

The Differential Geometry of the Orbit Space of Extended Affine Jacobi Group A_1

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Abstract. We define certain extensions of Jacobi groups of A_1 , prove an analogue of Chevalley theorem for their invariants, and construct a Dubrovin–Frobenius structure on its orbit space.

Key words: Dubrovin–Frobenius manifolds; Hurwitz spaces; extended Jacobi groups

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Dedicated to the memory of Professor Boris Dubrovin

1 Introduction

Dubrovin–Frobenius manifold is a geometric interpretation of a remarkable system of differential equations called WDVV equations [6]. Since early nineties, there has been a continuous exchange of ideas from fields that are not trivially related to each other, such as topological quantum field theory, non-linear waves, singularity theory, random matrices theory, integrable systems, and Painlevé equations. Dubrovin–Frobenius manifolds theory is a bridge between them.

1.1 Orbit space of reflection groups and its extensions

In [6], Dubrovin pointed out that WDVV solutions with certain good analytic properties are related with partition functions of TFT. Afterwards, Dubrovin conjectured that WDVV solutions with certain good analytic properties are in one to one correspondence with discrete groups. This conjecture is supported by ideas which come from singularity theory, because in this setting there exists an integrable systems/discrete group correspondence. Furthermore, in minimal models, such as Gepner chiral rings, there exists a correspondence between physical models and discrete groups. In [14], Hertling proved that a particular class of Dubrovin–Frobenius manifold, called polynomial Dubrovin–Frobenius manifold, is isomorphic to the orbit space of a finite Coxeter group, which are spaces such that their geometric structure is invariant under the finite Coxeter group. In [2, 3, 6, 7, 9, 10, 24], there are many examples of WDVV solutions that are associated with orbit spaces of natural extensions of finite Coxeter groups, such as extended affine Weyl groups, and Jacobi groups. Therefore, the construction of Dubrovin–Frobenius manifolds on orbit space of reflection groups and its extensions is a prospective project of the classification of WDVV solutions. In addition, WDDV solutions arising from orbit spaces may also have some applications in TFT or some combinatorial problem, because previously these relationships were demonstrated in some examples, such as the orbit space of the finite Coxeter group A_1 , and the extended affine Weyl group A_1 [8, 11].

1.2 Hurwitz space/orbit space correspondence

There are several other non-trivial connections that Dubrovin–Frobenius manifolds theory can make. For example, Hurwitz spaces is the one of the main sources of examples of Dubrovin–Frobenius manifolds. Hurwitz spaces H_{g,n_0,n_1,\dots,n_m} are moduli space of covering over \mathbb{CP}^1 with a fixed ramification profile. More specifically, H_{g,n_0,n_1,\dots,n_m} is moduli space of pairs

$$\{C_g, \lambda: C_g \mapsto \mathbb{CP}^1\},$$

where C_g is a compact Riemann surface of genus g and λ is meromorphic function with poles in

$$\lambda^{-1}(\infty) = \{\infty_0, \infty_1, \dots, \infty_m\}.$$

Moreover, λ has degree $n_i + 1$ near ∞_i . Hurwitz space, with a choice of a specific Abelian differential, called quasi-momentum or primary differential, give rise to a Dubrovin–Frobenius manifold; see section [6, 19] for details. In some examples, the Dubrovin–Frobenius structure of Hurwitz spaces are isomorphic to Dubrovin–Frobenius manifolds associated with orbit spaces of suitable groups. For instance, the orbit space of the finite Coxeter group A_1 is isomorphic to the Hurwitz space $H_{0,1}$. Furthermore, orbit space of the extended affine Weyl group \tilde{A}_1 and of the Jacobi group $\mathcal{J}(A_1)$ are isomorphic to the Hurwitz spaces $H_{0,0,0}$ and $H_{1,1}$ respectively. Motivated by these examples, we construct the following diagram

$$\begin{array}{ccc} H_{0,1} \cong \text{orbit space of } A_1 & \xrightarrow{1} & H_{0,0,0} \cong \text{orbit space of } \tilde{A}_1 \\ \downarrow 2 & & \downarrow 4 \\ H_{1,1} \cong \text{orbit space of } \mathcal{J}(A_1) & \xrightarrow{3} & H_{1,0,0} \cong ? \end{array}$$

From the Hurwitz space side, the vertical lines 2 and 4 mean that we increase the genus by 1, and the horizontal lines mean that we split one pole of order 2 into two simple poles. From the orbit space side, the vertical line 2 means that we are doing an extension from the finite Coxeter group A_1 to the Jacobi group $\mathcal{J}(A_1)$; the line horizontal line 1 means that we are extending the orbit space of A_1 to the extended affine Weyl group \tilde{A}_1 . Therefore, one might ask if the line 3 and 4 would imply an orbit space interpretation of the Hurwitz space $H_{1,0,0}$. The main goal of this paper is to define a new class of groups such that its orbit space carries the Dubrovin–Frobenius structure of $H_{1,0,0}$. The new group is called extended affine Jacobi group A_n , and is denoted by $\mathcal{J}(\tilde{A}_1)$. This group is an extension of the Jacobi group $\mathcal{J}(A_1)$ and of the extended affine Weyl group \tilde{A}_1 .

1.3 Results

The main goal of this paper is to construct the Dubrovin–Frobenius structure of the Hurwitz space $H_{1,0,0}$ from the data of the group $\mathcal{J}(\tilde{A}_1)$. In other words, we derive the WDVV solution associated to the group $\mathcal{J}(\tilde{A}_1)$ without using the correspondent Hurwitz space construction. First of all, recall the definition of WDVV equation:

Definition 1.1. The function $F(t)$, $t = (t^1, t^2, \dots, t^n)$ is a solution of a WDVV equation if its third derivatives

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{1.1}$$

satisfy the following conditions:

- 1) $\eta_{\alpha\beta} = c_{1\alpha\beta}$ is constant nondegenerate matrix;

2) the function

$$c_{\alpha\beta}^{\gamma} = \eta^{\gamma\delta} c_{\alpha\beta\delta}$$

is structure constant of associative algebra;

3) $F(t)$ must be quasi-homogeneous function

$$F(c^{d_1} t^1, \dots, c^{d_n} t^n) = c^{d_F} F(t^1, \dots, t^n)$$

for any nonzero c and for some numbers d_1, \dots, d_n, d_F .

Our goal is to extract a WDVV equation from the data of a suitable group $\mathcal{J}(\tilde{A}_1)$. We define the group $\mathcal{J}(\tilde{A}_1)$. Recall that the group A_1 acts on $\mathbb{C} \ni v_0$ by reflections

$$v_0 \mapsto -v_0.$$

The group $\mathcal{J}(\tilde{A}_1)$ is an extension of the group A_1 in the following sense:

Proposition 1.2. *The group $\mathcal{J}(\tilde{A}_1) \ni (w, t, \gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v, \tau) = (u, v_0, v_2, \tau)$ as follows:*

$$\begin{aligned} w(u, v, \tau) &= (u, wv, \tau), \\ t(u, v, \tau) &= \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \tau, v + \lambda\tau + \mu, \tau \right), \\ \gamma(u, v, \tau) &= \left(u + \frac{c \langle v, v \rangle_{\tilde{A}_1}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \end{aligned}$$

where $w \in A_1$ acts by reflection in the first v_0 variables of $\mathbb{C}^2 \ni v = (v_0, v_2)$,

$$t = (\lambda, \mu) \in \mathbb{Z}^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \langle v, v \rangle_{\tilde{A}_1} = 2v_0^2 - 2v_2^2.$$

See Section 2.1 for details.

In order to define any geometric structure in an orbit space, first it is necessary to define a notion of invariant $\mathcal{J}(\tilde{A}_1)$ sections. For this purpose, we generalise the ring of invariant functions used in [2, 3] for the group $\mathcal{J}(A_1)$, which are called Jacobi forms. This notion was first defined in [12] by Eichler and Zagier for the group $\mathcal{J}(A_1)$, and it was further generalised for the group $\mathcal{J}(A_1)$ in [23] by Wirthmuller. Furthermore, an explicit base of generators were derived in [2, 3] by Bertola. The Jacobi forms used in this thesis are defined by:

Definition 1.3. The weak \tilde{A}_1 -invariant Jacobi forms of weight k , order l , and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+2} \oplus \mathbb{H} \ni (u, v_0, v_2, \tau) = (u, v, \tau)$ which satisfy

$$\begin{aligned} \varphi(w(u, v, \tau)) &= \varphi(u, v, \tau), & A_1\text{-invariant condition,} \\ \varphi(t(u, v, \tau)) &= \varphi(u, v, \tau), \\ \varphi(\gamma(u, v, \tau)) &= (c\tau + d)^{-k} \varphi(u, v, \tau), \\ E\varphi(u, v, \tau) &:= -\frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi(u, v, \tau) = m\varphi(u, v, \tau), & \text{Euler vector field.} \end{aligned} \tag{1.2}$$

Moreover, the weak \tilde{A}_1 -invariant Jacobi forms are meromorphic in the variable v_2 on a fixed divisor, in contrast with the Jacobi forms of the group $\mathcal{J}(A_1)$, which are holomorphic in each variable; see details on Section 2.2. The ring of weak \tilde{A}_1 -invariant Jacobi forms gives the notion

of the Euler vector field; indeed, the vector field defined in the last equation of (1.2) measures the degree of the Jacobi forms, which coincides with the index. The differential geometry of the orbit space of the group $\mathcal{J}(\tilde{A}_1)$ should be understood as the space such that its sections are written in terms of Jacobi forms. Then, in order for this statement to make sense, we must prove a Chevalley type theorem, which is:

Theorem 1.4. *The trigraded algebra of Jacobi forms $J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_1}$ is freely generated by 2 fundamental Jacobi forms (φ_0, φ_1) over the graded ring $E_{\bullet,\bullet}$*

$$J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = E_{\bullet,\bullet}[\varphi_0, \varphi_1],$$

where

$$E_{\bullet,\bullet} = J_{\bullet,\bullet,0}$$

is the ring of coefficients.

More specifically, the ring of function $E_{\bullet,\bullet}$ is the space of functions $f(v_2, \tau)$ such that, for fixed τ , the functions $\tau \mapsto f(v_2, \tau)$ is an elliptic function.

Moreover, (φ_0, φ_1) are given by

Corollary 1.5. *The function*

$$\left[e^{z \frac{\partial}{\partial p}} \left(e^{2\pi i u} \frac{\theta_1(v_0 + v_2 + p)\theta_1(-v_0 + v_2 + p)}{\theta_1(2v_2 + p)\theta_1'(0)} \right) \right] \Big|_{p=0} = \varphi_1^{\mathcal{J}(\tilde{A}_1)} + \varphi_0^{\mathcal{J}(\tilde{A}_1)} z + O(z^2),$$

generates the Jacobi forms $\varphi_0^{\mathcal{J}(\tilde{A}_1)}$ and $\varphi_1^{\mathcal{J}(\tilde{A}_1)}$, where

$$\varphi_0^{\mathcal{J}(\tilde{A}_1)} := \frac{\partial}{\partial p} (\hat{\varphi}_1^{\mathcal{J}(\tilde{A}_1)}) \Big|_{p=0}.$$

This lemma realises the functions $(\varphi_0, \varphi_1, v_2, \tau)$ as coordinates of the orbit space of $\mathcal{J}(\tilde{A}_1)$. The unit vector field is chosen to be

$$e = \frac{\partial}{\partial \varphi_0}, \tag{1.3}$$

because φ_0 is the basic generator with maximum weight degree; see the Section 2.2 for details.

The last component we need to construct is the intersection form of the orbit space of $\mathcal{J}(\tilde{A}_1)$. The natural candidate to be such a metric is the invariant metric of the group $\mathcal{J}(\tilde{A}_1)$, which given by

$$g = 2dv_0^2 - 2dv_2^2 + 2dud\tau. \tag{1.4}$$

From the data of the intersection form (1.4), is possible to derive a second flat metric of the orbit space $\mathcal{J}(\tilde{A}_1)$. The second metric is given by

$$\eta^* := \text{Lie}_e g^*,$$

and it is denoted by the Saito metric due to K. Saito, who was the first to define this metric for the case of finite Coxeter group [18]. One of the main technical problems of this paper is to prove that the Saito metric η^* is flat. At this point, we can state our main result.

Theorem 1.6. *A suitable covering of the orbit space $(\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H}) / \mathcal{J}(\tilde{A}_1)$ with the intersection form (1.4), unit vector field (1.3), and Euler vector field given by the last equation of (1.2) has a Dubrovin–Frobenius manifold structure. Moreover, a suitable covering of $\mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H} / \mathcal{J}(\tilde{A}_1)$ is isomorphic as Dubrovin–Frobenius manifold to a suitable covering of the Hurwitz space $H_{1,0,0}$.*

See Section 3.4 for details. In particular, we derive explicitly the WDVV solution associated with the orbit space of $\mathcal{J}(\tilde{A}_1)$, which is given by

$$F(t^1, t^2, t^3, t^4) = \frac{i}{4\pi} (t^1)^2 t^4 - 2t^1 t^2 t^3 - (t^2)^2 \log \left(t^2 \frac{\theta_1'(0, t^4)}{\theta_1(2t^3, t^4)} \right),$$

where

$$\theta_1(v, \tau) = 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi i \tau (n + \frac{1}{2})^2} \sin((2n + 1)v). \quad (1.5)$$

The results of this paper are important because of the following:

1. The Hurwitz spaces $H_{1,0,0}$ are classified by the group $\mathcal{J}(\tilde{A}_1)$, hence we increase the knowledge of the WDVV/discrete group correspondence. Recently, the case $\mathcal{J}(\tilde{A}_1)$ attracted the attention of experts, due to its application in integrable systems [5, 13, 16].
2. The orbit space construction of the group $\mathcal{J}(\tilde{A}_1)$ can be generalised to the group $\mathcal{J}(\tilde{A}_n)$; see the definition in [1]. Further, the same can be done to the other classical finite Coxeter groups as B_n, D_n . Hence, these orbit spaces could give rise to a new class of Dubrovin–Frobenius manifolds. Furthermore, the associated integrable hierarchies of this new class of Dubrovin–Frobenius manifolds could have applications in Gromov–Witten theory and combinatorics.

This paper is organised in the following way: In Section 2, we define extended affine Jacobi group $\mathcal{J}(\tilde{A}_1)$ and we prove some results related with its ring of invariant functions. In Section 3, we construct a Dubrovin–Frobenius structure on the orbit spaces of $\mathcal{J}(\tilde{A}_1)$ and compute its free-energy. Furthermore, we show that the orbit space of the group $\mathcal{J}(\tilde{A}_1)$ is isomorphic, as a Dubrovin–Frobenius manifold, to the Hurwitz–Frobenius manifold $\tilde{H}_{1,0,0}$ [6, 19]. See Theorem 3.7 for details.

2 Invariant theory of $\mathcal{J}(\tilde{A}_1)$

The focus of this section is to define a new extension of the finite Coxeter group A_1 such that it contains the affine Weyl group \tilde{A}_1 and the Jacobi group $\mathcal{J}(A_1)$. This new extension will be denoted by Extended affine Jacobi group $\mathcal{J}(\tilde{A}_1)$. Further, we prove that, from the data of the group $\mathcal{J}(\tilde{A}_1)$, we can reconstruct the Dubrovin–Frobenius structure of the Hurwitz space $H_{1,0,0}$ on the orbit space of $\mathcal{J}(\tilde{A}_1)$. The advantage of this orbit space construction is the Chevalley Theorem 2.25, which gives a global interpretation for orbit space of $\mathcal{J}(\tilde{A}_1)$. Furthermore, it attaches the group $\mathcal{J}(\tilde{A}_1)$ to the Hurwitz space $H_{1,0,0}$, and this fact might be useful in the general understanding of WDVV/group correspondence. These results are interesting because the Hurwitz space $H_{1,0,0}$ is well known to have a rich Dubrovin–Frobenius structure, called a tri-Hamiltonian structure [16] and [15]. This fact realises the orbit space of $\mathcal{J}(\tilde{A}_1)$ as suitable ambient space for Dubrovin–Frobenius submanifolds. Furthermore, it shows an interesting relationship between the integrable systems of the ambient space and the integrable systems of its Dubrovin–Frobenius submanifolds.

2.1 The group $\mathcal{J}(\tilde{A}_1)$

The main goal of this section is to motivate and to define the group $\mathcal{J}(\tilde{A}_1)$. In order to do that, it will be necessary to recall the definition of the group A_1 , and some of its extensions. Moreover, its goal is to understand how to derive WDDV solution starting from these groups.

