

Degree-One Rational Cherednik Algebras for the Symmetric Group

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Abstract. Drinfeld orbifold algebras deform skew group algebras in polynomial degree at most one and hence encompass graded Hecke algebras, and in particular symplectic reflection algebras and rational Cherednik algebras. We introduce parametrized families of Drinfeld orbifold algebras for symmetric groups acting on doubled representations that generalize rational Cherednik algebras by deforming in degree one. We characterize rich families of maps recording commutator relations with their linear parts supported only on and only off the identity when the symmetric group acts on the natural permutation representation plus its dual. This produces degree-one versions of \mathfrak{gl}_n -type rational Cherednik algebras. When the symmetric group acts on the standard irreducible reflection representation plus its dual there are no degree-one Lie orbifold algebra maps, but there is a three-parameter family of Drinfeld orbifold algebras arising from maps supported only off the identity. These provide degree-one generalizations of the \mathfrak{sl}_n -type rational Cherednik algebras $H_{0,c}$.

Key words: rational Cherednik algebra; skew group algebra; deformations; Drinfeld orbifold algebra; Hochschild cohomology; Poincaré–Birkhoff–Witt conditions; symmetric group

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1 Introduction

Skew group or smash-product algebras $S(V)\#G$ twist the symmetric algebra $S(V)$ of a finite-dimensional vector space V together with the action of the group algebra $\mathbb{C}G$ of a finite group G acting linearly on V . The center is the invariant polynomial ring $S(V)^G$ and there is a natural grading by polynomial degree, with elements in V of degree one and elements in $\mathbb{C}G$ of degree zero.

Utilizing parameter maps that originate as Hochschild 2-cocycles to explore formal deformations of $S(V)\#G$ has proven useful because although the resulting algebras are noncommutative they give rise to deformations of $S(V)^G$ (by examining centers), yet are easily described as quotient algebras. Both the polynomial degree and the support, i.e., which group elements appear in the nonzero image, are helpful descriptors for the parameter maps and hence the relations for the quotient algebras.

Degree-zero deformations of skew group algebras involve parameter maps that identify commutators of elements in V with certain elements of the group algebra. Several important families of these are of broad interest in noncommutative geometry, combinatorics, and representation theory and are the subject of an already extensive literature (see [12] and [13] and further references therein). By comparison, finding elements of degree one with which to identify com-

mutators of elements in V requires a more intricate analysis of which cocycles pass obstructions in cohomology in order to determine if the resulting deformations satisfy PBW properties [8, 18]. As a result, degree-one deformations are not as well understood or as often studied, yet could also be significant in giving insight into deformations of the invariant algebra $S(V)^G$ and in connection with singularities of orbifolds.

Degree-zero deformations of skew group algebras are called Drinfeld graded Hecke algebras in recognition of their origins in [4] (see also [15]). These include the important special cases when G acts on a symplectic vector space, and more particularly when G is a complex reflection group acting by the sum of a reflection representation and its dual (a doubled representation). The latter leads to the rational Cherednik algebras, first introduced in [3] as rational degenerations of double affine Hecke algebras and later highlighted as an important subfamily of the more general symplectic reflection algebras introduced in [6]. When built from an action of the symmetric group, rational Cherednik algebras model Hamiltonian reduction in quantum mechanics and are used to show integrability of Calogero–Moser systems [5].

Degree-one deformations of skew group algebras were termed Drinfeld orbifold algebras and characterized via explicit PBW conditions on parameter maps in [18], building on [1] and [2]. The conditions are also interpreted in Hochschild cohomology. In [8] we describe the Drinfeld orbifold algebras for S_n acting on its natural permutation representation, $W \cong \mathbb{C}^n$, by starting with candidate 2-cocycles and imposing the PBW conditions from [18]. Here we expand on that class of examples by considering S_n acting on both its doubled permutation representation $W^* \oplus W$ and the doubled representation $\mathfrak{h}^* \oplus \mathfrak{h}$, where $W = \mathfrak{h} \oplus \iota$ is the sum of the $(n-1)$ -dimensional irreducible standard and the trivial representations. This not only results in much richer families of algebras, but also yields degree-one generalizations of rational Cherednik algebras for these doubled representations.

More specifically, in [8] we describe all Drinfeld orbifold algebras where the linear parts of the maps recording commutator relations are supported only on or only off the identity in S_n , and show there are no such maps with linear part supported both on and off the identity. For the two doubled representations of S_n considered here we describe all degree-one families of Drinfeld orbifold algebras whose maps have linear part supported only on or only off the identity (Theorems 7.1 and 7.2). For maps with linear part supported both on and off the identity we provide a family of examples involving $W^* \oplus W$ (Theorem 5.10) and observe there are no corresponding such maps for the doubled standard representation $\mathfrak{h}^* \oplus \mathfrak{h}$ (Remark 6.6). We summarize our main results.

Theorem 1.1. *For the symmetric group S_n ($n \geq 3$) acting on $V \cong \mathbb{C}^{2n}$ by the doubled permutation representation, there is*

- (1) *a 17-parameter family of Lie orbifold algebras described by 22 homogeneous quadratic equations, and*
- (2) *a seven-parameter family of Drinfeld orbifold algebras described in terms of parameter maps with linear part supported only off the identity that are controlled by four homogeneous quadratic equations in six of the parameters.*

These are the only degree-one deformations of the skew group algebra for S_n acting by the doubled permutation representation whose parameter maps have linear part supported only on or only off the identity.

See Theorems 4.1 and 5.9 for more details about the maps, Theorems 7.1 and 7.2 for the resulting quotient algebras, and Table 2 in Section 7.1 for a summary.

Theorem 1.2. *For the symmetric group S_n ($n \geq 3$) acting on $V \cong \mathbb{C}^{2n-2}$ by the doubled standard representation, there are no degree-one Lie orbifold algebras, but there is a three-parameter*

family of Drinfeld orbifold algebras described by parameter maps with linear part supported only off the identity.

These are the only degree-one deformations of the skew group algebra for S_n acting by the doubled standard representation whose parameter maps have linear part supported only on or only off the identity.

See Theorems 6.3 and 6.4 for details, Theorem 7.3 for the resulting algebras, and Table 3 in Section 7.2 for a summary. The S_2 case in Theorems 1.1 and 1.2 can be analyzed in a similar way but there are some differences in the dimensions of spaces of pre-Drinfeld orbifold algebra maps and in the explicit PBW conditions.

