square_{\mathrm{AdS}^{15}(\mathbb{O})} f = \square_{\mathbb{O}\times\mathbb{O}} f^*_{/\mathrm{AdS}^{15}(\mathbb{O})},$$

where $\square_{\mathbb{O}\times\mathbb{O}} = \sum_{i=0}^{7}\left(\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2}\right)$ and for $x, y \in \mathbb{O}$ such that $\|y\|^2 - \|x\|^2 > 0$, $f^*(x,y) = f\left(\frac{x}{\sqrt{\|y\|^2-\|x\|^2}}, \frac{y}{\sqrt{\|y\|^2-\|x\|^2}}\right)$. For the specific choice of the function $f(x,y) = y_1$, one easily computes that $\square_{\mathbb{O}\times\mathbb{O}} f^*_{/\mathrm{AdS}^{15}(\mathbb{O})}(x,y) = 15y_1$, thus

$$\square_{\mathrm{AdS}^{15}(\mathbb{O})} f(x,y) = 15y_1.$$

For the point with coordinates

$$\left(\frac{\exp_p\left(\sum_{i=1}^{7}\theta_i Y_i\right)w}{\sqrt{1-\rho^2}}, \frac{\exp_p\left(\sum_{i=1}^{7}\theta_i Y_i\right)}{\sqrt{1-\rho^2}}\right) \in \mathrm{AdS}^{15}(\mathbb{O})$$

one has

$$y_1 = \frac{\cos\eta}{\sqrt{1-\rho^2}} = \cosh r \cos\eta.$$

We therefore deduce that

$$\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}(\cosh r\cos\eta) = 15\cosh r\cos\eta.$$

Using the formula (3.5), after a straightforward computation, this yields $g(r) = -\frac{1}{\cosh^2 r}$ and therefore

$$\begin{aligned}\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})} &= \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r} - \frac{1}{\cosh^2 r}\left(\frac{\partial^2}{\partial\eta^2} + 6\cot\eta\frac{\partial}{\partial\eta}\right) \\ &= \widetilde{\triangle}_{\mathbb{O}H^1} - \frac{1}{\cosh^2 r}\widetilde{\triangle}_{\mathbb{S}^7}.\end{aligned}$$

Finally, to conclude, one notes that the sub-Laplacian L is given by the difference between the Laplace–Beltrami operator of $\mathrm{AdS}^{15}(\mathbb{O})$ and the vertical Laplacian. Therefore by (3.1) and (3.2),

$$\widetilde{L} = \tilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})} + \tilde{\triangle}_{\mathbb{S}^7} = \frac{\partial^2}{\partial r^2} + (7\coth r + 7\tanh r)\frac{\partial}{\partial r} + \tanh^2 r\left(\frac{\partial^2}{\partial\eta^2} + 6\cot\eta\frac{\partial}{\partial\eta}\right). \quad \blacksquare$$

Remark 3.2. As a consequence of the previous result, we can check that the Riemannian measure of $\mathrm{AdS}^{15}(\mathbb{O})$ in the coordinates (r,η), which is the symmetric and invariant measure for $\tilde{L}$ is given by

$$\mathrm{d}\overline{\mu} = \frac{\pi^7}{90}\sinh^7 r\cosh^7 r\sin^6\eta\,\mathrm{d}r\,\mathrm{d}\eta. \tag{3.6}$$

(See also Remark 2 in [1], which corresponds to the case of the octonionic Hopf fibration.)

4 Integral representations of the subelliptic heat kernel

In this section, we give two integral representations of the subelliptic heat kernel associated with $\tilde{L}$. We denote by $p_t(r,\eta)$ the heat kernel of $\tilde{L}$ issued from the point $r = \eta = 0$ with respect to the measure (3.6). We remark that studying the subelliptic heat kernel associated with $\tilde{L}$ is enough to study the heat kernel of L, because due to (3.3) the heat kernel $h_t(w,\theta)$ of L issued from the point with cylindric coordinates $w = 0$, $\theta = 0$ is then given by

$$h_t(w,\theta) = p_t\left(\tanh^{-1}\|w\|, \|\theta\|\right).$$

4.1 First integral representation

We denote by s_t the heat kernel of the operator

$$\tilde{\triangle}_{\mathbb{S}^7} = \frac{\partial^2}{\partial\eta^2} + 6\cot\eta\frac{\partial}{\partial\eta}$$