The group A_n acts on the space $\Omega^{A_n} = \{(v_0, v_1, \dots, v_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^n v_i = 0\}$ by permutations:

$$(v_0, v_1, \dots, v_n) \mapsto (v_{i_0}, v_{i_1}, \dots, v_{i_n}). \quad (2.1)$$

Let us concentrate on the simplest possible case, i.e., $n = 1$. In this case, the action on $\mathbb{C} \cong \Omega^{A_1}$ is just:

$$v_0 \mapsto -v_0.$$

The understanding of the orbit space of A_1 requires a Chevalley theorem for the ring of invariants. The Chevalley theorem from the group A_n says that

Theorem 2.1 ([4]). *Let the Coxeter group A_n which acts on $\Omega^{A_n} \ni (v_0, v_1, \dots, v_n)$ as (2.1), then*

$$\mathbb{C}[v_0, v_1, \dots, v_n]^{A_n} \cong \mathbb{C}[a_2, a_3, \dots, a_{n+1}],$$

where a_i are weighted homogeneous polynomials of degree i .

In the A_1 case, the ring of invariants is just

$$\mathbb{C}[v_0^2] \cong \mathbb{C}[a_2],$$

then the orbit space of A_1 is just the

$$\text{Spec}(\mathbb{C}[v_0^2]).$$

In the papers [6, 7], it was demonstrated that \mathbb{C}/A_1 has structure of Dubrovin–Frobenius manifold. Furthermore, it is isomorphic to the Hurwitz space $H_{0,1}$, i.e., the space of rational functions with a double pole. The isomorphism can be realized by the following map:

$$[v_0] \mapsto \lambda^{A_1}(p, v_0) = (p - v_0)(p + v_0) = p^2 + a_2.$$

Note that the isomorphism works, because $\lambda^{A_1}(p, v_0)$ is invariant under the A_1 -action. Applying the methods developed in [6, 7], one can show that the WDVV solution associated with this orbit space is

$$F(t^1) = \frac{(t^1)^3}{6},$$

where t^1 is the flat coordinate of the metric η .

In [6, 10] it was also considered the extended affine A_1 that is denoted by \tilde{A}_1 . The action on

$$(L^{A_1} \otimes \mathbb{C}) \oplus \mathbb{C} = \left\{ (v_0, v_1, v_2) \in \mathbb{C}^3 : \sum_{i=0}^2 v_i = 0 \right\}$$

is

$$v_0 \mapsto \pm v_0 + \mu_0, \quad v_2 \mapsto v_2 + \mu_2,$$

where $\mu_0, \mu_2 \in \mathbb{Z}$.

A notion of the invariant ring for the group extended affine A_n was defined in [10], and Dubrovin and Zhang proved that this invariant ring for the case \tilde{A}_1 is isomorphic to

$$\mathbb{C}[e^{2\pi i v_2} \cos(2\pi i v_0), e^{2\pi i v_2}].$$

Therefore, the orbit space of \tilde{A}_1 is the weight projective variety associated with

$$\text{Spec}(\mathbb{C}[e^{2\pi i v_2} \cos(2\pi i v_0), e^{2\pi i v_2}]).$$

Further, a Dubrovin–Frobenius manifold structure was built on the orbit space of \tilde{A}_1 with the following WDVV solution:

$$F(t^1, t^2) = \frac{(t^1)^2 t^2}{2} + e^{t^2}. \quad (2.2)$$

The orbit space of \tilde{A}_1 is also associated with a Hurwitz space, but the relation is slightly less straightforward. The first step is to consider the following map:

$$[v_0, v_2] \mapsto \lambda^{\tilde{A}_1}(p, v_0, v_2) = e^p + e^{2\pi i v_2} \cos(2\pi i v_0) + e^{2\pi i v_2} e^{-p}.$$

The second is to consider the Legendre transformation of S_2 type [6, Appendix B and Chapter 5]. Consider

$$b = e^{2\pi i v_2} \cos(2\pi i v_0), \quad a = e^{2\pi i v_2},$$

and the following choice of primary differential $d\tilde{p}$ implicitly given by

$$dp = \frac{d\tilde{p}}{\tilde{p} - b}.$$

Then, in these new coordinates $\lambda^{\tilde{A}_1}$, is given by

$$\lambda(\tilde{p}, a, b) = \tilde{p} + \frac{a}{\tilde{p} - b}.$$

Hence, the orbit space of \tilde{A}_1 is isomorphic to the Hurwitz space $H_{0,0,0}$, i.e., space of fractional functions with two simple poles.

The next example of group to be considered is the Jacobi group $\mathcal{J}(A_1)$, which acts on

$$\Omega^{\mathcal{J}(A_1)} := (L^{A_1} \otimes \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{H} = \left\{ (v_0, v_1, u, \tau) \in \mathbb{C}^3 \oplus \mathbb{H} : \sum_{i=0}^1 v_i \in \mathbb{Z} + \tau\mathbb{Z} \right\}$$

as follows:

A_1 -action:

$$v_0 \mapsto -v_0, \quad u \mapsto u, \quad \tau \mapsto \tau. \quad (2.3)$$

Translation:

$$v_0 \mapsto v_0 + \mu_0 + \lambda_0 \tau, \quad u \mapsto u - \lambda_0 v_0 - \frac{\lambda_0^2}{2} \tau, \quad \tau \mapsto \tau, \quad (2.4)$$

where $\mu_0, \lambda_0 \in \mathbb{Z}$.

$\text{SL}_2(\mathbb{Z})$ -action:

$$v_0 \mapsto \frac{v_0}{c\tau + d}, \quad u \mapsto u - \frac{cv_0^2}{2(c\tau + d)}, \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad (2.5)$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$.

The notion of invariant ring of $\mathcal{J}(A_1)$ was first defined in [12]. However, the definitions stated in [2, 3, 23] are more suitable for this purpose.

Definition 2.2. The weak A_1 -invariant, Jacobi forms of weight k , and index m are holomorphic functions on $\Omega = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \ni (u, v_0, \tau)$ which satisfy

$$\begin{aligned} \varphi(u, -v_0, \tau) &= \varphi(u, v_0, \tau), & A_1\text{-invariant condition,} \\ \varphi\left(u - \lambda_0 v_0 - \frac{\lambda_0^2}{2}\tau, v_0 + \lambda_0\tau + \mu, \tau\right) &= \varphi(u, v_0, \tau), \\ \varphi\left(u + \frac{cv_0^2}{2(c\tau + d)}, \frac{v_0}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^k \varphi(u, v_0, \tau), \\ E\varphi(u, v_0, \tau) &:= \frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi(u, v_0, \tau) = m\varphi(u, v_0, \tau). \end{aligned}$$

Moreover, φ are locally bounded functions of v_0 as $\Im(\tau) \mapsto +\infty$ (weak condition).

The space of \tilde{A}_1 -invariant Jacobi forms of weight k , and index m is denoted by $J_{k,m}^{A_1}$, and $J_{\bullet,\bullet}^{\mathcal{J}(A_1)} = \bigoplus_{k,m} J_{k,m}^{A_1}$ is the space of Jacobi forms A_1 -invariant.

In [12], it was proved the following a version of the Chevalley theorem.

Theorem 2.3. Let $J_{\bullet,\bullet}^{\mathcal{J}(A_1)}$ the ring of Jacobi forms A_1 -invariant, then

$$J_{\bullet,\bullet}^{\mathcal{J}(A_1)} \cong M_{\bullet}[\varphi_0, \varphi_2],$$

where M_{\bullet} is the ring of holomorphic modular forms, and

$$\varphi_2 = e^{2\pi i u} \left(\frac{\theta_1(v_0, \tau)}{\theta_1'(0, \tau)} \right)^2, \quad \varphi_0 = \varphi_2 \wp(v_0, \tau),$$

θ_1 is the Jacobi θ_1 -function (1.5), and \wp is the Weierstrass P-function, which is defined as

$$\wp(v, \tau) = \frac{1}{v^2} + \sum_{m^2+n^2 \neq 0}^{\infty} \frac{1}{(v - m - n\tau)^2} - \frac{1}{(m + n\tau)^2}. \quad (2.6)$$

Note that this Chevalley theorem is slightly different from the others. The ring of the coefficients is the ring of holomorphic of modular forms, instead of just \mathbb{C} . The geometric interpretation of this fact is that the orbit space of $\mathcal{J}(A_1)$ is a line bundle, such that its base is family of elliptic curves E_{τ} quotient by the group A_1 parametrised by $\mathbb{H}/\text{SL}_2(\mathbb{Z})$. In [2] and [3], it was proved that orbit space of $\mathcal{J}(A_1)$ has a Dubrovin–Frobenius structure. Furthermore, the orbit space of $\mathcal{J}(A_1)$ is isomorphic to $H_{1,1}$, i.e., space of elliptic functions with one double pole. The explicit isomorphism is given by the map

$$[(u, v_0, \tau)] \mapsto \lambda^{\mathcal{J}(A_1)}(v, u, v_0, \tau) = e^{2\pi i u} \frac{\theta_1(v - v_0, \tau) \theta_1(v + v_0, \tau)}{\theta_1^2(v, \tau)}. \quad (2.7)$$

As in the A_1 case, the isomorphism is only possible, because the map (2.7) is invariant under (2.3)–(2.5). A WDVV solution for this case is the following:

$$F(t^1, t^2, \tau) = \frac{(t^1)^2 \tau}{2} + \frac{t^1 (t^2)^2}{2} - \frac{\pi i (t^2)^2}{48} E_2(\tau), \quad (2.8)$$

where

$$E_2(\tau) = 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(m + n\tau)^2}.$$

A remarkable fact in these orbit space constructions is its correspondence with Hurwitz spaces, which can be summarized by the following diagram:

$$\begin{array}{ccc} H_{0,1} \cong \mathbb{C}/A_1 & \xrightarrow{1} & H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1 \\ \downarrow 2 & & \downarrow 4 \\ H_{1,1} \cong (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H})/\mathcal{J}(A_1) & \xrightarrow{3} & H_{1,0,0} \cong ? \end{array}$$

The arrows of the diagram above have a double meaning. The first one is simply an extension of the group, the arrow 2 is ‘‘Jacobi’’ extension, and the arrow 1 is ‘‘affine’’ extension. The second meaning is related to the Hurwitz space side: the arrows 2 and 4 increase by one the genus, and the arrows 1 and 3 split a double pole in 2 simple poles. The missing part of the diagram is exactly the orbit space counter part of $H_{1,0,0}$. The diagram suggest that the new group should be an extension of the A_1 group, such that combine the groups \tilde{A}_1 , and $\mathcal{J}(A_1)$, furthermore, it should preserve $H_{1,0,0}$ in a similar way for what was done in (2.7). To construct the desired group, we start from the group $\mathcal{J}(A_1)$ and make an extension in order to incorporate the \tilde{A}_1 group. Concretely, we extend the domain $\Omega^{\mathcal{J}(A_1)}$ to

$$\Omega^{\mathcal{J}(\tilde{A}_1)} := \Omega^{A_1} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} = \{(v_0, v_1, v_2, u, \tau) \in \mathbb{C}^4 \oplus \mathbb{H} : v_0 + v_1 \in \mathbb{Z} \oplus \tau\mathbb{Z}\},$$

and we extend the group action $\mathcal{J}(A_1)$ to the following action:

A_1 -action:

$$v_0 \mapsto -v_0, \quad v_2 \mapsto v_2, \quad u \mapsto u, \quad \tau \mapsto \tau. \quad (2.9)$$

Translation:

$$\begin{aligned} v_0 &\mapsto v_0 + \mu_0 + \lambda_0\tau, & v_2 &\mapsto v_2 + \mu_2 + \lambda_2\tau, \\ u &\mapsto u - 2\lambda_0v_0 + 2\lambda_2v_2 - \lambda_0^2\tau + \lambda_2^2\tau + k, & \tau &\mapsto \tau, \end{aligned} \quad (2.10)$$

where $(\lambda_0, \lambda_2), (\mu_0, \mu_2) \in \mathbb{Z}^2$, and $k \in \mathbb{Z}$.

$\mathrm{SL}_2(\mathbb{Z})$ -action:

$$v_0 \mapsto \frac{v_0}{c\tau + d}, \quad v_2 \mapsto \frac{v_2}{c\tau + d}, \quad u \mapsto u + \frac{c(v_0^2 - v_2^2)}{(c\tau + d)}, \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad (2.11)$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$.

The group action (2.9), (2.10), and (2.11) is called extended affine Jacobi group A_1 , and is denoted by $\mathcal{J}(\tilde{A}_1)$.

Remark 2.4. The translations of the group \tilde{A}_1 are a subgroup of the translations of the group $\mathcal{J}(\tilde{A}_1)$. Therefore, it is in that sense that $\mathcal{J}(\tilde{A}_1)$ is a combination of \tilde{A}_1 and $\mathcal{J}(A_1)$.

In order to rewrite the action of $\mathcal{J}(\tilde{A}_1)$ in an intrinsic way, consider the A_1 in the following extended space

$$L^{\tilde{A}_1} = \left\{ (z_0, z_1, z_2) \in \mathbb{Z}^3 : \sum_{i=0}^3 z_i = 0 \right\}.$$

The action of A_1 on $L^{\tilde{A}_1}$ is given by

$$w(z_0, z_1, z_2) = (z_1, z_0, z_2)$$

permutations in the first two variables. Moreover, A_1 also acts on the complexification of $L^{\tilde{A}_1} \otimes \mathbb{C}$. Let us use the following identification $\mathbb{Z}^2 \cong L^{\tilde{A}_1}$, $\mathbb{C}^2 \cong L^{\tilde{A}_1} \otimes \mathbb{C}$, which is possible due to the maps

$$(v_0, v_2) \mapsto (v_0, -v_0, v_2), \quad (v_0, v_1, v_2) \mapsto (v_0, v_2).$$

The action of A_1 on $\mathbb{C}^2 \ni v = (v_0, v_2)$ is:

$$w(v) = w(v_0, v_2) = (-v_0, v_2).$$

Let the quadratic form $\langle \cdot, \cdot \rangle_{\tilde{A}_1}$ be given by

$$\langle v, v \rangle_{\tilde{A}_1} = v^T M_{\tilde{A}_1} v = v^T \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} v = 2v_0^2 - 2v_2^2. \quad (2.12)$$

Consider the following group $L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$ with the following group operation

$$\begin{aligned} \forall (\lambda, \mu, k), (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}, \\ (\lambda, \mu, k) \bullet (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{\tilde{A}_1}). \end{aligned}$$

Note that $\langle \cdot, \cdot \rangle_{\tilde{A}_1}$ is invariant under A_1 group, then A_1 acts on $L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$. Hence, we can take the semidirect product $A_1 \ltimes (L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z})$ given by the following product

$$\begin{aligned} \forall (w, \lambda, \mu, k), (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in A_1 \times L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}, \\ (w, \lambda, \mu, k) \bullet (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (w\tilde{w}, w\lambda + \tilde{\lambda}, w\mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{\tilde{A}_1}). \end{aligned}$$

Denoting $W(\tilde{A}_1) := A_1 \ltimes (L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z})$, we can define

Definition 2.5. The Jacobi group $\mathcal{J}(\tilde{A}_1)$ is defined as a semidirect product $W(\tilde{A}_1) \rtimes \text{SL}_2(\mathbb{Z})$. The group action of $\text{SL}_2(\mathbb{Z})$ on $W(\tilde{A}_1)$ is defined as

$$\begin{aligned} \text{Ad}_\gamma(w) &= w, \\ \text{Ad}_\gamma(\lambda, \mu, k) &= \left(a\mu - b\lambda, -c\mu + d\lambda, k + \frac{ac}{2} \langle \mu, \mu \rangle_{\tilde{A}_1} - bc \langle \mu, \lambda \rangle_{\tilde{A}_1} + \frac{bd}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \right) \end{aligned}$$

for $(w, t = (\lambda, \mu, k)) \in W(\tilde{A}_1)$, $\gamma \in \text{SL}_2(\mathbb{Z})$. Then the multiplication rule is given as follows

$$(w, t, \gamma) \bullet (\tilde{w}, \tilde{t}, \tilde{\gamma}) = (w\tilde{w}, t\text{Ad}_\gamma(w\tilde{t}), \gamma\tilde{\gamma}).$$

Then, the action of Jacobi group $\mathcal{J}(\tilde{A}_1)$ on $\Omega^{\mathcal{J}(\tilde{A}_1)} := \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \in (u, v, \tau)$ is described by the main three generators

$$\hat{w} = (w, 0, I_{\text{SL}_2(\mathbb{Z})}), \quad t = (I_{A_1}, \lambda, \mu, k, I_{\text{SL}_2(\mathbb{Z})}), \quad \gamma = \left(I_{A_1}, 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right),$$

which acts on $\Omega^{\mathcal{J}(\tilde{A}_1)}$ as follows

$$\begin{aligned} \hat{w}(u, v = (v_0, v_2), \tau) &= (u, -v_0, v_2, \tau), \\ t(u, v = (v_0, v_2), \tau) &= \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \tau + k, v_0 + \lambda_0 \tau + \mu_0, v_2 + \lambda_2 \tau + \mu_2, \tau \right), \\ \gamma(u, v = (v_0, v_2), \tau) &= \left(u + \frac{c \langle v, v \rangle_{\tilde{A}_1}}{2(c\tau + d)}, \frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \end{aligned}$$

where $\lambda, \mu, k \in L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$,

$$\lambda = (\lambda_0, \lambda_2), \quad \mu = (\mu_0, \mu_2).$$

In a more condensed form we have the following proposition.