The algebras in Theorems 1.1 and 1.2 specialize to the well-known rational Cherednik algebras for the symmetric group, described as of \mathfrak{gl}_n - and \mathfrak{sl}_n -type respectively in [9], and hence should be of substantial interest. In particular, Theorem 1.2 provides a degree one version of the \mathfrak{sl}_n -type rational Cherednik algebras $H_{0,c}$. We refer to the algebras as degree-one rational Cherednik algebras. Investigating their structure, properties, combinatorics, representation theory, geometric significance, and potential importance in physics should provide fertile ground for future research. It would also be natural to explore whether similar algebras exist for other complex reflection groups.

The paper is organized as follows. After a brief summary of preliminaries in Section 2 that apply to any finite group acting linearly on \mathbb{C}^n , we restrict to the setting of the symmetric group and the two doubled representations of interest, except as noted in Lemmas 3.1 and 5.1, Proposition 6.1, and Corollary 6.2. All pre-Drinfeld orbifold algebra maps for S_n acting by the doubled permutation representation are constructed in Section 3. We analyze when these lift in Sections 4 and 5, proving Theorems 4.1, 5.9, and 5.10 using computational details treated earlier in the two sections. In particular, Sections 4.1–4.3 provide explicit equations governing the parameter maps described in Theorem 4.1 and Section 4.5 provides some related algebraic varieties that may be of independent interest. Section 6 begins with Proposition 6.1 providing conditions under which we can combine Drinfeld orbifold algebra maps for subrepresentations into a map for their direct sum. Corollary 6.2 is then used with the results from Sections 4 and 5 to describe in Theorems 6.3 and 6.4 all Drinfeld orbifold algebra maps for S_n acting by the doubled standard representation on the subspace $\mathfrak{h}^* \oplus \mathfrak{h}$ when the linear part is supported only on or only off the identity. In Section 7 we present as quotients the resulting degree-one rational Cherednik algebras arising from the maps in Sections 4–6.

2 Preliminaries

Throughout this section, we let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$. All tensors will be over \mathbb{C} .

2.1 Skew group algebras

Let G be a finite group that acts on a \mathbb{C} -algebra R by algebra automorphisms, and write ${}^g s$ for the result of acting by $g \in G$ on $s \in R$. The skew group algebra $R\#G$ is the semi-direct product algebra $R \rtimes \mathbb{C}G$ with underlying vector space $R \otimes \mathbb{C}G$ and multiplication of simple tensors defined by

$$(r \otimes g)(s \otimes h) = r({}^g s) \otimes gh$$

for all $r, s \in R$ and $g, h \in G$. The skew group algebra becomes a G -module by letting G act diagonally on $R \otimes \mathbb{C}G$, with conjugation on the group algebra factor:

$${}^g(s \otimes h) = ({}^g s) \otimes ({}^g h) = ({}^g s) \otimes ghg^{-1}.$$

In working with elements of skew group algebras, we commonly omit tensor symbols unless the tensor factors are lengthy expressions.

If G acts linearly on a vector space $V \cong \mathbb{C}^n$, then G also acts on the tensor algebra $T(V)$ and symmetric algebra $S(V)$ by algebra automorphisms. Assign elements of V degree one and elements of G degree zero to make the skew group algebras $T(V)\#G$ and $S(V)\#G$ graded algebras.

2.2 Cochains

A k -cochain is a G -graded linear map $\mu = \sum_{g \in G} \mu_g g$ with components $\mu_g: \bigwedge^k V \rightarrow S(V)$. If each μ_g maps into V , then μ is called a *linear cochain*, and if each μ_g maps into \mathbb{C} , then μ is called a *constant cochain*.

We regard a map μ on $\bigwedge^k V$ as a multilinear alternating map on V^k and write $\mu(v_1, \dots, v_k)$ in place of $\mu(v_1 \wedge \dots \wedge v_k)$. Of course, if $\mu(v_1, \dots, v_k) = 0$, then μ is zero on any permutation of v_1, \dots, v_k . Also, if μ is zero on all k -tuples of basis vectors, then μ is zero on any k -tuple of vectors. We exploit these facts in the computations in Sections 4 and 5.

The *support of a cochain* μ is the set of group elements for which the component μ_g is not the zero map. For X a subset of G , we say a cochain μ is *supported only on X* if $\mu_g = 0$ for all g not in X . Similarly, we say μ is *supported only off X* if $\mu_g = 0$ for all g in X . At times, it is convenient to talk about support in a weaker sense, so we say μ is *supported on X* if $\mu_g \neq 0$ for some g in X and that μ is *supported off X* if $\mu_g \neq 0$ for some g not in X . (Hence, it is possible for a cochain to be simultaneously supported on and off of a set.) The *kernel of a cochain* μ is the set of vectors v_0 such that $\mu(v_0, v_1, \dots, v_{k-1}) = 0$ for all $v_1, \dots, v_{k-1} \in V$.

The group G acts on the components of a cochain. Specifically, for a group element h and component μ_g , the map ${}^h\mu_g$ is defined by $({}^h\mu_g)(v_1, \dots, v_k) = {}^h(\mu_g({}^{h^{-1}}v_1, \dots, {}^{h^{-1}}v_k))$. In turn, the group acts on the space of cochains by letting ${}^h\mu = \sum_{g \in G} {}^h\mu_g \otimes hgh^{-1}$. Thus μ is a *G -invariant cochain* if and only if ${}^h\mu_g = \mu_{hgh^{-1}}$ for all $g, h \in G$.

2.3 Drinfeld orbifold algebras

For a parameter map $\kappa = \kappa^L + \kappa^C$, where κ^L is a linear 2-cochain and κ^C is a constant 2-cochain, the quotient algebra

$$\mathcal{H}_\kappa = T(V)\#G / \langle vw - wv - \kappa^L(v, w) - \kappa^C(v, w) \mid v, w \in V \rangle$$

is called a *Drinfeld orbifold algebra* if the associated graded algebra $\text{gr } \mathcal{H}_\kappa$ is isomorphic to the skew group algebra $S(V)\#G$. The condition $\text{gr } \mathcal{H}_\kappa \cong S(V)\#G$ is called a *Poincaré–Birkhoff–Witt (PBW) condition*, in analogy with the PBW Theorem for universal enveloping algebras.

Further, if \mathcal{H}_κ is a Drinfeld orbifold algebra and t is a complex parameter, then

$$\mathcal{H}_{\kappa, t} := T(V)\#G[t] / \langle vw - wv - \kappa^L(v, w)t - \kappa^C(v, w)t^2 \mid v, w \in V \rangle$$

is called a *Drinfeld orbifold algebra over $\mathbb{C}[t]$* . In [18, Theorem 2.1], Shepler and Witherspoon make an explicit connection between the PBW condition and deformations in the sense of Gerstenhaber [10] by showing how to interpret Drinfeld orbifold algebras over $\mathbb{C}[t]$ as formal deformations of the skew group algebra $S(V)\#G$. For more on the broader context of formal deformations see [8, Section 4].