with respect to the reference measure $\sin^6\eta\,\mathrm{d}\eta$. The operator $\tilde{\triangle}_{\mathbb{S}^7}$ belongs to the family of Jacobi diffusion operators which have been extensively studied in the literature, see for instance the appendix in [5] and the references therein. In particular, the spectrum of $\tilde{\triangle}_{\mathbb{S}^7}$ is given by

$$\mathbf{Sp}\big(-\tilde{\triangle}_{\mathbb{S}^7}\big) = \{m(m+6),\, m\in\mathbb{N}\},$$

and the eigenfunction corresponding to the eigenvalue $m(m+6)$ is $P_m^{5/2,5/2}(\cos\eta)$ where $P_m^{5/2,5/2}$ is the Jacobi polynomial

$$P_m^{5/2,5/2}(x) = \frac{(-1)^m}{2^m m!\big(1-x^2\big)^{5/2}}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\big(1-x^2\big)^{5/2+m}.$$

As a consequence, one has the following spectral decomposition for the heat kernel:

$$s_t(\eta,u) = \frac{1}{\pi}\sum_{m=0}^{+\infty}\frac{2^{4m+7}m!(m+5)![(m+3)!]^2}{(2m+6)!(2m+5)!}\mathrm{e}^{-m(m+6)t}P_m^{5/2,5/2}(\cos\eta)P_m^{5/2,5/2}(\cos u).$$

Proposition 4.1. *For $r\geq 0$, $\eta\in[0,\pi]$, and $t>0$ we have*

$$p_t(r,\eta) = \int_0^\infty s_t(\eta,\mathrm{i}u)q_{t,15}(\cosh r\cosh u)\sinh^6 u\,\mathrm{d}u,$$

where

$$q_{t,15}(\cosh s) := \frac{\mathrm{e}^{-49t}}{(2\pi)^7\sqrt{4\pi t}}\left(-\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^7\mathrm{e}^{-s^2/4t} \tag{4.1}$$

is the Riemannian heat kernel on the 15*-dimensional real hyperbolic space* $\mathbb{H}^{15}$.

Proof. Since $\pi\colon \mathrm{AdS}^{15}(\mathbb{O})\to\mathbb{O}H^1$ is a (semi-Riemannian) totally geodesic submersion, the operators $\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}$ and $\widetilde{\triangle}_{\mathbb{S}^7}$ commute. Thus

$$\mathrm{e}^{t\tilde{L}} = \mathrm{e}^{t(\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}+\widetilde{\triangle}_{\mathbb{S}^7})} = \mathrm{e}^{t\widetilde{\triangle}_{\mathbb{S}^7}}\mathrm{e}^{t\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}.$$

We deduce that the heat kernel of $\tilde{L}$ can be written as

$$p_t(r,\eta) = \int_0^\pi s_t(\eta,u)p_t^{\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}(r,u)\sin^6 u\,\mathrm{d}u, \tag{4.2}$$

where s_t is the heat kernel of (3.4) with respect to the measure $\sin^6\eta\,\mathrm{d}\eta$, $\eta\in[0,\pi)$, and $p_t^{\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}(r,u)$ the heat kernel at $(0,0)$ of $\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}$ with respect to the measure in (3.6), i.e.,

$$\mathrm{d}\mu(r,u) = \frac{\pi^7}{90}\sinh^7 r\cosh^7 r\sin^6 u\,\mathrm{d}r\,\mathrm{d}u, \qquad r\in[0,\infty), \qquad u\in[0,\pi].$$

In order to write (4.2) more precisely, let us consider the analytic change of variables $\tau\colon(r,\eta)\to(r,\mathrm{i}\eta)$ that will be applied on functions of the type $f(r,\eta)=h(r)\mathrm{e}^{-\mathrm{i}\lambda\eta}$, with h smooth and compactly supported on $[0,\infty)$ and $\lambda>0$. Then as we saw in the proof of Proposition 3.1 one can see that

$$\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}(f\circ\tau)=\left(\widetilde{\triangle}_{\mathbb{H}^{15}}f\right)\circ\tau,$$

where

$$\widetilde{\triangle}_{\mathbb{H}^{15}}=\widetilde{\triangle}_{\mathbb{O}H^1}+\frac{1}{\cosh^2 r}\widetilde{\triangle}_P,\qquad \widetilde{\triangle}_P=\frac{\partial^2}{\partial\eta^2}+6\coth\eta\frac{\partial}{\partial\eta}.$$