Proposition 2.6. *The group $\mathcal{J}(\tilde{A}_1) \ni (\hat{w}, t, \gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v, \tau)$ as follows:*

$$\begin{aligned} \hat{w}(u, v, \tau) &= (u, w(v), \tau), \\ t(u, v, \tau) &= \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \tau + k, v + \lambda \tau + \mu, \tau \right), \\ \gamma(u, v, \tau) &= \left(u + \frac{c \langle v, v \rangle_{\tilde{A}_1}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \end{aligned} \quad (2.13)$$

Substituting (2.12) in (2.13), we get the transformation law (2.9), (2.10), and (2.11). The explanation of why (2.13) is that a group action for $\mathcal{J}(\tilde{A}_1)$ is just straightforward computations, but it is a bit long, so this part of the proof will be omitted.

2.2 Jacobi forms of $\mathcal{J}(\tilde{A}_1)$

In order to understand the differential geometry of orbit space, first we need to study the algebra of the invariant functions. Informally, every time that there is a group W acting on a vector space V , one could think of the orbit spaces V/W as V , but you should remember yourself one can only use the W -invariant sections of V . Hence, motivated by the definition of Jacobi forms of group A_n defined in [23], and used in the context of Dubrovin–Frobenius manifold in [2, 3], we present the following:

Definition 2.7. The weak \tilde{A}_1 -invariant Jacobi forms of weight k , order l , and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v_0, v_2, \tau) = (u, v, \tau)$ which satisfy

$$\begin{aligned} \varphi(w(u, v, \tau)) &= \varphi(u, v, \tau), \quad A_1\text{-invariant condition}, \\ \varphi(t(u, v, \tau)) &= \varphi(u, v, \tau), \\ \varphi(\gamma(u, v, \tau)) &= (c\tau + d)^{-k} \varphi(u, v, \tau), \\ E\varphi(u, v, \tau) &:= -\frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi(u, v_0, v_2, \tau) = m\varphi(u, v_0, v_2, \tau). \end{aligned} \quad (2.14)$$

Moreover,

- 1) φ is locally bounded functions of v_0 as $\Im(\tau) \mapsto +\infty$ (weak condition),
- 2) for fixed u, v_0, τ the function $v_2 \mapsto \varphi(u, v_0, v_2, \tau)$ is meromorphic with poles of order at most $l + 2m$ at in $v_2 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$,
- 3) for fixed $u, v_2 \neq 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$, τ the function $v_0 \mapsto \varphi(u, v_0, v_2, \tau)$ is holomorphic,
- 4) for fixed $u, v_0, v_2 \neq 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$. The function $\tau \mapsto \varphi(u, v_0, v_2, \tau)$ is holomorphic.

The space of \tilde{A}_1 -invariant Jacobi forms of weight k , order l , and index m are denoted by $J_{k,l,m}^{\tilde{A}_1}$, and $J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_1}$ is the space of Jacobi forms \tilde{A}_1 -invariant.

Remark 2.8. The condition $E\varphi(u, v_0, v_2, \tau) = m\varphi(u, v_0, v_2, \tau)$ implies that $\varphi(u, v_0, v_2, \tau)$ has the following form

$$\varphi(u, v_0, v_2, \tau) = f(v_0, v_2, \tau) e^{2\pi i m u}$$

and the function $f(v_0, v_2, \tau)$ has the following transformation law

$$\begin{aligned} f(v_0, v_2, \tau) &= f(-v_0, v_2, \tau), \\ f(v_0, v_2, \tau) &= e^{-2\pi i m \left(\langle \lambda, v \rangle + \frac{\langle \lambda, \lambda \rangle}{2} \tau \right)} f(v_0 + m_0 + n_0\tau, v_2 + m_2 + n_2\tau, \tau), \\ f(v_0, v_2, \tau) &= (c\tau + d)^{-k} e^{2\pi i m \left(\frac{c \langle v, v \rangle}{c\tau + d} \right)} f \left(\frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

The functions $f(v_0, v_2, \tau)$ are more closely related to the definition of Jacobi form of the Eichler–Zagier type [12]. The coordinate u works as kind of automorphic correction in this functions

$f(v_0, v_2, \tau)$. Further, the coordinate u will be crucial to construct an equivariant metric on the orbit space of $\mathcal{J}(\tilde{A}_1)$; see Section 3.

Remark 2.9. Note that the Jacobi forms in the Definition 2.2 are holomorphic, and in the Definition 2.7, the Jacobi forms are meromorphic in the variable v_2 .

The main result of this section is the following:

The ring of \tilde{A}_1 -invariant Jacobi forms is polynomial over a suitable ring $E_{\bullet, \bullet} := J_{\bullet, \bullet, 0}^{\mathcal{J}(\tilde{A}_1)}$ on suitable generators φ_0, φ_1 . Before stating precisely the theorem, I will define the objects $E_{\bullet, \bullet}, \varphi_0, \varphi_1$.

The ring $E_{\bullet, l} := J_{\bullet, l, 0}^{\mathcal{J}(\tilde{A}_1)}$ is the space of meromorphic Jacobi forms of index 0 with poles of order at most l at $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$, by definition. The sub-ring $J_{\bullet, 0, 0}^{\mathcal{J}(\tilde{A}_1)} \subset E_{\bullet, \bullet}$ has a nice structure, indeed:

Lemma 2.10. The sub-ring $J_{\bullet, 0, 0}^{\mathcal{J}(\tilde{A}_1)}$ is equal to $M_{\bullet} := \bigoplus M_k$, where M_k is the space of modular forms of weight k for the full group $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Using the Remark 2.8, we know that functions $\varphi(u, v_0, v_2, \tau) \in J_{\bullet, 0, 0}^{\mathcal{J}(\tilde{A}_1)}$ can not depend on u , so $\varphi(u, v_0, v_2, \tau) = \varphi(v_0, v_2, \tau)$. Moreover, for fixed v_2, τ the functions $v_0 \mapsto \varphi(v_0, v_2, \tau)$ are holomorphic elliptic functions. Therefore, by Liouville theorem, these function are constant in v_0 . Similar argument shows that these function do not depend on v_2 , because $l + 2m = 0$, i.e., there is no pole. Then, $\varphi = \varphi(\tau)$ are standard holomorphic modular forms. ■

Lemma 2.11. If $\varphi \in E_{\bullet, \bullet} = J_{\bullet, \bullet, 0}^{\mathcal{J}(\tilde{A}_1)}$, then φ depends only on the variables v_2, τ . Moreover, if $\varphi \in J_{0, l, 0}^{\mathcal{J}(\tilde{A}_1)}$ for fixed τ the function $v_2 \mapsto \varphi(v_2, \tau)$ is an elliptic function with poles of order at most l on $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$.

Proof. The proof is essentially the same of the Lemma 2.10, the only difference is that now we have poles at $v_2 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$. Hence, we have dependence on v_2 . ■

As a consequence of Lemma 2.11, the function $\varphi \in E_{k, l} = J_{k, l, 0}^{\mathcal{J}(\tilde{A}_1)}$ has the following form

$$\varphi(v_2, \tau) = f(\tau)g(v_2, \tau),$$

where $f(\tau)$ is holomorphic modular form of weight k , and for fixed τ , the function $v_2 \mapsto g(v_2, \tau)$ is an elliptic function of order at most l at the poles $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$.

At this stage, we are able to define φ_0, φ_1 . Note that a natural way to produce meromorphic Jacobi forms is by using rational functions of holomorphic Jacobi forms. Starting here, we will denote the Jacobi forms related with the Jacobi group $\mathcal{J}(A_1)$ with the upper index $\mathcal{J}(A_1)$, for instance

$$\varphi^{\mathcal{J}(A_1)},$$

and the Jacobi forms related with the Jacobi group $\mathcal{J}(\tilde{A}_1)$ with the upper index $\mathcal{J}(\tilde{A}_1)$

$$\varphi^{\mathcal{J}(\tilde{A}_1)}.$$

In [2], Bertola found a basis of the generators of the Jacobi form algebra by producing a holomorphic Jacobi form of type A_n as product of θ -functions

$$\varphi^{\mathcal{J}(A_n)} = e^{2\pi i u} \prod_{i=1}^{n+1} \frac{\theta_1(z_i, \tau)}{\theta_1'(0, \tau)}.$$

Afterwards, Bertola defined a recursive operator to produce the remaining basic generators. In order to recall the details, see [2]. Our strategy will follow the same logic of Bertola method; we use theta functions to produce a basic generator and thereafter, we produce a recursive operator to produce the remaining part.

Lemma 2.12. *Let be $\varphi_3^{\mathcal{J}(A_2)}(u_1, z_1, z_2, \tau)$ the holomorphic A_2 -invariant Jacobi form, which corresponds to the algebra generator of maximal weight degree, in this case degree 3. More explicitly,*

$$\varphi_3^{\mathcal{J}(A_2)} = e^{-2\pi i u_1} \frac{\theta_1(z_1, \tau) \theta_1(z_2, \tau) \theta_1(-z_1 - z_2, \tau)}{\theta_1'(0, \tau)^3}.$$

Let be $\varphi_2^{\mathcal{J}(A_1)}(u_2, z_3, \tau)$ the holomorphic A_1 -invariant Jacobi form, which corresponds to the algebra generator of maximal weight degree, in this case degree 2:

$$\varphi_2^{\mathcal{J}(A_1)} = e^{-2\pi i u_2} \frac{\theta_1(z_3, \tau)^2}{\theta_1'(0, \tau)^2}.$$

Then, the function

$$\varphi_1^{\mathcal{J}(\tilde{A}_1)} = \frac{\varphi_3^{\mathcal{J}(A_2)}}{\varphi_2^{\mathcal{J}(A_1)}} \quad (2.15)$$

is meromorphic Jacobi form of index 1, weight -1 , order 0.

Proof. For our convenience, we change the labels z_1, z_2, z_3 to $v_0 + v_2, -v_0 + v_2, 2v_2$, respectively. Then (2.15) has the following form

$$\varphi_1^{\mathcal{J}(\tilde{A}_1)}(u, v_0, v_2, \tau) = e^{-2\pi i u} \frac{\theta_1(v_0 + v_2, \tau) \theta_1(-v_0 + v_2, \tau)}{\theta_1'(0, \tau) \theta_1(2v_2, \tau)}. \quad (2.16)$$

Let us prove each item separately.

A_1 -invariant. The A_1 group acts on (2.16) by permuting its roots, thus (2.16) remains invariant under this operation.

Translation invariant. Recall that under the translation $v \mapsto v + m + n\tau$, the Jacobi theta function transforms as [2, 22]:

$$\theta_1(v_i + \mu_i + \lambda_i \tau, \tau) = (-1)^{\lambda_i + \mu_i} e^{-2\pi i (\lambda_i v_i + \frac{\lambda_i^2}{2} \tau)} \theta_1(v_i, \tau). \quad (2.17)$$

Then, substituting the transformation (2.17) into (2.16), we conclude that (2.16) remains invariant.

$SL_2(\mathbb{Z})$ -invariant. Under $SL_2(\mathbb{Z})$ -action the following function transform as

$$\frac{\theta_1\left(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d}\right)}{\theta_1'\left(0, \frac{a\tau+d}{c\tau+d}\right)} = (c\tau + d)^{-1} \exp\left(\frac{\pi i c v_i^2}{c\tau + d}\right) \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)}. \quad (2.18)$$

Then, substituting (2.18) in (2.16), we get

$$\varphi_1^{\mathcal{J}(\tilde{A}_1)} \mapsto \frac{\varphi_1^{\mathcal{J}(\tilde{A}_1)}}{c\tau + d}.$$

Index 1.

$$\frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi_1^{\mathcal{J}(\tilde{A}_1)} = \varphi_1^{\mathcal{J}(\tilde{A}_1)}.$$

Analytic behavior. Note that $\varphi_1^{\mathcal{J}(\tilde{A}_1)} \theta_1^2(2v_2, \tau)$ is holomorphic function in all the variables v_i . Therefore, $\varphi_1^{\mathcal{J}(\tilde{A}_1)}$ are holomorphic functions on the variables v_0 , and meromorphic function in the variable v_2 with poles on $\frac{j}{2} + \frac{l\tau}{2}$, $j, l = 0, 1$ of order 2, i.e., $l = 0$, since $m = 1$. ■

In order to define the desired recursive operator, it is necessary to enlarge the domain of the Jacobi forms from $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v_0, v_2, \tau)$ to $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{H} \ni (u, v_0, v_2, p, \tau)$. In addition, we define a lift of Jacobi forms defined in $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H}$ to $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{H}$ as

$$\varphi(u, v_0 + v_2, -v_0 + v_2, \tau) \mapsto \hat{\varphi}(p) := \varphi(u, v_0 + v_2 + p, -v_0 + v_2 + p, \tau).$$

A convenient way to do computations in these extended Jacobi forms is to use the following coordinates

$$s = u + g_1(\tau)p^2, \quad z_1 = v_0 + v_2 + p, \quad z_2 = -v_0 + v_2 + p, \quad z_3 = 2v_2 + p, \quad \tau = \tau.$$

The bilinear form $\langle v, v \rangle_{\tilde{A}_1}$ is extended to

$$\langle (z_1, z_2, z_3), (z_1, z_2, z_3) \rangle_E = z_1^2 + z_2^2 - z_3^2,$$

or equivalently,

$$\langle (v_0, v_2, p), (v_0, v_2, p) \rangle_E = 2v_0^2 - 2v_2^2 + p^2.$$

The action of the Jacobi group \tilde{A}_1 in this extended space is

$$\begin{aligned} \hat{w}_E(u, v, p, \tau) &= (u, w(v), p, \tau), \\ t_E(u, v, p, \tau) &= \left(u - \langle \lambda, v \rangle_E - \frac{1}{2} \langle \lambda, \lambda \rangle_E \tau + k, v + p + \lambda \tau + \mu, \tau \right), \\ \gamma_E(u, v, p, \tau) &= \left(u + \frac{c \langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

Proposition 2.13. *Let be $\varphi \in J_{k,m,\bullet}^{\mathcal{J}(\tilde{A}_1)}$, and $\hat{\varphi}$ the correspondent extended Jacobi form. Then,*

$$\left. \frac{\partial}{\partial p} (\hat{\varphi}) \right|_{p=0} \in J_{k-1,m,\bullet}^{\mathcal{J}(\tilde{A}_1)}.$$

Proof. *A_1 -invariant.* The vector field $\frac{\partial}{\partial p}$ in coordinates s, z_1, z_2, z_3, τ reads

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} + 2g_1(\tau)p \frac{\partial}{\partial u}.$$

Moreover, in the coordinates s, z_1, z_2, z_3, τ the A_1 group acts by permuting z_1 and z_2 . Then

$$\begin{aligned} \left. \frac{\partial}{\partial p} (\varphi(s, z_2, z_1, z_3, \tau)) \right|_{p=0} &= \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right) (\varphi(s, z_2, z_1, z_3, \tau)) \Big|_{p=0} \\ &= \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right) (\varphi(s, z_1, z_2, z_3, \tau)) \Big|_{p=0}. \end{aligned}$$

Translation invariant:

$$\begin{aligned} &\left. \frac{\partial}{\partial p} (\varphi(u - \langle \lambda, v \rangle_E - \langle \lambda, \lambda \rangle_E \tau, v + p + \lambda \tau + \mu, \tau)) \right|_{p=0} \\ &= \frac{\partial}{\partial p} \langle \lambda, v \rangle_E \Big|_{p=0} \varphi(u, v, \tau) + \frac{\partial \varphi}{\partial p} \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \tau + k, v + \lambda \tau + \mu, \tau \right) \\ &= \frac{\partial \varphi}{\partial p} \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2} \langle \lambda, \lambda \rangle_{\tilde{A}_1} \tau + k, v + \lambda \tau + \mu, \tau \right) = \frac{\partial \varphi}{\partial p} (u, v, \tau). \end{aligned}$$