2.4 Lie orbifold algebras

The parameter maps of Drinfeld orbifold algebras decompose as $\kappa = \sum_g \kappa_g g$. When κ is a parameter map for a Drinfeld orbifold algebra and the linear part $\kappa^L = \kappa_1^L$ is supported only on the

identity then the map gives rise to a Lie orbifold algebra (see [18, Section 4] and Definition 2.1). Lie orbifold algebras deform universal enveloping algebras twisted by a group action just as certain symplectic reflection algebras deform Weyl algebras twisted by a group action.

2.5 Drinfeld orbifold algebra maps

Though the defining PBW condition for a Drinfeld orbifold algebra \mathcal{H}_κ involves an isomorphism of algebras, Shepler and Witherspoon proved an equivalent characterization [18, Theorem 3.1] in terms of properties of the parameter map κ .

Definition 2.1. Let $\kappa = \kappa^L + \kappa^C$ where κ^L is a linear 2-cochain and κ^C is a constant 2-cochain, and let Alt_3 denote the alternating group on three elements. Let V^g denote the set of vectors in V that are fixed by group element g . We say κ is a *Drinfeld orbifold algebra map* if the following conditions are satisfied for all $g \in G$ and $v_1, v_2, v_3 \in V$:

$$\text{im } \kappa_g^L \subseteq V^g, \quad (2.1)$$

$$\text{the map } \kappa \text{ is } G\text{-invariant}, \quad (2.2)$$

$$\sum_{\sigma \in \text{Alt}_3} \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)})({}^g v_{\sigma(1)} - v_{\sigma(1)}) = 0 \quad \text{in } S(V), \quad (2.3)$$

$$\sum_{\sigma \in \text{Alt}_3} \sum_{xy=g} \kappa_x^L(v_{\sigma(1)} + {}^y v_{\sigma(1)}, \kappa_y^L(v_{\sigma(2)}, v_{\sigma(3)})) = 2 \sum_{\sigma \in \text{Alt}_3} \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)})({}^g v_{\sigma(1)} - v_{\sigma(1)}), \quad (2.4)$$

$$\sum_{\sigma \in \text{Alt}_3} \sum_{xy=g} \kappa_x^C(v_{\sigma(1)} + {}^y v_{\sigma(1)}, \kappa_y^L(v_{\sigma(2)}, v_{\sigma(3)})) = 0. \quad (2.5)$$

In the special case when the linear component κ^L of a Drinfeld orbifold algebra map is supported only on the identity, we call κ a *Lie orbifold algebra map*.

To simplify reference to the expressions appearing in the last three Drinfeld orbifold algebra map properties, we define operators ψ and ϕ that convert 2-cochains (such as κ^L and κ^C) into the 3-cochains we see evaluated within the properties.

Definition 2.2. Let μ denote a linear or constant 2-cochain and ν a linear 2-cochain. Define $\psi(\mu) = \sum_{g \in G} \psi_g g$ to be the 3-cochain with components $\psi_g: \bigwedge^3 V \rightarrow S(V)$ given by

$$\psi_g(v_1, v_2, v_3) = \sum_{\sigma \in \text{Alt}_3} \mu_g(v_{\sigma(1)}, v_{\sigma(2)})({}^g v_{\sigma(3)} - v_{\sigma(3)}).$$

Define $\phi(\mu, \nu) = \sum_{g \in G} \phi_g g$ to be the 3-cochain with components $\phi_g = \sum_{xy=g} \phi_{x,y}$, where $\phi_{x,y}: \bigwedge^3 V \rightarrow V \oplus \mathbb{C}$ is given by

$$\phi_{x,y}(v_1, v_2, v_3) = \sum_{\sigma \in \text{Alt}_3} \mu_x(v_{\sigma(1)} + {}^y v_{\sigma(1)}, \nu_y(v_{\sigma(2)}, v_{\sigma(3)})). \quad (2.6)$$

Thus $\phi(\mu, \nu)$ is G -graded with components ϕ_g and also $(G \times G)$ -graded with components $\phi_{x,y}$.

For the interested reader, we indicate in [8] how the maps ψ and ϕ relate to coboundary and bracket operations in Hochschild cohomology of a skew group algebra.

2.6 Drinfeld orbifold algebra maps (condensed definition)

Equipped with the definitions of ψ and ϕ , the properties of a Drinfeld orbifold map $\kappa = \kappa^L + \kappa^C$ (Definition 2.1) may be expressed succinctly:

- (2.1) $\text{im } \kappa_g^L \subseteq V^g$ for each g in G ,
- (2.2) the map κ is G -invariant,
- (2.3) $\psi(\kappa^L) = 0$,
- (2.4) $\phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C)$,
- (2.5) $\phi(\kappa^C, \kappa^L) = 0$.

Note that any G -invariant 2-cochain whose linear part is supported only on the identity trivially satisfies properties (2.1) and (2.3), so in this case it is enough to analyze conditions under which properties (2.4) and (2.5) hold (see Theorem 4.1 and Section 4).

Remark 2.3. If \mathcal{H}_κ is a Drinfeld orbifold algebra, then κ must satisfy conditions (2.2)–(2.5), but not necessarily the image constraint (2.1). However, [18, Theorem 7.2(ii)] guarantees there will exist a Drinfeld orbifold algebra $\mathcal{H}_{\tilde{\kappa}}$ such that $\mathcal{H}_{\tilde{\kappa}} \cong \mathcal{H}_\kappa$ as filtered algebras and $\tilde{\kappa}$ satisfies the image constraint $\text{im } \tilde{\kappa}_g^L \subseteq V^g$ for each g in G . Thus, in classifying Drinfeld orbifold algebras, it suffices to only consider Drinfeld orbifold algebra maps.

Theorem 2.4 ([18, Theorems 3.1 and 7.2(ii)]). *A quotient algebra \mathcal{H}_κ satisfies the PBW condition $\text{gr } \mathcal{H}_\kappa \cong S(V)\#G$ if and only if there exists a Drinfeld orbifold algebra map $\tilde{\kappa}$ such that $\mathcal{H}_\kappa \cong \mathcal{H}_{\tilde{\kappa}}$.*

2.7 Strategy

As described and utilized in [8], the process of determining the set of all Drinfeld orbifold algebra maps consists of two phases, and language from cohomology and deformation theory can be used to describe each phase. First, one finds all *pre-Drinfeld orbifold algebra maps*, i.e., all G -invariant linear 2-cochains satisfying the image condition (2.1) and the mixed Jacobi identity (2.3). To find such maps supported on transpositions (Proposition 3.4) we utilize a bijection between pre-Drinfeld orbifold algebra maps and a particular set of representatives of Hochschild cohomology classes (see Lemma 2.5). But to find such maps supported only on the identity (Proposition 3.2) we present a simpler argument based on Lemma 3.1 analyzing the eigenvector structure of images dependent on the group action on input vectors. In the second phase (Sections 4 and 5) we determine for which pre-Drinfeld orbifold algebra maps κ^L there exists a compatible G -invariant constant 2-cochain κ^C such that properties (2.4) and (2.5) hold. We say κ^C *clears the first obstruction* if property (2.4) holds and *clears the second obstruction* if property (2.5) holds. If a G -invariant constant 2-cochain κ^C clears both obstructions, then we say κ^L *lifts* to the Drinfeld orbifold algebra map $\kappa = \kappa^L + \kappa^C$.