Then, one deduces

$$\mathrm{e}^{t\widetilde{L}}(f\circ\tau)=\mathrm{e}^{t\widetilde{\triangle}_{\mathbb{S}^7}}\mathrm{e}^{t\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}(f\circ\tau)=\mathrm{e}^{t\widetilde{\triangle}_{\mathbb{S}^7}}\left(\left(\mathrm{e}^{t\widetilde{\triangle}_{\mathbb{H}^{15}}}f\right)\circ\tau\right)=\left(\mathrm{e}^{-t\widetilde{\triangle}_P}\mathrm{e}^{t\widetilde{\triangle}_{\mathbb{H}^{15}}}f\right)\circ\tau.$$

Now, since for every $f(r,\eta)=h(r)\mathrm{e}^{-\mathrm{i}\lambda\eta}$,

$$\left(\mathrm{e}^{t\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}f\right)(0,0)=\left(\mathrm{e}^{t\widetilde{\triangle}_{\mathbb{H}^{15}}}\right)\left(f\circ\tau^{-1}\right)(0,0),$$

one deduces that for a function h depending only on u,

$$\int_0^\pi h(u)p_t^{\widetilde{\square}_{\mathrm{AdS}^{15}(\mathbb{O})}}(r,u)\sin^6 u\,\mathrm{d}u=\int_0^\infty h(-\mathrm{i}u)q_{t,15}(\cosh r\cosh u)\sinh^6 u\,\mathrm{d}u.$$

Therefore, coming back to (4.2), one infers that using the analytic extension of s_t one must have

$$\int_0^\pi s_t(\eta,u)p_t^{\square_{\mathrm{AdS}^{15}(\mathbb{O})}}(r,u)\sin^6 u\,\mathrm{d}u=\int_0^\infty s_t(\eta,-\mathrm{i}u)q_{t,15}(\cosh r\cosh u)\sinh^6 u\,\mathrm{d}u,$$

where $q_{t,15}$ is the Riemannian heat kernel on the real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). ■

4.2 Second integral representation

Proposition 4.2. *For $r\geq 0$, $\eta\in[0,\pi]$, and $t>0$ we have*

$$p_t(r,\eta)=\int_0^\pi\int_0^\infty G_t(\eta,\varphi,u)q_{t,9}(\cosh r\cosh u)\sin^5\varphi\,\mathrm{d}u\,\mathrm{d}\varphi.$$

where $q_{t,9}$ is the 9-dimensional Riemannian heat kernel on the hyperbolic space $\mathbb{H}^9$:

$$q_{t,9}(\cosh s):=\frac{\mathrm{e}^{-16t}}{(2\pi)^4\sqrt{4\pi t}}\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\mathrm{e}^{-s^2/4t},$$

and

$$G_t(\eta,\varphi,u)=\frac{15}{8}\sum_{m\geq 0}\mathrm{e}^{-(m(m+6)+33)t}(\cos\eta+\mathrm{i}\sin\eta\cos\varphi)^m\cosh((m+3)u).\tag{4.3}$$

Proof. The strategy of the following method appeals to some results proved in [8]. Firstly, we decompose the subelliptic heat kernel in the η variable with respect to the basis of normalized eigenfunctions of $\tilde{\triangle}_{S^7}=\frac{\partial^2}{\partial\eta^2}+6\cot\eta\frac{\partial}{\partial\eta}$. Accordingly,

$$p_t(r,\eta)=\sum_{m\geq 0}f_m(t,r)h_m(\eta),$$

where for each m, h_m is given by

$$h_m(\eta)=\frac{15}{16}\int_0^\pi(\cos\eta+\mathrm{i}\sin\eta\cos\varphi)^m\sin^5\varphi\,\mathrm{d}\varphi$$

and $f_m(t,\cdot)$ solves the following heat equation

$$\frac{\partial}{\partial t}f_m(t,r)=\left(\frac{\partial^2}{\partial r^2}+(7\coth r+7\tanh r)\frac{\partial}{\partial r}-m(m+6)\tanh^2 r\right)f_m(t,r)$$
$$=\left(\frac{\partial^2}{\partial r^2}+(7\coth r+7\tanh r)\frac{\partial}{\partial r}+\frac{m(m+6)}{\cosh^2 r}-m(m+6)\right)f_m(t,r).$$