$SL_2(\mathbb{Z})$ -equivariant of weight k :

$$\begin{aligned} & \left. \frac{\partial}{\partial p} \left(\varphi \left(u + \frac{c\langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \right) \right|_{p=0} \\ &= \frac{c}{2(c\tau + d)} \frac{\partial}{\partial p} \langle v, v \rangle_E \Big|_{p=0} \varphi(u, v, \tau) + \frac{1}{c\tau + d} \frac{\partial \varphi}{\partial p} \left(u + \frac{c\langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \\ &= \frac{1}{c\tau + d} \frac{\partial \varphi}{\partial p} \left(u + \frac{c\langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = \frac{1}{(c\tau + d)^k} \frac{\partial \varphi}{\partial p}(u, v, \tau). \end{aligned}$$

Then,

$$\frac{\partial \varphi}{\partial p} \left(u + \frac{c\langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = \frac{1}{(c\tau + d)^{k-1}} \frac{\partial \varphi}{\partial p}(u, v, \tau).$$

Index 1:

$$\frac{1}{2\pi i} \frac{\partial}{\partial u} \frac{\partial}{\partial p} \hat{\varphi} = \frac{1}{2\pi i} \frac{\partial}{\partial p} \frac{\partial}{\partial u} \hat{\varphi} = \frac{\partial}{\partial p} \hat{\varphi}. \quad \blacksquare$$

Corollary 2.14. *The function*

$$\left[e^{z \frac{\partial}{\partial p}} \left(\frac{e^{2\pi i u} \theta_1(v_0 + v_2 + p) \theta_1(-v_0 + v_2 + p)}{\theta_1(2v_2 + p) \theta_1'(0)} \right) \right] \Big|_{p=0} = \varphi_1^{\mathcal{J}(\tilde{A}_1)} + \varphi_0^{\mathcal{J}(\tilde{A}_1)} z + O(z^2),$$

generates the Jacobi forms $\varphi_0^{\mathcal{J}(\tilde{A}_1)}$ and $\varphi_1^{\mathcal{J}(\tilde{A}_1)}$, where

$$\varphi_0^{\mathcal{J}(\tilde{A}_1)} := \frac{\partial}{\partial p} (\hat{\varphi}_1^{\mathcal{J}(\tilde{A}_1)}) \Big|_{p=0}.$$

Proof. Acting $\frac{\partial}{\partial p}$ k times in $\varphi_1^{\mathcal{J}(\tilde{A}_1)}$, we have

$$\left[\frac{\partial^k}{\partial p^k} \left(\frac{e^{2\pi i u} \theta_1(v_0 + v_2 + p) \theta_1(-v_0 + v_2 + p)}{\theta_1(2v_2 + p) \theta_1'(0)} \right) \right] \Big|_{p=0} \in J_{1-k, 1, \bullet}^{\mathcal{J}(\tilde{A}_1)}. \quad \blacksquare$$

Corollary 2.15. *The generating function can be written as*

$$\begin{aligned} & \left[e^{z \frac{\partial}{\partial p}} \left(\frac{e^{2\pi i u} \theta_1(v_0 + v_2 + p) \theta_1(-v_0 + v_2 + p)}{\theta_1(2v_2 + p) \theta_1'(0)} \right) \right] \Big|_{p=0} \\ &= e^{-2\pi i(u + i g_1(\tau) z^2)} \frac{\theta_1(z - v_0 + v_2, \tau) \theta_1(z + v_0 + v_2, \tau)}{\theta_1'(0) \theta_1(z + 2v_2)}. \end{aligned}$$

Proof.

$$\begin{aligned} & \left[e^{z \frac{\partial}{\partial p}} \left(\frac{e^{2\pi i u} \theta_1(v_0 + v_2 + p) \theta_1(-v_0 + v_2 + p)}{\theta_1'(0) \theta_1(2v_2 + p)} \right) \right] \Big|_{p=0} \\ &= \left[e^{z \frac{\partial}{\partial p}} \left(e^{-2\pi i(s + i g_1(\tau) p^2)} \frac{\theta_1(v_0 + v_2 + p) \theta_1(-v_0 + v_2 + p)}{\theta_1(2v_2 + p) \theta_1'(0)} \right) \right] \Big|_{p=0} \\ &= e^{-2\pi i(u + i g_1(\tau) z^2)} \frac{\theta_1(z - v_0 + v_2, \tau) \theta_1(z + v_0 + v_2, \tau)}{\theta_1'(0) \theta_1(z + 2v_2)}. \quad \blacksquare \end{aligned}$$

The next lemma is one of the main points of inquiry in this section, because this lemma identifies the orbit space of the group $\mathcal{J}(\tilde{A}_1)$ with the Hurwitz space $H_{1,0,0}$. This relationship is possible due to the construction of the generating function of the Jacobi forms of type \tilde{A}_1 , which can be completed to be the Landau–Ginzburg superpotential of $H_{1,0,0}$ as follows:

$$\begin{aligned} & e^{-2\pi i(u+ig_1(\tau)z^2)} \frac{\theta_1(z-v_0+v_2, \tau)\theta_1(z+v_0+v_2, \tau)}{\theta_1'(0)\theta_1(z+2v_2)} \\ & \mapsto e^{-2\pi iu} \frac{\theta_1(v-v_0+v_2, \tau)\theta_1(v+v_0+v_2, \tau)}{\theta_1(v\tau)\theta_1(v+2v_2, \tau)}. \end{aligned}$$

Lemma 2.16. *There exists a local isomorphism between $\Omega/\mathcal{J}(\tilde{A}_1)$ and $H_{1,0,0}$.*

Proof. The correspondence is realized by the map

$$[(u, v_0, v_2, \tau)] \longleftrightarrow \lambda(v) = e^{-2\pi iu} \frac{\theta_1(v-v_0, \tau)\theta_1(v+v_0, \tau)}{\theta_1(v-v_2, \tau)\theta_1(v+v_2, \tau)}, \quad (2.19)$$

where $\theta_1(v, \tau)$ is the Jacobi θ_1 -function defined on (1.5).

It is necessary to prove that the map is well defined and one to one.

Well defined. Note that the map does not depend on the choice of the representative of $[(u, v_0, v_2, \tau)]$ if the function (2.19) is invariant under the action of $\mathcal{J}(\tilde{A}_1)$. Therefore, let us prove the invariance of the map (2.19).

A_1 -invariant. The A_1 group acts on (2.19) by permuting its roots, thus (2.19) remains invariant under this operation.

Translation invariant. Recall that under the translation $v \mapsto v+m+n\tau$, the Jacobi θ -function transforms as [22]:

$$\theta_1(v_i + \mu_i + \lambda_i\tau, \tau) = (-1)^{\lambda_i + \mu_i} e^{-2\pi i(\lambda_i v_i + \frac{\lambda_i^2}{2}\tau)} \theta_1(v_i, \tau). \quad (2.20)$$

Then, substituting the transformation (2.20) into (2.19), we conclude that (2.19) remains invariant.

$SL_2(\mathbb{Z})$ -invariant. Under $SL_2(\mathbb{Z})$ -action the following function transforms to

$$\frac{\theta_1\left(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d}\right)}{\theta_1'(0, \frac{a\tau+d}{c\tau+d})} = (c\tau+d)^{-1} \exp\left(\frac{\pi i c v_i^2}{c\tau+d}\right) \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)}. \quad (2.21)$$

Then, substituting the transformation (2.21) into (2.19), we conclude that (2.19) remains invariant.

Injectivity. Note that for fixed v, v_0, v_2, u , the function $\tau \mapsto f(\tau) := \lambda(v, v_0, v_2, u, \tau)$ is a modular form with character [12]. This is clear because $\lambda(v, v_0, v_2, u, \tau)$ is rational function of $\theta_1(z, \tau)$, which is modular form with character for special values of z [12]. If $\lambda(v, v_0, v_2, u, \tau) = \lambda(v, \hat{v}_0, \hat{v}_2, \hat{u}, \hat{\tau})$, then for fixed $v, v_0, v_2, u, \hat{v}_0, \hat{v}_2, \hat{u}$, we have $f(\tau) = f(\hat{\tau})$; in particular, $f(\tau), f(\hat{\tau})$ have the same vanishing order, and this implies that $\tau, \hat{\tau}$ belongs to the same $SL_2(\mathbb{Z})$ orbit.

Two elliptic functions are equal if they have the same zeros and poles with multiplicity mod $\mathbb{Z} \oplus \tau\mathbb{Z}$. So, for a fixed τ in the $SL_2(\mathbb{Z})$ orbit

$$\hat{v}_0 = v_0 + \lambda_0\tau + \mu_0, \quad \hat{v}_2 = v_2 + \lambda_2\tau + \mu_2, \quad (\lambda_i, \mu_i) \in \mathbb{Z}^2.$$

Furthermore, for two different representations of the same $SL_2(\mathbb{Z})$ orbit, but considering fixed cells, we have

$$\hat{v}_0 = \frac{v_0}{c\tau+d}, \quad \hat{v}_2 = \frac{v_2}{c\tau+d}, \quad \hat{\tau} = \frac{a\tau+b}{c\tau+d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Since, $\lambda(v, v_0, v_2, u, \tau)$ is invariant under translations, and $\text{SL}_2(\mathbb{Z})$, for $\hat{\tau} = \tau$, we have

$$\hat{u} = u - \langle \lambda, v \rangle_{\tilde{A}_1} - \langle \lambda, \lambda \rangle_{\tilde{A}_1} \frac{\tau}{2} + k.$$

For $\hat{\tau} = \frac{a\tau+b}{c\tau+d}$,

$$\hat{u} = u - \frac{c\langle v, v \rangle_{\tilde{A}_1}}{2(c\tau+d)} + k,$$

where $k \in \mathbb{Z}$.

Surjectivity. Any elliptic function can be written as rational functions of Weierstrass σ -function up to a multiplication factor [22]; by using the formula

$$\sigma(v_i, \tau) = \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)} \exp(-2\pi i g_1(\tau) v_i^2), \quad g_1(\tau) = \frac{\eta'(\tau)}{\eta(\tau)},$$

where $\eta(\tau)$ is the Dedekind η -function, we get the desire result. ■

Remark 2.17. Lemma 2.16 is a local biholomorphism of manifolds, but this does not necessarily means isomorphism of Dubrovin–Frobenius structure. On a Hurwitz space, there may exist in several inequivalent Dubrovin–Frobenius structures. For instance, in [17] Romano constructed two generalised WDDV solution on the Hurwitz space $H_{1,0,0}$. Furthermore, in [2] and [3], Bertola constructed two different Dubrovin–Frobenius structures on the orbit space of the Jacobi group G_2 . The Dubrovin–Frobenius structure of this orbit space will be constructed only in Section 3.

Remark 2.18. Lemma 2.16 associates a group to $H_{1,0,0}$, and this could be useful for the general understanding of the WDDV solutions/discrete group correspondence [6].

Corollary 2.19. *The functions $(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ obtained by the formula*

$$\begin{aligned} \lambda^{\tilde{A}_1} &= e^{-2\pi i u} \frac{\theta_1(v - v_0, \tau) \theta_1(v + v_0, \tau)}{\theta_1(v - v_2, \tau) \theta_1(v + v_2, \tau)} \\ &= \varphi_1^{\tilde{A}_1} [\zeta(v - v_2, \tau) - \zeta(v + v_2, \tau) + 2\zeta(v_2, \tau)] + \varphi_0^{\tilde{A}_1} \end{aligned} \quad (2.22)$$

are Jacobi forms of weight 0, -1 respectively, index 1, and order 0. More explicitly,

$$\begin{aligned} \varphi_1^{\tilde{A}_1} &= \frac{\theta_1(v_0 + v_2, \tau) \theta_1(-v_0 + v_2, \tau)}{\theta_1'(0, \tau) \theta_1(2v_2, \tau)} e^{-2\pi i u}, \\ \varphi_0^{\tilde{A}_1} &= -\varphi_1^{\tilde{A}_1} [\zeta(v_0 - v_2, \tau) - \zeta(v_0 + v_2, \tau) + 2\zeta(v_2, \tau)], \end{aligned} \quad (2.23)$$

where $\zeta(v, \tau)$ is the Weierstrass ζ -function for the lattice $(1, \tau)$, i.e.,

$$\zeta(v, \tau) = \frac{1}{v} + \sum_{m^2+n^2 \neq 0}^{\infty} \frac{1}{v - m - n\tau} + \frac{1}{m + n\tau} + \frac{v}{(m + n\tau)^2}.$$

Proof. Let us prove each item separately.

A_1 -invariant, translation invariant. The first line of (2.22) are A_1 -invariant, and translation invariant by the lemma (2.16). Then, by the Laurent expansion of $\lambda^{\tilde{A}_1}$, we have that $\varphi_i^{\tilde{A}_1}$ are A_1 -invariant, and translation invariant.

$\mathrm{SL}_2(\mathbb{Z})$ -equivariant. The first line of (2.22) are $\mathrm{SL}_2(\mathbb{Z})$ -invariant, but the Weierstrass ζ -functions of the second line of (2.22) have the following transformation law

$$\zeta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\zeta(z, \tau).$$

Then, $\varphi_i^{\tilde{A}_1}$ must have the following transformation law:

$$\begin{aligned}\varphi_0^{\tilde{A}_1}\left(u + \frac{c\langle v, v \rangle_{\tilde{A}_1}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \varphi_0^{\tilde{A}_1}(u, v, \tau), \\ \varphi_1^{\tilde{A}_1}\left(u + \frac{c\langle v, v \rangle_{\tilde{A}_1}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^{-k} \varphi_1^{\tilde{A}_1}(u, v, \tau).\end{aligned}$$

Index 1:

$$\frac{1}{2\pi i} \frac{\partial}{\partial u} \lambda^{\tilde{A}_1} = \lambda^{\tilde{A}_1}.$$

Then

$$\frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi_i^{\tilde{A}_1} = \varphi_i^{\tilde{A}_1}.$$

Analytic behavior. Note that $\lambda^{\tilde{A}_1} \theta_1^2(2v_2, \tau)$ is holomorphic function in all the variables v_i . Therefore, $\varphi_i^{\tilde{A}_1}$ are holomorphic functions on the variables v_0 , and meromorphic function in the variable v_2 with poles on $\frac{j}{2} + \frac{l\tau}{2}$, $j, l = 0, 1$ of order 2, i.e., $l = 0$, since $m = 1$ for all $\varphi_i^{\tilde{A}_1}$.

To prove the formula (2.23) let us compute the following limit:

$$\lim_{z \rightarrow v_2} \lambda^{\tilde{A}_1} v_2 = \varphi_1^{\tilde{A}_1} = e^{-2\pi i u} \frac{\theta_1(v_0 + v_2, \tau) \theta_1(-v_0 + v_2, \tau)}{\theta_1'(0, \tau) \theta_1(2v_2, \tau)}.$$

Let us also compute the zeros of $\lambda^{\tilde{A}_1}$

$$\lambda^{\tilde{A}_1}(v_0) = 0 = \varphi_1^{\tilde{A}_1} [\zeta(v_0 - v_2, \tau) - \zeta(v_0 + v_2, \tau) + 2\zeta(v_2, \tau)] + \varphi_0^{\tilde{A}_1}. \quad \blacksquare$$

Lemma 2.20. *The functions $\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}$ are algebraically independent over the ring $E_{\bullet, \bullet}$.*

Proof. If $P(X, Y)$ is any polynomial in $E_{\bullet, \bullet}(X, Y)$, such that $P(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}) = 0$, then, the fact $\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}$ have an index that implies that each homogeneous component $P_d(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ has to vanish identically. Defining $p_d\left(\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}\right) := \frac{P_d(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})}{(\varphi_1^{\tilde{A}_1})^d}$, we have that $p_d\left(\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}\right)$ is identically 0 iff

$\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}$ is constant (belongs to $E_{\bullet, \bullet}$), but

$$\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}} = \frac{\wp'(v_2, \tau)}{\wp(v_0, \tau) - \wp(v_2, \tau)} \neq a(v_2, \tau),$$

where $a(v_2, \tau)$ is any function belongs to $E_{\bullet, \bullet}$. Then, $\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}$ are algebraically independent over the ring $E_{\bullet, \bullet}$.

Recall that $\wp(v, \tau)$ is the Weierstrass P-function (2.6). \blacksquare

Consider the Landau–Ginzburg superpotential (2.24) for the $\mathcal{J}(A_2)$ case below.