2.8 Hochschild cohomology to pre-DOA maps

We briefly recall how Hochschild cohomology can be used in general to find linear and constant 2-cochains κ that are both G -invariant and satisfy property (2.3). For more detailed background discussion about the connections to deformations and further references see [8].

For an algebra A over \mathbb{C} with bimodule M , the *Hochschild cohomology* of A with coefficients in M is $\text{HH}^\bullet(A, M) := \text{Ext}_{A \otimes A^{\text{op}}}^\bullet(A, M)$, which is abbreviated to $\text{HH}^\bullet(A)$ if $M = A$. For any finite group G acting linearly on a vector space $V \cong \mathbb{C}^n$, and for $A = S(V)\#G$, using results of Ştefan [19] yields the following, where R^G denotes the set of elements in R fixed by every g in G ,

$$\text{HH}^\bullet(S(V)\#G) \cong \text{HH}^\bullet(S(V), S(V)\#G)^G \cong (H^\bullet)^G.$$

Here H^\bullet is the G -graded vector space $H^\bullet = \bigoplus_{g \in G} H_g^\bullet$ with components

$$H_g^{p,d} = S^d(V^g) \otimes \bigwedge^{p-\text{codim}(V^g)} (V^g)^* \otimes \bigwedge^{\text{codim}(V^g)} ((V^g)^*)^\perp \otimes \mathbb{C}g,$$

first described independently by Farinati [7] and Ginzburg–Kaledin [11]. Note that H^\bullet is tri-graded by cohomological degree p , homogeneous polynomial degree d , and group element g .

Since the exterior factors of $H_g^{p,d}$ can be identified with a subspace of $\bigwedge^p V^*$, and since $S^d(V^g) \otimes \bigwedge^p V^* \otimes \mathbb{C}g \cong \text{Hom}(\bigwedge^p V, S^d(V^g)g)$, the space H^\bullet may be identified with a subspace of the cochains introduced earlier in this section. The next lemma records the relationship between properties (2.2) and (2.3) of a Drinfeld orbifold algebra map and Hochschild cohomology. When $d = 1$, the lemma is a restatement of [18, Theorem 7.2(i) and (ii)]. When $d = 0$, the lemma is a restatement of [17, Corollary 8.17(ii)]. It is also possible to give a linear algebraic proof in the spirit of [16, Lemma 1.8].

Lemma 2.5. *For a 2-cochain $\kappa = \sum_{g \in G} \kappa_g g$ with $\text{im } \kappa_g \subseteq S^d(V^g)$ for each $g \in G$, the following are equivalent:*

(a) *The map κ is G -invariant and satisfies the mixed Jacobi identity, i.e., for all $v_1, v_2, v_3 \in V$*

$$[v_1, \kappa(v_2, v_3)] + [v_2, \kappa(v_3, v_1)] + [v_3, \kappa(v_1, v_2)] = 0 \quad \text{in } S(V)\#G,$$

where $[\cdot, \cdot]$ denotes the commutator in $S(V)\#G$.

(b) *For all $g, h \in G$ and $v_1, v_2, v_3 \in V$:*

- (i) ${}^h(\kappa_g(v_1, v_2)) = \kappa_{hgh^{-1}}({}^h v_1, {}^h v_2)$, and
- (ii) $\kappa_g(v_1, v_2)({}^g v_3 - v_3) + \kappa_g(v_2, v_3)({}^g v_1 - v_1) + \kappa_g(v_3, v_1)({}^g v_2 - v_2) = 0$.

(c) *The map κ is an element of*

$$(H^{2,d})^G = \left(\bigoplus_{g \in G} \left(S^d(V^g)g \otimes \bigwedge^{2-\text{codim}(V^g)} (V^g)^* \otimes \bigwedge^{\text{codim}(V^g)} ((V^g)^*)^\perp \right) \right)^G.$$

Remark 2.6. Part (b(ii)) of Lemma 2.5 is $2\psi(\kappa) = 0$. Part (c) of Lemma 2.5 shows that κ can only be supported on elements g with $\text{codim } V^g \in \{0, 2\}$ since negative exterior powers are zero and an element g with codimension one acts nontrivially on $H_g^{2,d}$.

3 Pre-Drinfeld orbifold algebra maps

Except as noted in Lemmas 3.1 and 5.1, Proposition 6.1, and Corollary 6.2, for the rest of the paper, let $G = S_n$ be the symmetric group with $n \geq 3$, let $W \cong \mathbb{C}^n$ denote its natural permutation representation, and consider the doubled permutation representation of S_n on $V = W^* \oplus W \cong \mathbb{C}^{2n}$. Let $\mathcal{B}_y = \{y_1, \dots, y_n\}$ be the standard basis for W and $\mathcal{B}_x = \{x_1, \dots, x_n\}$ be the corresponding dual basis for W^* . Then the action of $\sigma \in S_n$ is given by ${}^\sigma y_i = y_{\sigma(i)}$ and ${}^\sigma x_i = x_{\sigma(i)}$. Recall that $W^* \cong \mathfrak{h}^* \oplus \iota^*$, where S_n acts trivially on the 1-dimensional subspace ι^* of W^* spanned by $x_{[n]} = \sum_{i=1}^n x_i$ and by the standard reflection representation on its $(n-1)$ -dimensional orthogonal complement \mathfrak{h}^* , and similarly $W \cong \mathfrak{h} \oplus \iota$. In Remark 4.4 and Section 6 we also consider the doubled standard representation of S_n on the subspace $\mathfrak{h}^* \oplus \mathfrak{h}$ spanned by

$$\left\{ \bar{x}_i := x_i - \frac{1}{n}x_{[n]}, \bar{y}_i := y_i - \frac{1}{n}y_{[n]} \mid 1 \leq i \leq n \right\} \quad (3.1)$$

or by $\{x_i - x_j, y_i - y_j \mid 1 \leq i, j \leq n\}$.