We consider then the operator

$$L_m:=\frac{\partial^2}{\partial r^2}+(7\coth r+7\tanh r)\frac{\partial}{\partial r}+\frac{m(m+6)}{\cosh^2 r}+49,$$

which was studied in [8, p. 229]. From [8, Theorem 2], with $\alpha=3+\frac{m}{2},\beta=-\frac{m}{2}$, we deduce that the solution to the wave Cauchy problem associated with the subelliptic Laplacian is given $f\in C_0^\infty\left(\mathbb{O}H^1\right)$ by

$$\cos\left(s\sqrt{-L_m}\right)(f)(w)=\frac{-\sinh s}{(2\pi)^4}\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\int_{\mathbb{O}H^1}K_m(s,w,y)f(y)\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8},$$

where

$$K_m(s,w,y)=\frac{(1-\overline{\langle w,y\rangle})^{3+m/2}}{(1-\langle w,y\rangle)^{m/2}}\frac{1}{\cosh^3(d(w,y))\sqrt{\cosh^2(s)-\cosh^2(d(w,y))}}$$
$$\times\,{}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{\cosh(d(w,y))-\cosh(s)}{2\cosh(d(w,y))}\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and $\mathrm{d}y$ stands for the Lebesgue measure in $\mathbb{R}^8$. Using the spectral formula

$$\mathrm{e}^{tL}=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\mathrm{e}^{-s^2/(4t)}\cos\left(s\sqrt{-L}\right)\mathrm{d}s,$$

which holds for any non positive self-adjoint operator, we deduce that the solution to the heat Cauchy problem associated with L_m:

$$\mathrm{e}^{tL_m}(f)(w)=\frac{\mathrm{e}^{-m(m+6)t-7^2t}}{\sqrt{4\pi t}(2\pi)^4}\int_{\mathbb{R}}\mathrm{d}s(-\sinh s)\mathrm{e}^{-s^2/(4t)}$$
$$\times\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\int_{\mathbb{O}H^1}K_m(s,w,y)f(y)\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8}.$$

Performing integration by parts 4-times,

$$\int_{\mathbb{R}}\mathrm{d}s(-\sinh s)\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\mathrm{e}^{-s^2/(4t)}\int_{\mathbb{O}H^1}K_m(s,w,y)f(y)\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8}$$
$$=\int_{\mathbb{O}H^1}f(y)\frac{\mathrm{d}y}{(1-||y||^2)^8}\int_{\mathbb{R}}\mathrm{d}s(-\sinh s)K_m(s,w,y)\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\mathrm{e}^{-s^2/4t}$$
$$=2\int_{\mathbb{O}H^1}f(y)\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8}\int_{d(w,y)}^{\infty}\mathrm{d}(\cosh(s))K_m(s,w,y)\left(\frac{1}{\sinh s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^4\mathrm{e}^{-s^2/4t}.$$

Thus we get

$$\mathrm{e}^{tL_m}(f)(0)=2\mathrm{e}^{-(m(m+6)+33)t}\int_{\mathbb{O}H^1}f(y)\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8}\int_{d(0,y)}^{\infty}\mathrm{d}(\cosh s)K_m(s,0,y)q_{t,9}(\cosh s).$$

As a result, the subelliptic heat kernel of L_m reads

$$\frac{\mathrm{d}y}{\left(1-||y||^2\right)^8}\int_{d(0,y)}^{\infty}\mathrm{d}(\cosh s)K_m(s,0,y)q_{t,9}(\cosh s)$$
$$=\mathrm{d}r\sinh^7 r\cosh^7 r\int_r^{\infty}\mathrm{d}(\cosh s)K_m(s,0,y)q_{t,9}(\cosh s).$$

By changing the variable $\cosh s=\cosh r\cosh u$ for $u\geq 0$, the last expression becomes

$$\mathrm{d}r\sinh^7 r\cosh^7 r\int_0^{\infty}{}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-\cosh u}{2}\right)q_{t,9}(\cosh r\cosh u)\,\mathrm{d}u.$$

Therefore $p_t(r,\eta)$ has the integral representation

$$2\sum_{m\geq 0}\mathrm{e}^{-(m(m+6)+33)t}h_m(\eta)\int_0^{\infty}{}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-\cosh u}{2}\right)q_{t,9}(\cosh r\cosh u)\,\mathrm{d}u.$$

Now, notice that ${}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-\cosh u}{2}\right)$ is simply the Cheybyshev polynomial of the first kind

$$T_{m+3}(x)={}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-x}{2}\right),$$

for all $x\in\mathbb{C}$. Therefore, one has

$${}_2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-\cosh u}{2}\right)=T_{m+3}(\cosh u)=\cosh((m+3)u),$$

and the proof is over. ■

Acknowledgements

F.B. is partially funded by the NSF grant DMS-1901315.

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