Theorem 2.21 ([2]). *The ring of A_2 -invariant Jacobi forms is free module of rank 3 over the ring of modular forms, moreover there exist a formula for its generators given by*

$$\begin{aligned}\lambda^{A_2} &= e^{-2\pi i u_2} \frac{\theta_1(z + v_0 + v_2, \tau) \theta_1(z - v_0 + v_2, \tau) \theta_1(z - 2v_2)}{\theta_1^3(z, \tau)} \\ &= -\frac{\varphi_3^{A_2}}{2} \wp'(z, \tau) + \varphi_2^{A_2} \wp(z, \tau) + \varphi_0^{A_2}.\end{aligned}\quad (2.24)$$

Lemma 2.22. *Let $\{\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}\}$ be set of functions given by the formula (2.22), and $\{\varphi_0^{A_2}, \varphi_2^{A_2}, \varphi_3^{A_2}\}$ given by (2.24), then*

$$\begin{aligned}\varphi_3^{A_2} &= \varphi_1^{\tilde{A}_1} \varphi_2^{A_1}, \\ \varphi_2^{A_2} &= \varphi_0^{\tilde{A}_1} \varphi_2^{A_1} + a_2(v_2, \tau) \varphi_j^{\tilde{A}_1} \varphi_2^{A_1}, \\ \varphi_0^{A_2} &= a_0(v_2, \tau) \varphi_0^{\tilde{A}_1} \varphi_2^{A_1} + b_0(v_2, \tau) \varphi_2^{\tilde{A}_1} \varphi_2^{A_1},\end{aligned}$$

where

$$\varphi_2^{A_1} := \frac{\theta_1^2(2v_2, \tau)}{\theta_1^2(0, \tau)} e^{2\pi i(-u_2 + u_1)}$$

and a_i, b_i are elliptic functions on v_2 .

Proof. Note the following relation

$$\frac{\lambda^{A_2}}{\lambda^{\tilde{A}_1}} = \frac{\theta_1(z - 2v_2, \tau) \theta_1(z + 2v_2, \tau)}{\theta_1^2(z, \tau)} e^{2\pi i(-u_2 + u_1)} = \varphi_2^{A_1} \wp(z, \tau) - \varphi_2^{A_1} \wp(2v_2, \tau).$$

Hence,

$$\begin{aligned}-\frac{\varphi_3^{A_2}}{2} \wp'(z, \tau) + \varphi_2^{A_2} \wp(z, \tau) + \varphi_0^{A_2} &= (\varphi_1^{\tilde{A}_1} [\zeta(z, \tau) - \zeta(z + 2v_2, \tau) + 2\zeta(v_2, \tau)] + \varphi_0^{\tilde{A}_1}) \\ &\quad \times (\varphi_2^{A_1} \wp(z, \tau) - \varphi_2^{A_1} \wp(2v_2, \tau)).\end{aligned}$$

Then, the desired result is obtained by doing a Laurent expansion in the variable z on both sides of the equality. \blacksquare

Corollary 2.23.

$$E_{\bullet, \bullet}[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}] = E_{\bullet, \bullet} \left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}} \right].$$

Moreover, we have the following lemma:

Lemma 2.24. *Let be $\varphi \in J_{\bullet, \bullet, m}^{\tilde{A}_1}$, then $\varphi \in E_{\bullet, \bullet} \left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}} \right]$.*

Proof. Let be $\varphi \in J_{\bullet, \bullet, m}^{\tilde{A}_1}$, then the function $\frac{\varphi}{(\varphi_1^{\tilde{A}_1})^m}$ is an elliptic function on the variables (v_0, v_2) with poles on $v_0 - v_2, v_0 + v_2, 2v_2$ due to the zeros of $\varphi_1^{\tilde{A}_1}$ and the poles of φ , which are by definition in $2v_2$. Expanding the function $\frac{\varphi}{(\varphi_1^{\tilde{A}_1})^m}$ in the variables v_0, v_2 we get

$$\frac{\varphi}{(\varphi_1^{\tilde{A}_1})^m} = \sum_{i=-1}^m a^i \wp^{(i)}(v_0 + v_2) + \sum_{i=-1}^m b^i \wp^{(i)}(-v_0 + v_2) + c(v_2, \tau), \quad (2.25)$$

where $\wp^{-1}(v) := \zeta(v)$, and $c(v_2, \tau)$ is an elliptic function in the variable v_2 .

However, the function $\frac{\varphi}{(\varphi_1^{A_1})^m}$ is invariant under the permutations of the variables v_0 , so the equation (2.25) is

$$\frac{\varphi}{(\varphi_1^{A_1})^m} = \sum_{i=-1}^m a^i (\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2)) + c(v_2, \tau). \quad (2.26)$$

Now we complete this function to A_2 -invariant function by summing and subtracting the following function in equation (2.26)

$$f(v_2, \tau) = \sum_{i=-1}^m a^i \wp^{(i)}(2v_2).$$

Hence,

$$\frac{\varphi}{(\varphi_1^{A_1})^m} = \sum_{i=-1}^m a^i (\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2) + \wp^{(i)}(2v_2)) + g(v_2, \tau). \quad (2.27)$$

Multiplying both side of the equation (2.27) by $\varphi_1^{A_1}$, we get

$$\varphi = \left(\sum_{i=-1}^m a^i (\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2) + \wp^{(i)}(2v_2)) \right) (\varphi_3^{A_2})^m + g(v_2, \tau) (\varphi_3^{A_2})^m.$$

To finish the proof, we will show that

$$\left(\sum_{i=-1}^m a^i (\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2) + \wp^{(i)}(2v_2)) \right) (\varphi_3^{A_2})^m$$

is a weak holomorphic Jacobi form of type A_2 . To finish the proof, note the following:

1. The functions

$$(\varphi_3^{A_2})^m (\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2) + \wp^{(i)}(2v_2)) \quad (2.28)$$

are A_2 -invariant by construction.

2. The functions (2.28) are invariant under the action of $(\mathbb{Z} \oplus \tau\mathbb{Z})^2$, because $\varphi_3^{A_2}$ is invariant, and

$$\wp^{(i)}(v_0 + v_2) + \wp^{(i)}(-v_0 + v_2) + \wp^{(i)}(2v_2) \quad (2.29)$$

are elliptic functions.

3. The functions (2.28) are equivariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, because $\varphi_3^{A_2}$ is equivariant, and (2.29) are elliptic functions.
4. The function $\varphi_3^{A_2}$ has zeros on $v_0 - v_2$, $v_0 + v_2$, $2v_2$ of order m , and (2.29) has poles on $v_0 - v_2$, $v_0 + v_2$, $2v_2$ of order $i + 2 \leq m$. So, the functions (2.28) are holomorphic.

Hence,

$$\varphi \in E_{\bullet, \bullet} \left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}} \right]. \quad \blacksquare$$

At this stage, the principal theorem can be stated in a precise way as follows.

Theorem 2.25. *The trigraded algebra of Jacobi forms $J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_1}$ is freely generated by 2 fundamental Jacobi forms $(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ over the graded ring $E_{\bullet,\bullet}$*

$$J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = E_{\bullet,\bullet} \left[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1} \right].$$

Proof.

$$J_{\bullet,\bullet,\bullet}^{\tilde{A}_1} \subset E_{\bullet,\bullet} \left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}} \right] = E_{\bullet,\bullet} \left[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1} \right] \subset J_{\bullet,\bullet,\bullet}^{\tilde{A}_1} \quad \blacksquare$$

Remark 2.26. The structural difference between the Chevalley theorems of the groups $J(A_1)$ and $\mathcal{J}(\tilde{A}_1)$ lies in the ring of coefficients. The ring of coefficients of Jacobi forms with respect to $J(A_1)$ are modular forms, and the ring of coefficients of Jacobi forms with respect to $\mathcal{J}(\tilde{A}_1)$ are the ring of elliptic functions with poles on $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \pmod{\mathbb{Z} \oplus \tau\mathbb{Z}}$, for fixed τ . See Lemma 2.11.

Remark 2.27. The geometry of $\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)$ is similar to $\Omega^{\mathcal{J}(A_1)}/\mathcal{J}(A_1)$. Indeed, the orbit space of $\mathcal{J}(\tilde{A}_1)$ is locally a line bundle over a family of two elliptic curves, $E_\tau/A_1 \otimes E_\tau$, where the first one is quotient by A_1 , and both are parametrised by $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

3 Frobenius structure on the orbit space of $\mathcal{J}(\tilde{A}_1)$

In this section, a Dubrovin–Frobenius manifold structure will be constructed on the orbit space of $\mathcal{J}(\tilde{A}_1)$. More precisely, it will define the data $(\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1), g^*, e, E)$, with the intersection form g^* , unit vector field e , and Euler vector field E . This data will be written naturally in terms of the invariant functions of $\mathcal{J}(\tilde{A}_1)$. Thereafter, it will be proved that this data is enough to the construction of the Dubrovin–Frobenius structure.

3.1 Intersection form

The first step to be done is to construct the intersection form. It will be shown that such metric can be constructed by using just the data of the group $\mathcal{J}(\tilde{A}_1)$. The strategy is to combine the intersection form of the group \tilde{A}_1 and $\mathcal{J}(A_1)$. Recall that the intersection form of the group \tilde{A}_1 [6, 10] is

$$ds^2 = 2dv_0^2 - 2dv_2^2,$$

and the intersection form of $\mathcal{J}(A_1)$ [2, 3, 6] is

$$ds^2 = dv_0^2 + 2dud\tau.$$

Therefore, the natural candidate to be the intersection form of $\mathcal{J}(\tilde{A}_1)$ is

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2dud\tau.$$

The following lemma proves that this metric is invariant metric of the group $\mathcal{J}(\tilde{A}_1)$. To be precise, the metric will be invariant under the action of A_1 , and translations, and equivariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.1. *The metric*

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2dud\tau \tag{3.1}$$

is invariant under the transformations (2.9), (2.10). Moreover, the transformations (2.11) determine a conformal transformation of the metric ds^2 , i.e.,

$$2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto \frac{2dv_0^2 - 2dv_2^2 + 2dud\tau}{(c\tau + d)^2}.$$

Proof. Under (2.9), (2.10), the differentials transform to

$$\begin{aligned} dv_0 &\mapsto -dv_0, & dv_0 &\mapsto dv_0 + \lambda_0 d\tau, & dv_2 &\mapsto dv_2 + \lambda_2 d\tau, \\ du &\mapsto du - \lambda_0^2 d\tau - 2\lambda_0 dv_0 + \lambda_2^2 d\tau + 2\lambda_2 dv_2, & d\tau &\mapsto d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} dv_0^2 &\mapsto dv_0^2, & dv_0^2 &\mapsto dv_0^2 + 2\lambda_0 dv_0 d\tau + \lambda_0^2 d\tau^2, & dv_2^2 &\mapsto dv_2^2 + 2\lambda_2 dv_2 d\tau + \lambda_2^2 d\tau^2, \\ 2dud\tau &\mapsto 2dud\tau - 2\lambda_0^2 d\tau^2 - 4\lambda_0 dv_0 d\tau + 2\lambda_2^2 d\tau^2 + 4\lambda_2 dv_2 d\tau. \end{aligned}$$

So,

$$2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto 2dv_0^2 - 2dv_2^2 + 2dud\tau.$$

Let us show that the metric has conformal transformation under the transformations (2.11)

$$\begin{aligned} dv_0 &\mapsto \frac{dv_0}{c\tau + d} - \frac{v_0 d\tau}{(c\tau + d)^2}, & dv_2 &\mapsto \frac{dv_2}{c\tau + d} - \frac{v_2 d\tau}{(c\tau + d)^2}, & d\tau &\mapsto \frac{d\tau}{(c\tau + d)^2}, \\ du &\mapsto du + \frac{c(2v_0 dv_0 - 2v_2 dv_2)}{c\tau + d} - \frac{c(v_0^2 - v_2^2)d\tau}{(c\tau + d)^2}. \end{aligned}$$

Then,

$$\begin{aligned} dv_0^2 &\mapsto \frac{dv_0^2}{(c\tau + d)^2} - \frac{2v_0 dv_0 d\tau}{(c\tau + d)^3} + \frac{v_0^2 d\tau^2}{(c\tau + d)^4}, & dv_2^2 &\mapsto \frac{dv_2^2}{(c\tau + d)^2} - \frac{2v_2 dv_2 d\tau}{(c\tau + d)^3} + \frac{v_2^2 d\tau^2}{(c\tau + d)^4}, \\ 2dud\tau &\mapsto \frac{2dud\tau}{(c\tau + d)^2} + \frac{c(4v_0 dv_0 - 4v_2 dv_2)d\tau}{(c\tau + d)^3} - \frac{c(2v_0^2 - 2v_2^2)d\tau^2}{(c\tau + d)^4}. \end{aligned}$$

Then,

$$2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto \frac{2dv_0^2 - 2dv_2^2 + 2dud\tau}{(c\tau + d)^2}. \quad \blacksquare$$

3.2 Euler and unit vector field

The next step is to construct a two vector field, which is a intrinsic object of the orbit space $\mathcal{J}(\tilde{A}_1)$. The first one is the Euler vector

$$E = -\frac{1}{2\pi i} \frac{\partial}{\partial u}, \quad (3.2)$$

which was already defined in the last equation of (2.14). Therefore, it is an already intrinsic object, since it comes from the definition of meromorphic Jacobi forms associated with $\mathcal{J}(\tilde{A}_1)$. In the invariant coordinates, the vector field (3.2) reads as

$$E = \varphi_0 \frac{\partial}{\partial \varphi_0} + \varphi_1 \frac{\partial}{\partial \varphi_1}.$$

The second one is given by the coordinates $(\varphi_0, \varphi_1, v_2, \tau)$ as

$$e = \frac{\partial}{\partial \varphi_0}, \quad (3.3)$$

and it is denoted by the unit vector field. This object is intrinsic to the orbit space of $\mathcal{J}(\tilde{A}_1)$, because it is written in terms of a meromorphic Jacobi forms associated with $\mathcal{J}(\tilde{A}_1)$.

3.3 Flat coordinates of the Saito metric

In order to construct the Dubrovin–Frobenius structure, it will be necessary to introduce the coordinates (t^1, t^2, t^3, t^4) .

Lemma 3.2. *There is a change of coordinates in $\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)$, given by*

$$t^1 = \varphi_0 + 2t^2 \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)}, \quad t^2 = \varphi_1, \quad t^3 = v_2, \quad t^4 = \tau.$$

Proof. Note that the function (2.19) can be parametrised by (t^1, t^2, t^3, t^4) as follows:

$$\begin{aligned} \lambda &= \varphi_0 + \varphi_1[\zeta(v - v_2|\tau) - \zeta(v + v_2|\tau) + 2\zeta(v_2)] \\ &= \varphi_0 + \varphi_1 \left[\frac{\theta'_1(v - v_2|\tau)}{\theta_1(v - v_2|\tau)} - \frac{\theta'_1(v + v_2|\tau)}{\theta_1(v + v_2|\tau)} + 2 \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} \right] \\ &= \varphi_0 + 2 \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} + \varphi_1 \left[\frac{\theta'_1(v - v_2|\tau)}{\theta_1(v - v_2|\tau)} - \frac{\theta'_1(v + v_2|\tau)}{\theta_1(v + v_2|\tau)} \right] \\ &= t^1 + t^2 \left[\frac{\theta'_1(v - t^3|t^4)}{\theta_1(v - t^3|t^4)} - \frac{\theta'_1(v + t^3|t^4)}{\theta_1(v + t^3|t^4)} \right] \end{aligned}$$

from the first line to the second line, the following equation was used:

$$\zeta(v - v_2, \tau) = \frac{\theta'_1(v - v_2|\tau)}{\theta_1(v - v_2|\tau)} + 4\pi i g_1(\tau)(v - v_2).$$

In this way, (t^1, t^2, t^3, t^4) are local coordinates of $\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)$ due to Lemma 2.16. ■

The side back effect of the coordinates (t^1, t^2, t^3, t^4) is the fact that they are not globally single valued functions on the quotient.