In this section we identify all pre-Drinfeld orbifold algebra maps for S_n with $n \geq 3$ acting by the doubled permutation representation on $W^* \oplus W$. That is, we find all linear 2-cochains κ^L satisfying the image condition (2.1), the G -invariance condition (2.2), and the mixed Jacobi identity $\psi(\kappa^L) = 0$ (2.3). To organize computations we make use of Lemma 2.5 relating Hochschild cohomology and pre-Drinfeld orbifold algebra maps.

By Remark 2.6, we need only consider group elements whose fixed point space has codimension zero or two. Thus for S_n acting by the doubled permutation representation we consider two cases: κ^L supported only on the identity and κ^L supported only on the set of transpositions (which act as reflections on W and bireflections on $W^* \oplus W$).

3.1 Pre-Drinfeld orbifold algebra maps supported only on the identity

We first prove a lemma that describes all G -invariant maps $\kappa_1: \bigwedge^2 V \rightarrow V \oplus \mathbb{C}$, where G is any finite group and V is a permutation representation of G . Since properties (2.1) and (2.3) are trivially satisfied when $g = 1$, this will produce pre-Drinfeld orbifold algebra maps supported only on the identity. Recall that κ_1 is G -invariant if and only if

$$\kappa_1({}^g u, {}^g v) = {}^g(\kappa_1(u, v)) \quad (3.2)$$

for all g in G and $u, v \in V$. The following lemma shows that how G acts on a set of representative basis vector pairs determines a G -invariant linear cochain.

Lemma 3.1. *Suppose G is a finite group acting on a complex vector space V by a permutation representation. If κ_1^L is G -invariant, then the following two conditions hold for all g in G and all basis vector pairs v_i and v_j .*

- (i) *If g swaps v_i and v_j , then $\kappa_1^L(v_i, v_j)$ is an eigenvector of g with eigenvalue -1 .*
- (ii) *If g fixes v_i and v_j , then the vector $\kappa_1^L(v_i, v_j)$ is in the fixed space V^g .*

Suppose every ordered pair v, w of basis vectors can be related to some v_i, v_j in a set S of representative basis vector pairs by $v = {}^g v_i$ and $w = {}^g v_j$ or $v = {}^g v_j$ and $w = {}^g v_i$ for some $g \in G$. If $\kappa_1^L: S \rightarrow V$ satisfies (i) and (ii) for all representative pairs in S , then there is a unique way to extend κ_1^L to be G -invariant on $\bigwedge^2 V$.

Proof. Assume κ_1^L is G -invariant. By (3.2), if g fixes both v_i and v_j , then $\kappa_1^L(v_i, v_j)$ must be an element of V^g . And if g swaps v_i and v_j , then due to skew-symmetry, $\kappa_1^L(v_i, v_j)$ must be a (-1) -eigenvector of g .

Suppose $\kappa_1^L: \bigwedge^2 V \rightarrow V$ satisfies (i) and (ii) for a set of representative basis vector pairs. If $v = {}^g v_i = {}^h v_i$ and $w = {}^g v_j = {}^h v_j$ for some representative pair v_i, v_j and some $g, h \in G$, then $h^{-1}g$ fixes the basis vector pair v_i, v_j . Hence by (ii), $\kappa_1^L(v_i, v_j)$ is in $V^{h^{-1}g}$ and ${}^g \kappa_1^L(v_i, v_j) = {}^h \kappa_1^L(v_i, v_j)$. If instead $v = {}^g v_i = {}^h v_j$ and $w = {}^g v_j = {}^h v_i$, then $h^{-1}g$ swaps v_i and v_j . Hence by (i), ${}^g \kappa_1^L(v_i, v_j) = {}^h \kappa_1^L(v_j, v_i)$. These imply that the unique way to extend κ_1^L to be G -invariant is well-defined. \blacksquare

We now apply this to the doubled permutation representation of S_n on $W^* \oplus W$ equipped with the basis $\mathcal{B}_x \cup \mathcal{B}_y$, where $\mathcal{B}_y = \{y_1, \dots, y_n\}$ is the standard basis for W and $\mathcal{B}_x = \{x_1, \dots, x_n\}$ is the corresponding dual basis for W^* . The following proposition summarizes the definitions of all G -invariant skew-symmetric bilinear maps, i.e., describes $(H_1^{2,0} \oplus H_1^{2,1})^G$.

Proposition 3.2. *Let S_n ($n \geq 3$) act by the doubled permutation representation on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ equipped with basis $\mathcal{B}_x \cup \mathcal{B}_y$. The S_n -invariant linear and constant 2-cochains $\kappa_1^L: \bigwedge^2 V \rightarrow V$ and $\kappa_1^C: \bigwedge^2 V \rightarrow \mathbb{C}$ are as given in Definition 3.7 in terms of complex parameters a_k, b_k for $1 \leq k \leq 7$ and α, β in \mathbb{C} respectively.*

Proof. *Linear cochains.* If $V \cong \mathbb{C}^{2n}$ is the doubled permutation representation of S_n and $\kappa_1^L: \bigwedge^2 V \rightarrow V$ is an S_n -invariant map, then the value on any pair of basis vectors in $\mathcal{B}_x \cup \mathcal{B}_y = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ can be obtained by acting by an appropriate permutation on one of the representative values $\kappa_1^L(x_1, x_2)$, $\kappa_1^L(y_1, y_2)$, $\kappa_1^L(x_1, y_1)$, or $\kappa_1^L(x_1, y_2)$.

Consider $\kappa_1^L(x_1, x_2)$. The permutation $\sigma = (12)$ swaps x_1 and x_2 , so by Lemma 3.1, $\kappa_1^L(x_1, x_2)$ must be a (-1) -eigenvector of (12) , i.e., a linear combination of the vectors $x_1 - x_2$ and $y_1 - y_2$. Both of these vectors are fixed by the group $S_{\{3, \dots, n\}}$ of permutations that fix both x_1 and x_2 . Thus, for a choice of complex parameters a_1 and b_1 , we let

$$\kappa_1^L(x_1, x_2) = a_1(x_1 - x_2) + b_1(y_1 - y_2).$$

A similar argument shows we can let

$$\kappa_1^L(y_1, y_2) = a_2(x_1 - x_2) + b_2(y_1 - y_2)$$

for some choice of complex parameters a_2 and b_2 .

Consider $\kappa_1^L(x_1, y_1)$. There are no permutations that will swap x_1 and y_1 . The group of permutations that fix both x_1 and y_1 is $S_{\{2, \dots, n\}}$, so by Lemma 3.1, $\kappa_1^L(x_1, y_1)$ must be an element of the subspace

$$V^{S_{\{2, \dots, n\}}} = \text{Span} \{x_1, x_{[n]}, y_1, y_{[n]}\}.$$

We define $\kappa_1^L(x_1, y_1)$ to be a linear combination of the basis elements, using complex parameters a_3 , a_4 , b_3 , and b_4 as weights. Orbiting yields the definition

$$\kappa_1^L(x_i, y_i) = a_3x_i + a_4x_{[n]} + b_3y_i + b_4y_{[n]}$$

for $1 \leq i \leq n$.

Consider $\kappa_1^L(x_1, y_2)$. There are no permutations that will swap x_1 and y_2 . The group of permutations that fix both x_1 and y_2 is $S_{\{3, \dots, n\}}$, so once again by Lemma 3.1, $\kappa_1^L(x_1, y_2)$ must be an element of the subspace

$$V^{S_{\{3, \dots, n\}}} = \text{Span} \{x_1, x_2, x_{[n]}, y_1, y_2, y_{[n]}\}.$$

We define $\kappa_1^L(x_1, y_2)$ to be a linear combination of the basis elements using complex parameters a_5 , a_6 , a_7 , b_5 , b_6 , and b_7 as weights. Orbiting yields the definition

$$\kappa_1^L(x_i, y_j) = a_5x_i + a_6x_j + a_7x_{[n]} + b_5y_i + b_6y_j + b_7y_{[n]}$$

for $1 \leq i, j \leq n$ with $i \neq j$.

Constant cochains. By comparison there is only a two-parameter family of G -invariant constant cochains. First, using (3.2), if a constant cochain $\kappa_1^C: \bigwedge^2 V \rightarrow \mathbb{C}$ is S_n -invariant and some element $g \in S_n$ swaps v_i and v_j , then $\kappa_1^C(v_i, v_j) = 0$. Thus $\kappa_1^C(x_i, x_j) = \kappa_1^C(y_i, y_j) = 0$. Also due to S_n -invariance, the value of $\kappa_1^C(x_i, y_j)$ only depends on whether $i = j$ or $i \neq j$, so for $\alpha, \beta \in \mathbb{C}$, we can let $\kappa_1^C(x_i, y_i) = \alpha$ and $\kappa_1^C(x_i, y_j) = \beta$ for $i \neq j$. This shows we have a 2-dimensional space of S_n -invariant maps κ_1^C . ■

Remark 3.3. It is also possible to confirm the dimensions for the linear and constant invariant cochains by using the equivalences from Lemma 2.5 and calculating inner products of characters. If $n \geq 3$, then, omitting details, we find for the linear cochains,

$$\dim (H_1^{2,1})^{S_n} = \dim \left(V \otimes \bigwedge^2 V^* \right)^{S_n} = \langle \chi_\iota, \chi_V \chi_{\bigwedge^2 V} \rangle = \langle \chi_V, \chi_{\bigwedge^2 V} \rangle = 14,$$

and for the constant cochains,

$$\dim (H_1^{2,0})^{S_n} = \dim \left(\bigwedge^2 V^* \right)^{S_n} = \langle \chi_\iota, \chi_{\bigwedge^2 V} \rangle = 2,$$

as expected.

3.2 Pre-Drinfeld orbifold algebra maps supported only off the identity

The following proposition describes $(H_g^{2,0} \oplus H_g^{2,1})^G$ where g is a transposition.

Proposition 3.4. *Let S_n ($n \geq 3$) act by the doubled permutation representation on $V = W^* \oplus W \cong \mathbb{C}^{2n}$, equipped with the basis $\mathcal{B}_x \cup \mathcal{B}_y$. The S_n -invariant linear and constant 2-cochains that satisfy the mixed Jacobi identity and are supported only on transpositions are the maps of the form given in Definition 3.8.*

Proof. We find the centralizer invariants $(H_g^{2,0} \oplus H_g^{2,1})^{Z(g)}$ when g is a transposition. Let $g = (12)$ and first note that the centralizer of g is $Z(g) = \langle (12) \rangle \times S_{\{3, \dots, n\}}$, the fixed point space of g is

$$V^g = \text{Span}\{x_1 + x_2, x_3, \dots, x_n, y_1 + y_2, y_3, \dots, y_n\},$$

and the orthogonal complement is

$$(V^g)^\perp = \text{Span}\{x_1 - x_2, y_1 - y_2\}.$$

A basis for $\bigwedge^2((V^g)^\perp)^*$ is the volume form

$$\text{vol}_g^\perp := (x_1^* - x_2^*) \wedge (y_1^* - y_2^*).$$

Note that $Z(g)$ acts trivially on vol_g^\perp , so

$$(H_g^{2,0})^{Z(g)} = H_g^{2,0} = \bigwedge^2((V^g)^\perp)^* \otimes \mathbb{C}g = \text{Span}\{\text{vol}_g^\perp \otimes (12)\}$$

and

$$\begin{aligned} (H_g^{2,1})^{Z(g)} &= V^{Z(g)} \otimes \bigwedge^2((V^g)^\perp)^* \otimes \mathbb{C}g \\ &= \text{Span}\left\{v \otimes \text{vol}_g^\perp \otimes (12) \mid v \in \left\{x_1 + x_2, \sum_{i=3}^n x_i, y_1 + y_2, \sum_{i=3}^n y_i\right\}\right\}. \end{aligned}$$

After orbiting the centralizer invariants to obtain G -invariants (see the end of Section 4.1 in [8] for more detail), these yield the description in Definition 3.8 for the cochain $\kappa_{\text{ref}} = \sum_{(ij) \in S_n} (\kappa_{(ij)}^C + \kappa_{(ij)}^L) \otimes (ij)$ supported off the identity. ■

3.3 Pre-Drinfeld orbifold algebra maps

By Lemma 2.5 and Remark 2.6, the polynomial degree one elements of Hochschild 2-cohomology found in Propositions 3.2 and 3.4 provide a description of all pre-Drinfeld orbifold algebra maps.

Corollary 3.5. *The pre-Drinfeld orbifold algebra maps for S_n ($n \geq 3$) acting by the doubled permutation representation on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ are the linear 2-cochains $\kappa^L = \kappa_1^L + \kappa_{\text{ref}}^L$ for κ_1^L described in terms of the parameters $a_1, \dots, a_7, b_1, \dots, b_7$ as in Definition 3.7 and κ_{ref}^L controlled by the parameters a, a^\perp, b, b^\perp as in Definition 3.8.*

In Theorems 4.1 and 5.9 we will characterize when the maps κ_1^L and κ_{ref}^L lift separately to Drinfeld orbifold algebra maps and in Theorem 5.10 we will show it is also possible to lift $\kappa_1^L + \kappa_{\text{ref}}^L$. Any two lifts of a particular pre-Drinfeld orbifold algebra map must differ by a constant 2-cochain that satisfies the mixed Jacobi identity. Lemma 2.5 and the results in this section yield the following corollary describing these maps.