Lemma 3.3. *The coordinates (t^1, t^2, t^3, t^4) have the following transformation laws under the action of the group $\mathcal{J}(\tilde{A}_1)$: they are invariant under (2.9). They transform as follows under (2.10):*

$$t^1 \mapsto t^1 - \lambda_2 t^2, \quad t^2 \mapsto t^2, \quad t^3 \mapsto t^3 + \mu_2 + \lambda_2 t^4, \quad t^4 \mapsto t^4.$$

Moreover, they transform as follows under (2.11)

$$t^1 \mapsto t^1 + \frac{2ct^2 t^3}{ct^4 + d}, \quad t^2 \mapsto \frac{t^2}{ct^4 + d}, \quad t^3 \mapsto \frac{t^3}{ct^4 + d}, \quad t^4 \mapsto \frac{at^4 + b}{ct^4 + d}.$$

Proof. The invariance under (2.9) is clear, since only t^1 depends on v_0 , and its dependence is given by φ_0 , which is invariant under (2.9). Let us check how t^α transforms under (2.10), (2.11): Since $t^3 = v_2$, $t^4 = \tau$, we have the desired transformations law defined as $\mathcal{J}(\tilde{A}_1)$. The coordinate $t^2 = \varphi_1$ is a invariant under (2.10) and transforms as modular form of weight -1 under (2.10). The only non-trivial term is t^1 , because it contains the term $\frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)}$, which transforms as follows under (2.10), (2.11) [22]

$$\frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} \mapsto \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} - 2\pi i n_2, \quad \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} \mapsto (c\tau + d) \frac{\theta'_1(v_2|\tau)}{\theta_1(v_2|\tau)} + 2\pi i c t^3.$$

The proof is completed when we do the rescaling from t^1 to $\frac{t^1}{2\pi i}$. ■

In order to make the coordinates (t^1, t^2, t^3, t^4) being well defined, it will be necessary to define them in a suitable covering over $\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)$. It is clear that the multivaluedness comes from the coordinates t^3, t^4 essentially. Therefore, the problem is solved by defining a suitable covering over the orbit space of $\mathcal{J}(\tilde{A}_1)$. This can be done by fixing a lattice $(1, t^4)$ and a representative of orbit given by the action

$$t^3 \mapsto t^3 + \mu_2 + \lambda_2 t^4. \quad (3.4)$$

In order to also realise the coordinates (u, v_0, v_2, τ) as globally well-behaviour in the covering of the orbit space of $\mathcal{J}(\tilde{A}_1)$, we also forget the A_1 -action by fixing a representative of each orbit. Therefore, in the following covering the problem

$$\widetilde{\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)} := \Omega^{\mathcal{J}(\tilde{A}_1)}/\mathbb{Z} \oplus \tau\mathbb{Z}, \quad (3.5)$$

where $\mathbb{Z} \oplus \tau\mathbb{Z}$ acts on $\Omega^{\mathcal{J}(\tilde{A}_1)}$ as

$$u \mapsto u + \lambda_0 \tau + \mu_0, \quad v_0 \mapsto v_0 - 2\lambda_0 v_0 - n_0^2 \tau, \quad v_2 \mapsto v_2, \quad \tau \mapsto \tau.$$

In the covering (3.5) the coordinates t^α , and the intersection form g^* are globally single valued. Hence, we have the necessary conditions to have Dubrovin–Frobenius manifold, since its geometry structure should be globally well defined. Note that, $\Omega^{\mathcal{J}(\tilde{A}_1)}/\mathcal{J}(\tilde{A}_1)$ has the structure of Twisted Frobenius manifold [6].

Remark 3.4. (t^1, t^2) lives in an enlargement of the algebra of $E_{\bullet,\bullet}[\varphi_0, \varphi_1]$. The extended algebra is the same as $E_{\bullet,\bullet}[\varphi_0, \varphi_1]$, but it is necessary to add the function $\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)}$ in the ring of coefficients $E_{\bullet,\bullet}$.

Remark 3.5. Note that a covering in the orbit space corresponds to a covering in the Hurwitz space. The fixation of a lattice in the orbit space of $\mathcal{J}(\tilde{A}_1)$ is equivalent to a choice of homology basis in the Hurwitz space $H_{1,0,0}$. Moreover, a choice of the representative of the action (3.4) in the variable v_2 is a choice of logarithm root in the Hurwitz space $H_{1,0,0}$. Furthermore, fixing a representative of the A_1 -action is to choice a pole or equivalently to choice a sheet in the Hurwitz space $H_{1,0,0}$.

Remark 3.6. The Dubrovin–Frobenius structure in a Hurwitz space is based on an open dense domain of a solution of a Darboux–Egoroff system [6, 19]. Hence, it is a local construction. Indeed, the canonical coordinates associated to the Hurwitz spaces are local coordinates even in the covering space described in Remark 3.5. The construction of the orbit space of $\mathcal{J}(\tilde{A}_1)$ complements the construction of the Hurwitz space $H_{1,0,0}$, because now, there exists global object where the local Dubrovin–Frobenius structure of $H_{1,0,0}$ lives. Indeed, the coordinates $(\varphi_0, \varphi_1, v_2, \tau)$ are global coordinates for the covering space (3.5), and from this fact, we derive that the Dubrovin–Frobenius structure is globally well defined in the covering. This is possible because we realise the group $\mathcal{J}(\tilde{A}_1)$ as a monodromy of orbit space $\mathcal{J}(\tilde{A}_1)$, and we know how the group $\mathcal{J}(\tilde{A}_1)$ acts on the Dubrovin–Frobenius structure.

3.4 Construction of WDVV solution

Theorem 3.7. *There exists Dubrovin–Frobenius structure on the manifold $\widetilde{\Omega/\mathcal{J}(\tilde{A}_1)}$ with the intersection form (3.1), the Euler vector field (3.2), and the unity vector field (3.3). Moreover, $\widetilde{\Omega/\mathcal{J}(\tilde{A}_1)}$ is isomorphic as Dubrovin–Frobenius manifold to $\tilde{H}_{1,0,0}$.*

Proof. The first step to be done is the computation of the intersection form in coordinates (t^1, t^2, t^3, t^4) . Hence, consider the transformation formula of ds^2 :

$$g^{\alpha\beta}(t) = \frac{\partial t^\alpha}{\partial x^i} \frac{\partial t^\beta}{\partial x^j} g^{ij}, \quad (3.6)$$

where $x^1 = u$, $x^2 = v_0$, $x^3 = v_2$, $x^4 = \tau$.

From the expression:

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2dud\tau = g_{ij}dx^i dx^j,$$

we have

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$(g^{ij}) = (g_{ij})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

To compute $g^{\alpha\beta}(t)$, let us write t^α in terms of x^i :

$$t^4 = \tau, \quad t^3 = v_2, \quad t^2 = -\frac{\theta_1(v_0 + v_2, \tau)\theta_1(v_0 - v_2, \tau)}{\theta_1(2v_2, \tau)\theta_1'(0, \tau)} e^{-2\pi i u},$$

using the following formulae [22]:

$$\begin{aligned} \frac{\wp'(v_2)}{\wp(v_0) - \wp(v_2)} &= \zeta(v_0 - v_2, \tau) - \zeta(v_0 + v_2, \tau) + 2\zeta(v_2, \tau), \\ \wp(v_0, \tau) - \wp(v_2, \tau) &= -\frac{\sigma(v_0 + v_2, \tau)\sigma(v_0 - v_2, \tau)}{\sigma^2(v_0, \tau)\sigma^2(v_2, \tau)}, \quad \frac{\sigma(2v_2, \tau)}{\sigma^4(v_2, \tau)} = -\wp'(v_2, \tau), \end{aligned}$$

it is possible to rewrite t^1 in a more suitable way

$$\begin{aligned} t^1 &= -t^2[\zeta(v_0 - v_2, \tau) - \zeta(v_0 + v_2, \tau) + 2\zeta(v_2, \tau)] + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \\ &= -t^2 \frac{\wp'(v_2, \tau)}{\wp(v_0, \tau) - \wp(v_2, \tau)} + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \\ &= -t^2 \frac{\wp'(v_2, \tau)\theta_1^2(v_2, \tau)\theta_1^2(v_0, \tau)}{\theta_1(v_0 + v_2, \tau)\theta_1(v_0 - v_2, \tau)\theta_1'(0, \tau)^2} + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \\ &= -\frac{\wp'(v_2, \tau)\theta_1^2(v_2, \tau)\theta_1^2(v_0, \tau)}{\theta_1(2v_2, \tau)\theta_1'(0, \tau)^3} e^{-2\pi i u} + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} = \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)}. \end{aligned}$$

To summarize

$$\begin{aligned} t^1 &= \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + 2t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)}, \\ t^2 &= -\frac{\theta_1(v_0 + v_2, \tau)\theta_1(v_0 - v_2, \tau)}{\theta_1(2v_2, \tau)\theta_1'(0, \tau)} e^{-2\pi i u}, \quad t^3 = v_2, \quad t^4 = \tau. \end{aligned}$$

Computing $g^{\alpha\beta}$ according to (3.6)

$$g^{\alpha\beta} = \frac{1}{2} \frac{\partial t^\alpha}{\partial v_0} \frac{\partial t^\beta}{\partial v_0} - \frac{1}{2} \frac{\partial t^\alpha}{\partial v_2} \frac{\partial t^\beta}{\partial v_2} + \frac{\partial t^\alpha}{\partial u} \frac{\partial t^\beta}{\partial \tau} + \frac{\partial t^\alpha}{\partial \tau} \frac{\partial t^\beta}{\partial u}.$$

Trivially, we get

$$g^{44} = g^{34} = 0, \quad g^{33} = -\frac{1}{2}, \quad g^{24} = -2\pi i t^2, \quad g^{14} = -2\pi i t^1.$$

The following non-trivial terms are computed in Appendix A:

$$\begin{aligned} g^{23} &= -\frac{t^1}{2} + t^2 \frac{\theta'_1(2t^3, \tau)}{\theta_1(2t^3, \tau)}, & g^{13} &= -2\pi i t^2 \frac{\partial}{\partial \tau} \left(\log \frac{\theta'_1(0, \tau)}{\theta_1(2t^3, \tau)} \right), \\ g^{22} &= 2(t^2)^2 \left[\frac{\theta''_1(2t^3, \tau)}{\theta_1(2t^3, \tau)} - \frac{\theta_1'^2(2t^3, \tau)}{\theta_1^2(2t^3, \tau)} \right], \\ g^{12} &= -2\pi i (t^2)^2 \left[\frac{\partial^2}{\partial t^3 \partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta_1(2t^3, \tau)} \right) \right) \right], \end{aligned} \quad (3.7)$$

$$\begin{aligned} g^{11} &= -4(t^2)^2 \frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} \frac{\partial}{\partial t^3} \left(\frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} \right) \left[2 \frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} - 2 \frac{\theta'_1(2t^3, \tau)}{\theta_1(2t^3, \tau)} \right] \\ &\quad + 8 \frac{\theta_1'^2(t^3, \tau)}{\theta_1^2(t^3, \tau)} (t^2)^2 \left[\frac{\theta''_1(2t^3, \tau)}{\theta_1(2t^3, \tau)} - \frac{\theta_1'^2(2t^3, \tau)}{\theta_1^2(2t^3, \tau)} \right] - 2(t^2)^2 \left[\frac{\partial}{\partial t^3} \left(\frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} \right) \right]^2 \\ &\quad - 16\pi i (t^2)^2 \frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(t^3, \tau)}{\theta_1(t^3, \tau)} \right). \end{aligned} \quad (3.8)$$

Differentiating $g^{\alpha\beta}$ w.r.t. t^1 we obtain a constant matrix η^* :

$$(\eta^{\alpha\beta}) = \frac{\partial}{\partial t^1} (g^{\alpha\beta}) = \begin{bmatrix} 0 & 0 & 0 & -2\pi i \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -2\pi i & 0 & 0 & 0 \end{bmatrix}.$$

So t^1, t^2, t^3, t^4 are the flat coordinates.

The next step is to calculate the matrix $F^{\alpha\beta}$ using the formula

$$F^{\alpha\beta} = \frac{g^{\alpha\beta}}{\deg(g^{\alpha\beta})}. \quad (3.9)$$

We can compute $\deg(g^{\alpha\beta})$ using the fact that we compute $\deg(t^\alpha)$. Indeed,

$$E = -\frac{1}{2\pi i} \frac{\partial}{\partial u}.$$

This implies that

$$\deg(t^1) = \deg(t^2) = 1, \quad \deg(t^3) = \deg(t^4) = 0.$$

Then, the function F is obtained from the equation

$$\frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} = \eta_{\alpha\alpha'} \eta_{\beta\beta'} F^{\alpha'\beta'}.$$

Computing

$$F^{\alpha 4} = \frac{g^{\alpha 4}}{\deg(g^{\alpha 4})},$$

we derive

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}.$$

Hence,

$$F = \frac{i}{4\pi} (t^1)^2 t^4 - 2t^1 t^2 t^3 + f(t^2, t^3, t^4). \quad (3.10)$$

Substituting F^{23} and F^{13} in (3.10)

$$F = \frac{i}{4\pi} (t^1)^2 t^4 - 2t^1 t^2 t^3 - (t^2)^2 \log \left(\frac{\theta'_1(0, t^4)}{\theta_1(2t^3, t^4)} \right) + h(t^2) + A_{\alpha\beta} t^\alpha t^\beta + C_\alpha t^\alpha + D,$$

where $A_{\alpha\beta}$, C_α , C_α are constants. Note that F^{22} , F^{12} contains the same information, furthermore, there is no information in F^{33} , F^{34} , F^{44} because

$$\deg(g^{33}) = \deg(g^{34}) = \deg(g^{44}) = 0.$$

However, $h(t^2)$ can be computed by using g^{33}

$$g^{33} = -\frac{1}{2} = E^\epsilon \eta^{3\mu} \eta^{3\lambda} c_{\epsilon\mu\lambda} = \frac{t^2}{4} c_{222}.$$

Using the formula (1.1), we have

$$F(t^1, t^2, t^3, t^4) = \frac{i}{4\pi} (t^1)^2 t^4 - 2t^1 t^2 t^3 - (t^2)^2 \log \left(t^2 \frac{\theta'_1(0, t^4)}{\theta_1(2t^3, t^4)} \right). \quad (3.11)$$

The remaining part of proof is to show that the equation (3.11) satisfies WDDV equations. Let us prove it step by step.

1. *Commutative of the algebra.* Defining the structure constant of the algebra as

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma},$$

commutative is straightforward.

2. *Normalization.* Using equation (3.11), we obtain

$$c_{1\alpha\beta}(t) = \frac{\partial^3 F}{\partial t^1 \partial t^\beta \partial t^\gamma} = \eta_{\alpha\beta}.$$

3. *Quasi homogeneity.* Applying the Euler vector field in the function (3.11), we have

$$E(F) = 2F - 2t^2.$$

4. *Associativity.* In order to prove that the algebra is associativity, we will first shown that the algebra is semisimple. First of all, note that the multiplication by the Euler vector field is equivalent to the intersection form. Indeed,

$$E \bullet \partial_\alpha = t^\sigma c_{\sigma\alpha}^\beta \partial_\beta = t^\sigma \partial_\sigma (\eta^{\beta\mu} \partial_\alpha \partial_\mu F) \partial_\beta = (d_\alpha - d_\beta) \eta^{\beta\mu} \partial_\alpha \partial_\mu F \partial_\beta = \eta_{\alpha\mu} g^{\mu\beta} \partial_\beta. \quad (3.12)$$

Therefore, the multiplication by the Euler vector field is semisimple if the following polynomial

$$\det(\eta_{\alpha\mu}g^{\mu\beta} - u\delta_\alpha^\beta) = 0, \quad (3.13)$$

has only simple roots; since $\det(\eta_{\alpha\mu}) \neq 0$, the equation (3.13) is equivalent to

$$\det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = 0.$$

Using that $\eta^{\alpha\beta} = \partial_1 g^{\alpha\beta}$, we have that

$$\det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = \det(g^{\alpha\beta}(t^1 - u, t^2, t^3, t^4)) = 0.$$

So, this is enough to compute $\det g^{\alpha\beta}$. In particular, computing $\det g$ in the coordinates $(\varphi_0, \varphi_1, v_2, \tau)$. Recall that

$$\begin{aligned} g^{\alpha\beta} &= g(dt^\alpha, dt^\beta), & \text{in coordinates } (t^1, t^2, t^3, t^4), \\ g^{lm} &= g(dv_l, dv_m), & \text{in coordinates } (u, v_0, v_2, \tau), \\ g^{ij} &= g(d\varphi_i, d\varphi_j), & \text{in coordinates } (\varphi_0, \varphi_1, v_2, \tau). \end{aligned}$$

Then,

$$\det g^{ij} = \det\left(\frac{\partial\varphi_i}{\partial v_l}\right) \det\left(\frac{\partial\varphi_j}{\partial v_m}\right) \det(g^{lm}).$$

Remark 3.8. The coordinates (u, v_0, v_2, τ) are defined away from the submanifold defined by $\det g = 0$. Therefore, we have to change coordinates to compute the roots of $\det g = 0$.

Hence, it is enough to compute the $\det\left(\frac{\partial\varphi_i}{\partial v_l}\right)$

$$\begin{aligned} \det\left(\frac{\partial\varphi_i}{\partial v_l}\right) &= \begin{vmatrix} \frac{\partial\varphi_0}{\partial v_0} & \frac{\partial\varphi_0}{\partial v_2} & \frac{\partial\varphi_0}{\partial\tau} & -2\pi i\varphi_0 \\ \frac{\partial\varphi_1}{\partial v_0} & \frac{\partial\varphi_1}{\partial v_2} & \frac{\partial\varphi_1}{\partial\tau} & -2\pi i\varphi_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \\ &= -2\pi i\varphi_0\varphi_1 \left[2\frac{\theta'_1(v_0)}{\theta_1(v_0)} - \frac{\theta'_1(-v_0 + v_2)}{\theta_1(-v_0 + v_2)} + \frac{\theta'_1(v_0 + v_2)}{\theta_1(v_0 + v_2)} \right] \\ &= -2\pi i e^{-4\pi i u} \frac{\theta_1(2v_0)}{\theta_1(2v_2)\theta_1(0)^2}. \end{aligned} \quad (3.14)$$

Then, equation (3.14) has four distinct roots $v_0 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$. Hence, the following system of equations

$$\det(g^{\alpha\beta}(t^1, t^2, t^3, t^4)) = 0, \quad \det(\eta^{\alpha\beta}(t^1, t^2, t^3, t^4)) \neq 0, \quad (3.15)$$

implies in existence of four functions $y_i(t^2, t^3, t^4)$ such that

$$t^1 = y_i(t^2, t^3, t^4), \quad i = 1, 2, 3, 4.$$

Sending $t^1 \mapsto t^1 - u$ in (3.15), we obtain

$$u^i = t^1 - y^i(t^2, t^3, t^4), \quad i = 1, 2, 3, 4.$$

The multiplication by the Euler vector field

$$g_j^i = \eta_{jk} g^{ki}, \quad \text{in canonical coordinates } (u^1, u^2, u^3, u^4)$$

is diagonal, so

$$g^{ij} = u^i \eta^{ij} \delta_{ij},$$

where η^{ij} is the canonical coordinates (u^1, u^2, u^3, u^4) , and the unit vector field have the following form:

$$\frac{\partial}{\partial t^1} = \sum_{i=1}^4 \frac{\partial u_i}{\partial t^1} \frac{\partial}{\partial u_i} = \sum_{i=1}^4 \frac{\partial}{\partial u_i}.$$

Moreover, since

$$[E, e] = \left[t^1 \frac{\partial}{\partial t^1} + t^2 \frac{\partial}{\partial t^2}, \frac{\partial}{\partial t^1} \right] = -e,$$

the Euler vector field in the coordinates (u^1, u^2, u^3, u^4) takes the following form:

$$E = \sum_{i=1}^4 u^i \frac{\partial}{\partial u_i}.$$

Using the relationship (3.12) between the coordinates (u^1, u^2, u^3, u^4) , we have

$$u^i \eta^{ij} \delta_{ij} = u^l \eta^{im} \eta^{jn} c_{lmn}, \quad (3.16)$$

differentiating both side of the equation (3.16) with respect t^1

$$c_{ij}^k = \delta_{ij},$$

which proves that the algebra is associative and semisimple.

Therefore, we proved that the equation (3.11) satisfies the WDVV equation. Moreover, the function F (3.11) is exactly the Free energy of the Dubrovin–Frobenius manifold of the Hurwitz space $\tilde{H}_{1,0,0}$. Hence, the covering of orbit space of $\mathcal{J}(\tilde{A}_1)$ and the covering over the Hurwitz space $H_{1,0,0}$ are isomorphic as a Dubrovin–Frobenius manifold, because they have the same WDVV solution. ■

Remark 3.9. Even though the Dubrovin–Frobenius structure constructed in a suitable covering of the orbit space of $\mathcal{J}(\tilde{A}_1)$ is isomorphic as a Dubrovin–Frobenius manifold to a suitable covering of the Hurwitz space $H_{1,0,0}$, this fact does not mean that the construction presented in this paper is equivalent to the Hurwitz space construction, because:

1. The constructions start with different hypotheses. Indeed, we derive a WDVV solution in the Hurwitz space framework from the data of the Hurwitz space itself, and through the choice of a suitable primary differential; see [6] and [19] for the definition. On another hand, the orbit space construction is derived from the data of the group $\mathcal{J}(\tilde{A}_1)$.
2. The Hurwitz space construction is based on domain of a solution of a Darboux–Egorrof system. The coordinate system associated with this system of equation is called canonical coordinates. Therefore, the Hurwitz space construction is a local construction, since it is based in a local solution of a system of equations. The orbit space construction, in the other hand, is built based on the invariant coordinates $(\varphi_0, \varphi_1, v_2, \tau)$, which some how have a global meaning. Furthermore, note that the existence of the invariant coordinates $(\varphi_0, \varphi_1, v_2, \tau)$ is not guaranteed in the Hurwitz space construction.

3. The intersection form (3.1), the Euler vector field (3.2), and the unity vector field (3.3) are intrinsic objects of the orbit space $\mathcal{J}(\hat{A}_1)$. Therefore, the Theorem 3.7 derives the WDVV solution (3.11) by using the equation (3.9) without using the correspondence with the Hurwitz space $H_{1,0,0}$. This argument was already used in the introduction of [7] to demonstrate the difference between the Hurwitz space construction on the $H_{0,n}$ and the orbit space construction of the orbit space of A_n .

Remark 3.10. The WDVV solution (3.11) was presented on p. 28 of [8]. However, there is a typo in the last term of the WDVV solution in the paper [8]. The WDVV solution in a correct form can also be found in [5] and [13].

4 Conclusion

The WDVV solution of $H_{1,0,0}$, which is (3.11), contains the term $\log\left(\frac{\theta'_1(0,t^4)}{\theta_1(2t^3,t^4)}\right)$ on the two exceptional variables (t^3, t^4) . This is a reflection of how the ring of invariants affects the WDVV solution. The same pattern is obtained in $\mathcal{J}(A_1)$, and \hat{A}_1 . The equation (2.8) contains $E_2(\tau)$ which is a quasi modular form, and the equation (2.2) contains e^{t^2} . These facts could be useful in regards to the understanding of the WDVV/groups correspondence.

The arrows of the diagram of in Section 2.1 may have a third meaning, which is an embedding of Dubrovin–Frobenius submanifolds [20, 21] in to the ambient space $H_{1,0,0}$. The fact that $H_{1,0,0}$ contains three Dubrovin–Frobenius submanifolds is not an accident. This comes from the tri-Hamiltonian structure that $H_{1,0,0}$ has [15, 16]. In a subsequent publication, we will study the Dubrovin–Frobenius manifolds of $H_{1,0,0}$, and its associated integrable systems.

A Appendix

Computing g^{12} :

$$\begin{aligned} g^{23} &= -\frac{1}{2} \frac{\partial t^2}{\partial v_2} = -\frac{t^2}{2} \left[-\frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2\frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\ &= -\frac{t^2}{2} \left[-\frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] - t^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \\ &= -\frac{1}{2\varphi'(v_2, \tau)} [-\zeta(v_0 - v_2, \tau) + \zeta(v_0 + v_2, \tau) - 2\zeta(v_2, \tau)] - t^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \\ &= \frac{1}{2} \frac{1}{\varphi(z_0, \tau) - \varphi(z_2, \tau)} - t^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} = -\frac{t^1}{2} + t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)}. \end{aligned}$$

Computing g^{13} :

$$\begin{aligned} g^{13} &= -\frac{1}{2} \frac{\partial t^1}{\partial v_2} = -\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{1}{\varphi(z_0) - \varphi(z_2)} - \frac{\partial t^2}{\partial v_2} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - t^2 \left[\frac{\theta''_1(v, \tau)}{\theta_1(v, \tau)} - \frac{\theta_1''(v, \tau)}{\theta_1^2(v, \tau)} \right] \\ &= -\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{1}{\varphi(z_0) - \varphi(z_2)} - t^1 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \\ &\quad - t^2 \left[\frac{\theta''_1(v, \tau)}{\theta_1(v, \tau)} - \frac{\theta_1''(v, \tau)}{\theta_1^2(v, \tau)} \right] \\ &= -2t^2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 2t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - t^2 \left[\frac{\theta''_1(v, \tau)}{\theta_1(v, \tau)} - \frac{\theta_1''(v, \tau)}{\theta_1^2(v, \tau)} \right] \end{aligned}$$

$$= -t^2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 2t^2 \frac{\theta_1'(2v_2, \tau)}{\theta_1(2v_2, \tau)} \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} - t^2 \frac{\theta_1''(v, \tau)}{\theta_1(v, \tau)}. \quad (\text{A.1})$$

To simplify this expression we need the following lemma.

Lemma A.1 ([2]). *When $x + y + z = 0$ holds*

$$\begin{aligned} & \frac{\theta_1''(x, \tau)}{\theta_1(x, \tau)} + \frac{\theta_1''(y, \tau)}{\theta_1(y, \tau)} - 2 \frac{\theta_1'(x, \tau)}{\theta_1(x, \tau)} \frac{\theta_1'(y, \tau)}{\theta_1(y, \tau)} \\ &= 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta_1'(0, \tau)}{\theta_1(x-y, \tau)} \right) \right) + 2 \frac{\theta_1'(x-y, \tau)}{\theta_1(x-y, \tau)} \left[\frac{\theta_1'(x, \tau)}{\theta_1(x, \tau)} - \frac{\theta_1'(y, \tau)}{\theta_1(y, \tau)} \right]. \end{aligned} \quad (\text{A.2})$$

Proof. Applying the formulas

$$\zeta(v, \tau) = \frac{\theta_1'(v, \tau)}{\theta_1(v, \tau)} + 4\pi i g_1(\tau) v, \quad \wp(v, \tau) = -\frac{\theta_1''(v, \tau)}{\theta_1(v, \tau)} + \left(\frac{\theta_1'(v, \tau)}{\theta_1(v, \tau)} \right)^2 - 4\pi i g_1(\tau),$$

in the identity [22]

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 = \wp(x) + \wp(y) + \wp(z),$$

we get

$$\begin{aligned} & \left(\frac{\theta_1'(x, \tau)}{\theta_1(x, \tau)} + \frac{\theta_1'(y, \tau)}{\theta_1(y, \tau)} + \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)} \right)^2 \\ &= -12\pi i g_1(\tau) - \frac{\theta_1''(x, \tau)}{\theta_1(x, \tau)} + \frac{\theta_1''(x, \tau)}{\theta_1^2(x, \tau)} - \frac{\theta_1''(y, \tau)}{\theta_1(y, \tau)} + \frac{\theta_1''(y, \tau)}{\theta_1^2(y, \tau)} - \frac{\theta_1''(z, \tau)}{\theta_1(z, \tau)} + \frac{\theta_1''(z, \tau)}{\theta_1^2(z, \tau)}. \end{aligned}$$

Simplifying

$$\begin{aligned} & 2 \frac{\theta_1'(x-y, \tau)}{\theta_1(x-y, \tau)} \left[\frac{\theta_1'(x, \tau)}{\theta_1(x, \tau)} - \frac{\theta_1'(y, \tau)}{\theta_1(y, \tau)} \right] + 2 \frac{\theta_1'(x, \tau)}{\theta_1(x, \tau)} \frac{\theta_1'(y, \tau)}{\theta_1(y, \tau)} \\ &= 3 \frac{\eta}{\omega} - \frac{\theta_1''(x, \tau)}{\theta_1(x, \tau)} - \frac{\theta_1''(y, \tau)}{\theta_1(y, \tau)} - \frac{\theta_1''(z, \tau)}{\theta_1(z, \tau)}, \end{aligned}$$

using the fact that

$$4\pi i \frac{\partial_\tau \theta_1'(0, \tau)}{\theta_1'(0, \tau)} = -12\pi i g_1(\tau), \quad \frac{\partial^2}{\partial^2 v} \theta_1(v, \tau) = 4\pi i \frac{\partial}{\partial \tau} \theta_1(v, \tau), \quad (\text{A.3})$$

and doing the substitution $y \mapsto -y$, $z \mapsto x - y$, we get the desired identity. \blacksquare

Substituting in the lemma $x = v_2$, $y = -v_2$ we get

$$2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} + 2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} = 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta_1'(0, \tau)}{\theta_1(2v_2, \tau)} \right) \right) + 4 \frac{\theta_1'(2v_2, \tau)}{\theta_1(2v_2, \tau)} \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)}. \quad (\text{A.4})$$

Substituting (A.4) in (A.1)

$$g^{13} = -2\pi i t^2 \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta_1'(0, \tau)}{\theta_1(2v_2, \tau)} \right) \right).$$

Computing g^{22} :

$$g^{22} = \frac{1}{2} \left(\frac{\partial t^2}{\partial v_0} \right)^2 - \frac{1}{2} \left(\frac{\partial t^2}{\partial v_2} \right)^2 + 2 \frac{\partial t^2}{\partial u} \frac{\partial t^2}{\partial \tau} = \frac{1}{2} \left(\frac{\partial t^2}{\partial v_0} \right)^2 - \frac{1}{2} \left(\frac{\partial t^2}{\partial v_2} \right)^2 - 4\pi i t^2 \frac{\partial t^2}{\partial \tau}.$$

First, we separately compute $\frac{\partial t^2}{\partial v_2}$, $\frac{\partial t^2}{\partial v_0}$, $\frac{\partial t^2}{\partial \tau}$

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial t^2}{\partial v_0} \right)^2 &= \frac{(t^2)^2}{2} \left[\frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} + \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \right]^2, \\ -\frac{1}{2} \left(\frac{\partial t^2}{\partial v_2} \right)^2 &= -\frac{(t^2)^2}{2} \left[-\frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2\frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right]^2, \\ -4\pi i t^2 \frac{\partial t^2}{\partial \tau} &= -4\pi i \frac{(t^2)^2}{2} \left[\frac{\partial_\tau \theta_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} + \frac{\partial_\tau \theta_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} - \frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\ &\quad - 4\pi i \frac{(t^2)^2}{2} \left[-\frac{\partial_\tau \theta'_1(0, \tau)}{\theta'_1(0, \tau)} \right]. \end{aligned}$$

Summing the equations we get

$$\begin{aligned} g^{22} &= \frac{(t^2)^2}{2} \left[4 \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \right] \\ &\quad + \frac{(t^2)^2}{2} \left[4 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \left[-\frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \right] - 4 \frac{\theta_1''(2v_2, \tau)}{\theta_1^2(2v_2, \tau)} \right] \\ &\quad + \frac{(t^2)^2}{2} \left[-2 \frac{\theta_1''(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2 \frac{\theta_1''(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} - 8\pi i \left[-\frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\partial_\tau \theta'_1(0, \tau)}{\theta'_1(0, \tau)} \right] \right], \end{aligned}$$

where was used (A.3). Substituting in Lemma A.1 $x = v_0 + v_2$, $y = v_0 - v_2$ we get

$$\begin{aligned} &\frac{\theta_1''(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta_1''(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2 \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \\ &= 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta(2v_2, \tau)} \right) \right) + 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \left[\frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \right]. \end{aligned}$$

Substituting the last identity in g^{22} we get

$$g^{22} = 2(t^2)^2 \left[\frac{\theta_1''(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\theta_1'^2(2v_2, \tau)}{\theta_1^2(2v_2, \tau)} \right].$$

Computing g^{12} :

$$\begin{aligned} g^{12} &= \frac{1}{2} \frac{\partial t^1}{\partial v_0} \frac{\partial t^2}{\partial v_0} - \frac{1}{2} \frac{\partial t^1}{\partial v_2} \frac{\partial t^2}{\partial v_2} + \frac{\partial t^1}{\partial u} \frac{\partial t^2}{\partial \tau} + \frac{\partial t^2}{\partial u} \frac{\partial t^1}{\partial \tau} \\ &= \frac{1}{2} \frac{\partial t^1}{\partial v_0} \frac{\partial t^2}{\partial v_0} - \frac{1}{2} \frac{\partial t^1}{\partial v_2} \frac{\partial t^2}{\partial v_2} - 2\pi i t^2 \frac{\partial t^1}{\partial \tau} - 2\pi i t^1 \frac{\partial t^2}{\partial \tau}. \end{aligned}$$