Corollary 3.6. *For S_n ($n \geq 3$) acting on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ by the doubled permutation representation, the S_n -invariant constant 2-cochains satisfying the mixed Jacobi identity are the maps $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C$ with κ_1^C given in terms of parameters α and β in Definition 3.7 and κ_{ref}^C described using parameter c in Definition 3.8.*

3.4 Definitions of linear and constant cochains

For convenience we collect here the definitions of the components of the maps determined in Propositions 3.2 and 3.4 and that will be needed to lift κ_1^L in Section 4 and κ_{ref}^L in Section 5.

Some parts of the descriptions below involve sums of basis vectors over subsets of $[n] = \{1, \dots, n\}$. For $I \subseteq [n]$ let $v_I = \sum_{i \in I} v_i$, where v stands for x or y and at times we omit the set braces in I . Let v_I^\perp denote the complementary vector $v_{[n]} - v_I$. In all three definitions, S_n ($n \geq 3$) acts by the doubled permutation representation on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ equipped with basis $\mathcal{B}_x \cup \mathcal{B}_y = \{x_1, \dots, x_n, y_1, \dots, y_n\}$.

Definition 3.7 (cochains supported only on the identity). Given complex parameters a_k, b_k for $1 \leq k \leq 7$ and α, β in \mathbb{C} , let $\kappa_1^L: \Lambda^2 V \rightarrow V$ and $\kappa_1^C: \Lambda^2 V \rightarrow \mathbb{C}$ be the S_n -invariant maps defined by

$$\kappa_1^L(x_i, x_j) = a_1(x_i - x_j) + b_1(y_i - y_j), \quad (3.3)$$

$$\kappa_1^L(y_i, y_j) = a_2(x_i - x_j) + b_2(y_i - y_j), \quad (3.4)$$

$$\kappa_1^L(x_i, y_i) = a_3x_i + a_4x_{[n]} + b_3y_i + b_4y_{[n]}, \quad (3.5)$$

$$\kappa_1^L(x_i, y_j) = a_5x_i + a_6x_j + a_7x_{[n]} + b_5y_i + b_6y_j + b_7y_{[n]}, \quad (3.6)$$

and

$$\kappa_1^C(x_i, x_j) = \kappa_1^C(y_i, y_j) = 0, \quad \kappa_1^C(x_i, y_i) = \alpha, \quad \kappa_1^C(x_i, y_j) = \beta,$$

where $1 \leq i \neq j \leq n$.

Definition 3.8 (cochains supported only on transpositions). Let $a, a^\perp, b, b^\perp, c$ be complex parameters and let T be the set of transpositions in S_n . Define a linear 2-cocycle $\kappa_{\text{ref}}^L = \sum_{g \in T} \kappa_g^L g$, where for $g = (rs)$, the component $\kappa_g^L: \Lambda^2 V \rightarrow V$ is defined for $1 \leq i, j \leq n$ by

$$\kappa_g^L(x_i, x_j) = \kappa_g^L(y_i, y_j) = 0$$

and

$$\kappa_g^L(x_i, y_j) = \begin{cases} ax_{r,s} + a^\perp x_{r,s}^\perp + by_{r,s} + b^\perp y_{r,s} & \text{if } i = j \text{ is in } \{r, s\}, \\ -(ax_{r,s} + a^\perp x_{r,s}^\perp + by_{r,s} + b^\perp y_{r,s}) & \text{if } \{i, j\} = \{r, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the $g = (rs)$ component of the constant 2-cocycle $\kappa_{\text{ref}}^C = \sum_{g \in T} \kappa_g^C g$ is defined for $1 \leq i, j \leq n$ by

$$\kappa_g^C(x_i, x_j) = \kappa_g^C(y_i, y_j) = 0$$

and

$$\kappa_g^C(x_i, y_j) = \begin{cases} c & \text{if } i = j \text{ is in } \{r, s\}, \\ -c & \text{if } \{i, j\} = \{r, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lastly, we define a constant 2-cochain $\kappa_{3\text{-cyc}}^C$ which we use to lift κ_{ref}^L in Section 5.1. The map $\kappa_{3\text{-cyc}}^C$ is not a Hochschild 2-cocycle but rather is based on the form of $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ in Propositions 5.3 and 5.4 and is constructed to ensure $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C)$ as in Proposition 5.5, thereby clearing the first obstruction to lifting κ_{ref}^L . The cochain $\kappa_{3\text{-cyc}}^C$ will also clear the second obstruction to lifting κ_{ref}^L , as verified in Lemma 5.6.

Definition 3.9 (cochains supported only on 3-cycles). Define an S_n -invariant map $\kappa_{3\text{-cyc}}^C = \sum_{g \in S_n} \kappa_g^C g$ with component maps $\kappa_g^C: \Lambda^2 V \rightarrow \mathbb{C}$. If g is not a 3-cycle, let $\kappa_g^C \equiv 0$. For a 3-cycle $g = (i j k)$, define the outcome of κ_g^C on a pair of basis vectors to be zero unless the indices are two distinct elements of $\{i, j, k\}$, in which case the outcome is defined by the following (and skew-symmetry):

$$\kappa_g^C(x_i, x_j) = \kappa_g^C(x_j, x_k) = \kappa_g^C(x_k, x_i) = (b^\perp - b)^2$$

and

$$\kappa_g^C(y_i, y_j) = \kappa_g^C(y_j, y_k) = \kappa_g^C(y_k, y_i) = (a^\perp - a)^2$$

and

$$\begin{aligned} \kappa_g^C(x_i, y_j) &= \kappa_g^C(y_j, x_k) = \kappa_g^C(x_k, y_i) = \kappa_g^C(y_i, x_j) = \kappa_g^C(x_j, y_k) = \kappa_g^C(y_k, x_i) \\ &= -(a^\perp - a)(b^\perp - b). \end{aligned}$$

4 Lie orbifold algebra maps that deform $S(W^* \oplus W) \# S_n$

In Section 3, as summarized in Proposition 3.2 and Definition 3.7, we determined the pre-Drinfeld orbifold algebra maps κ_1^L supported only on the identity. Here we find conditions under which these maps also endow V with a Lie algebra structure — i.e., under which they lift to Lie orbifold algebra maps because there exists a constant 2-cochain κ^C such that $\kappa = \kappa_1^L + \kappa^C$ also satisfies the remaining properties (2.4) and (2.5).