We have that

$$\begin{aligned} \frac{\partial t^1}{\partial v_0} &= 2 \frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + 2 \frac{\partial t^2}{\partial v_0} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)}, \\ \frac{\partial t^1}{\partial v_2} &= -2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + 2 \frac{\partial t^2}{\partial v_2} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2t^2 \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right], \\ \frac{\partial t^1}{\partial \tau} &= 2 \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + 2 \frac{\partial t^2}{\partial \tau} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2t^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right). \end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2} \frac{\partial t^1}{\partial v_0} \frac{\partial t^2}{\partial v_0} &= t^2 \left[\frac{\theta'_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} + \frac{\theta'_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} \right] \frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} + \left(\frac{\partial t^2}{\partial v_0} \right)^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)}, \\
-\frac{1}{2} \frac{\partial t^1}{\partial v_2} \frac{\partial t^2}{\partial v_2} &= -t^2 \left[-\frac{\theta'_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} + \frac{\theta'_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} \right] \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \\
&\quad - t^2 \left[-2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} - \left(\frac{\partial t^2}{\partial v_2} \right)^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \\
&\quad - t^2 \frac{\partial t^2}{\partial v_2} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right], \\
-2\pi i t^1 \frac{\partial t^2}{\partial \tau} &= -2\pi i \left[\frac{\partial_\tau \theta_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} + \frac{\partial_\tau \theta_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} \right] t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \\
&\quad - 2\pi i \left[-\frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\partial_\tau \theta'_1(0, \tau)}{\theta'_1(0, \tau)} \right] t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} - 4\pi i t^2 \frac{\partial t^2}{\partial \tau} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)}, \\
-2\pi i t^2 \frac{\partial t^1}{\partial \tau} &= -4\pi i t^2 \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} - 4\pi i t^2 \frac{\partial t^2}{\partial \tau} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \\
&\quad - 4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right).
\end{aligned}$$

Let us separate g^{12} in three terms

$$g^{12} = (1) + (2) + (3),$$

where

$$\begin{aligned}
(1) &= t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[\frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \left(\frac{\theta'_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} + \frac{\theta'_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} \right) \right] \\
&\quad + t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \left(-\frac{\theta'_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} + \frac{\theta'_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right) \right] \\
&\quad + t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-2\pi i \left(\frac{\partial_\tau \theta_1(v_0+v_2, \tau)}{\theta_1(v_0+v_2, \tau)} + \frac{\partial_\tau \theta_1(v_0-v_2, \tau)}{\theta_1(v_0-v_2, \tau)} \right) \right] \\
&\quad + t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-2\pi i \left(-\frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\partial_\tau \theta'_1(0, \tau)}{\theta'_1(0, \tau)} \right) \right] \\
&\quad + t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-4\pi i \left(\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \right], \\
(2) &= \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \left[\left(\frac{\partial t^2}{\partial v_0} \right)^2 - \left(\frac{\partial t^2}{\partial v_2} \right)^2 - 8\pi i t^2 \frac{\partial t^2}{\partial \tau} \right] \\
&= 4 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} (t^2)^2 \left[\frac{\theta''_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\theta_1''(2v_2, \tau)}{\theta_1^2(2v_2, \tau)} \right],
\end{aligned}$$

where was used the previous computation of g^{22}

$$(3) = -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) - t^2 \frac{\partial t^2}{\partial v_2} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right].$$

To simplify the expression (1) we need to use the Lemma A.1 with the following substitutions

$x = v_0, y = v_2$

$$\frac{\theta''_1(v_0, \tau)}{\theta_1(v_0, \tau)} + \frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)}$$

$$= 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta(v_0 - v_2, \tau)} \right) \right) + 2 \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \left[\frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right]. \quad (\text{A.5})$$

Using the substitutions $x = v_0$, $y = -v_2$

$$\begin{aligned} & \frac{\theta''_1(v_0, \tau)}{\theta_1(v_0, \tau)} + \frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2 \frac{\theta'_1(v_0, \tau) \theta'_1(v_2, \tau)}{\theta_1(v_0, \tau) \theta_1(v_2, \tau)} \\ &= 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta(v_0 + v_2, \tau)} \right) \right) + 2 \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \left[\frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} + \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right]. \end{aligned} \quad (\text{A.6})$$

Summing (A.5) with (A.6)

$$\begin{aligned} & 2 \frac{\theta''_1(v_0, \tau)}{\theta_1(v_0, \tau)} + 2 \frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta(v_0 - v_2, \tau)} \right) \right) - 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta(v_0 + v_2, \tau)} \right) \right) \\ &= 2 \frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \left(\frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} + \frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \right) \\ & \quad + 2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \left(-\frac{\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \right). \end{aligned}$$

Substituting in (1) we get

$$\begin{aligned} (1) &= t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-2 \frac{\theta'_1(2v_2, \tau) \theta'_1(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1(v_2, \tau)} \right] \\ & \quad + t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-2\pi i \left(-\frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} + \frac{\partial_\tau \theta'_1(0, \tau)}{\theta'_1(0, \tau)} \right) + 8\pi i \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \\ &= t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[-2 \frac{\theta'_1(2v_2, \tau) \theta'_1(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1(v_2, \tau)} - 2\pi i \frac{\partial}{\partial \tau} \left(\log \frac{\theta'_1(0, \tau)}{\theta_1(2v_2, \tau)} \right) + 2 \frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right]. \end{aligned}$$

Using the identity (A.2), we get

$$(1) = t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right].$$

We compute (3)

$$\begin{aligned} (3) &= -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) - t^2 \frac{\partial^2}{\partial v_2} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \\ &= -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) - t^2 \left(t^1 - 2t^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right) \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \\ &= -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) + 2(t^2)^2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \\ & \quad - t^2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] - 2(t^2)^2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right]. \end{aligned}$$

The result implies

$$\begin{aligned} (1) + (3) &= -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \\ & \quad - 2(t^2)^2 \left[\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right]. \end{aligned}$$

Computing g^{12} :

$$g^{12} = -4\pi i (t^2)^2 \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) - 2(t^2)^2 \left[\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta''_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\ + 4 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} (t^2)^2 \left[\frac{\theta''_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right].$$

To simplify this expression we need to prove one more lemma.

Lemma A.2.

$$2 \frac{\theta'''_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2 \frac{\theta''_1(v_2, \tau) \theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 4 \frac{\theta_1^3(v_2, \tau)}{\theta_1^3(v_2, \tau)} \\ = 4\pi i \frac{\partial^2}{\partial v_2 \partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta_1(2v_2, \tau)} \right) \right) + 8 \frac{\theta''_1(2v_2, \tau) \theta'_1(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1(v_2, \tau)} - 8 \frac{\theta_1^2(2v_2, \tau) \theta'_1(v_2, \tau)}{\theta_1^2(2v_2, \tau) \theta_1(v_2, \tau)} \\ + 4 \frac{\theta'_1(2v_2, \tau) \theta''_1(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1(v_2, \tau)} - 4 \frac{\theta'_1(2v_2, \tau) \theta_1^2(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1^2(v_2, \tau)}. \quad (\text{A.7})$$

Proof. Differentiating the identity with respect to v_2 we obtain (A.7). ■

Computing g^{12} :

$$g^{12} = (t^2)^2 \left[-\frac{\theta'''_1(v_2, \tau)}{\theta_1(v_2, \tau)} + \frac{\theta'_1(v_2, \tau) \theta''_1(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 2 \frac{\theta'_1(v_2, \tau) \theta''_1(v_2, \tau)}{\theta_1(v_2, \tau) \theta_1(v_2, \tau)} + 2 \frac{\theta_1^3(v_2, \tau)}{\theta_1^3(v_2, \tau)} \right] \\ + (t^2)^2 \left[2 \frac{\theta'_1(2v_2, \tau) \theta''_1(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau) \theta_1^2(v_2, \tau)}{\theta_1(2v_2, \tau) \theta_1^2(v_2, \tau)} + 4 \frac{\theta'_1(v_2, \tau) \theta''_1(2v_2, \tau)}{\theta_1(v_2, \tau) \theta_1(2v_2, \tau)} \right] \\ + (t^2)^2 \left[-4 \frac{\theta'_1(v_2, \tau) \theta_1^2(2v_2, \tau)}{\theta_1(v_2, \tau) \theta_1^2(2v_2, \tau)} \right].$$

Applying (A.7), we get

$$g^{12} = -2\pi i (t^2)^2 \left[\frac{\partial^2}{\partial v_2 \partial \tau} \left(\log \left(\frac{\theta'_1(0, \tau)}{\theta_1(2v_2, \tau)} \right) \right) \right].$$

Computing g^{11} :

$$g^{11} = \frac{1}{2} \left(\frac{\partial t^1}{\partial v_0} \right)^2 - \frac{1}{2} \left(\frac{\partial t^1}{\partial v_2} \right)^2 + 2 \frac{\partial t^1}{\partial u} \frac{\partial t^1}{\partial \tau} = \frac{1}{2} \left(\frac{\partial t^1}{\partial v_0} \right)^2 - \frac{1}{2} \left(\frac{\partial t^1}{\partial v_2} \right)^2 - 4\pi i t^1 \frac{\partial t^1}{\partial \tau}.$$

Computing $\frac{1}{2} \left(\frac{\partial t^1}{\partial v_0} \right)^2$, $\frac{1}{2} \left(\frac{\partial t^1}{\partial v_2} \right)^2$ and $-4\pi i t^1 \frac{\partial t^1}{\partial \tau}$:

To simplify the computation let us define

$$A := \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_2, \tau)} e^{-2\pi i u}.$$

Then,

$$\frac{1}{2} \left(\frac{\partial t^1}{\partial v_0} \right)^2 = 2 \frac{\theta_1^2(v_0, \tau)}{\theta_1^2(v_0, \tau)} A^2 + 4A \frac{\theta'_1(v_0, \tau)}{\theta_1(v_0, \tau)} \frac{\partial t^2}{\partial v_0} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2 \left(\frac{\partial t^2}{\partial v_0} \right)^2 \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)}, \\ -\frac{1}{2} \left(\frac{\partial t^1}{\partial v_2} \right)^2 = -2 \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} A^2 + 2A \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \left[2 \frac{\partial t^2}{\partial v_2} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} + 2t^2 \left[\frac{\theta''_1(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \right]$$

$$\begin{aligned}
& -2 \left(\frac{\partial t^2}{\partial v_2} \right)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 4t^2 \frac{\partial t^2}{\partial v_2} \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \\
& - 2(t^2)^2 \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right]^2, \\
-4\pi i t^1 \frac{\partial t^1}{\partial \tau} &= -8\pi i A^2 \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] - 8\pi i A \frac{\partial t^2}{\partial \tau} \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \\
& - 8\pi i A t^2 \frac{\partial}{\partial \tau} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right) - 16\pi i A t^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \\
& - 16\pi i t^2 \frac{\partial t^2}{\partial \tau} \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 16\pi i (t^2)^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial \tau} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right).
\end{aligned}$$

Then, we have

$$g^{11} = (1) + (2) + (3) + (4) + (5),$$

where

$$\begin{aligned}
(1) &= A^2 \left[2 \frac{\theta_1'^2(v_0, \tau)}{\theta_1^2(v_0, \tau)} - 2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 8\pi i \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \right] \\
&= A^2 \left[2 \frac{\theta_1'^2(v_0, \tau)}{\theta_1^2(v_0, \tau)} - 2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} - 2 \frac{\theta_1''(v_0, \tau)}{\theta_1(v_0, \tau)} + 2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \\
&= 2A^2 [\wp(v_0) - \wp(v_2)] = 2 \frac{16\omega^4}{[\wp(v_0) - \wp(v_2)]^2} [\wp(v_0) - \wp(v_2)] = 32 \frac{\omega^4}{\wp(v_0) - \wp(v_2)}, \\
(2) &= -8\pi i t^2 A \frac{\partial}{\partial \tau} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right) + 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[2 \frac{\theta_1'(v_0, \tau)}{\theta_1(v_0, \tau)} \left[\frac{\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \right] \right] \\
&+ 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \left[\frac{-\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2 \frac{\theta_1'(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \right] \\
&+ 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[2 \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] - 8\pi i \left[\frac{\partial_\tau \theta_1(v_0, \tau)}{\theta_1(v_0, \tau)} - \frac{\partial_\tau \theta_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \right] \\
&+ 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[-4\pi i \left[\frac{\partial_\tau \theta_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} + \frac{\partial_\tau \theta_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} - \frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\partial_\tau \theta_1'(0, \tau)}{\theta_1'(0, \tau)} \right] \right].
\end{aligned}$$

Using (A.2),

$$\begin{aligned}
& 2 \frac{\theta_1''(v_0, \tau)}{\theta_1(v_0, \tau)} + 2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta_1'(0, \tau)}{\theta(v_0 - v_2, \tau)} \right) \right) - 4\pi i \frac{\partial}{\partial \tau} \left(\log \left(\frac{\theta_1'(0, \tau)}{\theta(v_0 + v_2, \tau)} \right) \right) \\
&= 2 \frac{\theta_1'(v_0, \tau)}{\theta_1(v_0, \tau)} \left(\frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} + \frac{\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} \right) \\
&+ 2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \left(-\frac{\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} \right), \\
(2) &= -8\pi i t^2 A \frac{\partial}{\partial \tau} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right) + 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[-4 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\theta_1'(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&+ 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[2 \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] + 4 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \\
&+ 2A t^2 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[-4\pi i \frac{\partial_\tau \theta_1'(0, \tau)}{\theta_1'(0, \tau)} + 4\pi i \frac{\partial_\tau \theta_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right].
\end{aligned}$$

Using again (A.2),

$$\begin{aligned}
(2) &= -8\pi it^2 A \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) + 8At^2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right], \\
(3) &= 4 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[\frac{1}{2} \left(\frac{\partial t^2}{\partial v_0} \right)^2 - \frac{1}{2} \left(\frac{\partial t^2}{\partial v_2} \right)^2 - 4\pi it^2 \frac{\partial t^2}{\partial \tau} \right] \\
&= 8 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} (t^2)^2 \left[\frac{\theta_1''(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\theta_1'^2(2v_2, \tau)}{\theta_1^2(2v_2, \tau)} \right], \\
(4) &= -2(t^2)^2 \left[\frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \right]^2 - 16\pi i (t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right), \\
(5) &= -4(t^2) \frac{\partial t^2}{\partial v_2} \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \\
&= -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[\frac{-\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&= -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[\frac{-\theta'_1(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta'_1(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right] \\
&\quad - 4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&= -4(t^2) A \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \\
&\quad - 4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right].
\end{aligned}$$

Summing (2) and (5)

$$\begin{aligned}
(2) + (5) &= -8\pi it^2 A \frac{\partial}{\partial \tau} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) + 8At^2 \frac{\theta_1''(v_2, \tau)}{\theta_1^2(v_2, \tau)} \left[\frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} - \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} \right] \\
&\quad - 4(t^2) A \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \\
&\quad - 4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&= -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&\quad + At^2 \left[-2 \frac{\theta_1'''(v_2, \tau)}{\theta_1(v_2, \tau)} + 6 \frac{\theta_1''(v_2, \tau) \theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} - 4 \frac{\theta_1'^3(v_2, \tau)}{\theta_1^3(v_2, \tau)} \right] \\
&= -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] + 2At^2 \wp'(v_2) \\
&= -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] - 32 \frac{\omega^4}{\wp(v_0) - \wp(v_2)}.
\end{aligned}$$

Summing (1) and (2) + (5)

$$(1) + (2) + (5) = -4(t^2)^2 \frac{\theta_1''(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta'_1(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta'_1(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right].$$

From the above results, we find

$$g^{11} = (1) + (2) + (5) + (3) + (4)$$

$$\begin{aligned}
&= -4(t^2)^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial v_2} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \left[2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} - 2 \frac{\theta_1'(2v_2, \tau)}{\theta_1(2v_2, \tau)} \right] \\
&\quad + 8 \frac{\theta_1'^2(v_2, \tau)}{\theta_1^2(v_2, \tau)} (t^2)^2 \left[\frac{\theta_1''(2v_2, \tau)}{\theta_1(2v_2, \tau)} - \frac{\theta_1'^2(2v_2, \tau)}{\theta_1^2(2v_2, \tau)} \right] \\
&\quad - 2(t^2)^2 \left[\frac{\partial}{\partial v_2} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right) \right]^2 - 16\pi i (t^2)^2 \frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \frac{\partial}{\partial \tau} \left(\frac{\theta_1'(v_2, \tau)}{\theta_1(v_2, \tau)} \right).
\end{aligned}$$

Summarizing, we have proved the identities (3.7) and (3.8).

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