Our main goal is to write down conditions on the parameters involved in the definitions of κ_1^L , κ_1^C , and κ_{ref}^C such that properties (2.4) and (2.5) hold, or in other words, such that $\phi(\kappa_1^L, \kappa_1^L) = 2\psi(\kappa_1^C + \kappa_{\text{ref}}^C)$ and $\phi(\kappa_1^C + \kappa_{\text{ref}}^C, \kappa_1^L) = 0$. Since $2\psi(\kappa_1^C + \kappa_{\text{ref}}^C) = 0$, we have that $\kappa_1^C + \kappa_{\text{ref}}^C$ clears both the first and second obstructions and the map κ_1^L gives rise to a Lie orbifold algebra if and only if $\phi(\kappa_1^L, \kappa_1^L) = \phi(\kappa_1^C + \kappa_{\text{ref}}^C, \kappa_1^L) = 0$. We use this to arrive at characterizing PBW conditions on parameters as summarized in the proof of Theorem 4.1, which states that κ_1^L can be lifted to $\kappa = \kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ precisely when a list of 22 homogeneous quadratic conditions in 17 parameters hold.

It will be convenient along the way to also consider $\phi(\kappa_{\text{ref}}^L, \kappa_1^L)$, for use in Theorem 5.10, by using $*$ to denote either C or L and x to denote either a transposition or the identity and computing, for $v_1, v_2, v_3 \in V$,

$$\phi_{x,1}^*(v_1, v_2, v_3) := \kappa_x^*(v_1, \kappa_1^L(v_2, v_3)) + \kappa_x^*(v_2, \kappa_1^L(v_3, v_1)) + \kappa_x^*(v_3, \kappa_1^L(v_1, v_2))$$

as uniformly as possible. This notation omits a factor of two (and hence differs from that in [8]) because $\psi(\kappa_1^C + \kappa_{\text{ref}}^C) = 0$ means the factor of 2 is irrelevant to clearing the first obstruction and it is also irrelevant to clearing the second obstruction.

First note that due to bilinearity and skew-symmetry it suffices to compute $\phi_{x,1}^*$, with x equal to the identity or a transposition, on basis triples of six main types for $1 \leq i, j, k \leq n$, where $n \geq 3$.

1. All basis vectors in W or in W^* and i, j, k distinct: $(x_i, x_j, x_k), (y_i, y_j, y_k)$.
2. Two basis vectors in W or in W^* and i, j, k distinct: $(x_i, x_j, y_k), (y_i, y_j, x_k)$.
3. Two basis vectors in W or W^* and i, j distinct: $(x_i, x_j, y_j), (y_i, y_j, x_j)$.

This is done in the next three subsections.

4.1 All basis vectors in W or in W^* and three distinct indices

For any distinct indices i, j, k with $1 \leq i, j, k \leq n$, we have

$$\phi_{x,1}^*(x_i, x_j, x_k) = \kappa_x^*(x_i, \kappa_1^L(x_j, x_k)) + \kappa_x^*(x_j, \kappa_1^L(x_k, x_i)) + \kappa_x^*(x_k, \kappa_1^L(x_i, x_j)).$$

Using bilinearity, skew-symmetry, and Definitions 3.7 and 3.8 of κ_1 and κ_{ref} yields for x either the identity or any transposition,

$$\begin{aligned} & \kappa_x^*(x_i, x_j - x_k) + \kappa_x^*(x_j, x_k - x_i) + \kappa_x^*(x_k, x_i - x_j) \\ &= 2[\kappa_x^*(x_i, x_j) + \kappa_x^*(x_j, x_k) + \kappa_x^*(x_k, x_i)] = 0, \end{aligned}$$

and

$$\kappa_x^*(x_i, y_j - y_k) + \kappa_x^*(x_j, y_k - y_i) + \kappa_x^*(x_k, y_i - y_j) = 0.$$

Combining these shows that $\phi_{x,1}^*(x_i, x_j, x_k) = 0$ and similarly $\phi_{x,1}^*(y_i, y_j, y_k) = 0$, for any (distinct i, j, k with) $1 \leq i, j, k \leq n$, for x either the identity or a transposition, and with $*$ = C or $*$ = L . Thus this case imposes no restrictions on any parameters.

4.2 Two basis vectors in W or W^* and three distinct indices

For any distinct indices i, j, k with $1 \leq i, j, k \leq n$, using the definition of κ_1^L , bilinearity, and skew-symmetry yields

$$\begin{aligned} \phi_{x,1}^*(x_i, x_j, y_k) &= 2a_5\kappa_x^*(x_i, x_j) + a_6\kappa_x^*(x_i - x_j, x_k) + a_7\kappa_x^*(x_i - x_j, x_{[n]}) \\ &\quad - b_1\kappa_x^*(y_i - y_j, y_k) + b_5(\kappa_x^*(x_i, y_j) - \kappa_x^*(x_j, y_i)) \\ &\quad + (b_6 - a_1)\kappa_x^*(x_i - x_j, y_k) + b_7\kappa_x^*(x_i - x_j, y_{[n]}). \end{aligned}$$

When $x = 1$ and $*$ = C , since $\kappa_1^C(v, w) = 0$ when $v, w \in W$ or $v, w \in W^*$, we have

$$\phi_{1,1}^C(x_i, x_j, y_k) = b_5(\beta - \beta) + (b_6 - a_1)(\beta - \beta) + b_7(\alpha - \alpha + (n-1)(\beta - \beta)) = 0,$$

and when $x = 1$ and $*$ = L using the definition of κ_1^L yields

$$\phi_{1,1}^L(x_i, x_j, y_k) = \gamma_1(x_i - x_j) + \gamma_2(y_i - y_j),$$

where

$$\begin{aligned} \gamma_1 &= a_1(a_5 + a_6 + na_7) - b_1a_2 - b_5a_6 + a_5(b_5 + b_6 + nb_7) + b_7(a_3 - a_5 - a_6), \\ \gamma_2 &= b_1(a_5 + a_6 + na_7) - b_1b_2 + b_1a_5 + b_5(b_5 - a_1 + nb_7) + b_7(b_3 - b_5 - b_6). \end{aligned}$$

When $x = g$ is a transposition, by Definition 3.8 we have that

$$\begin{aligned} \phi_{g,1}^*(x_i, x_j, y_k) &= (b_6 - a_1)\kappa_g^*(x_i - x_j, y_k) \\ &= \begin{cases} \pm(b_6 - a_1)\kappa_g^*(x_l, y_k), & \text{if } g = (lk) \text{ with } l = i \text{ or } l = j \text{ respectively,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and we define

$$\gamma_3 = -(b_6 - a_1)\kappa_{(jk)}^C(x_j, y_k) = c(b_6 - a_1).$$

Interchanging the roles of x and y and recomputing yields that for distinct i, j, k with $1 \leq i, j, k \leq n$,

$$\begin{aligned} \phi_{x,1}^*(y_i, y_j, x_k) &= -2b_6\kappa_x^*(y_i, y_j) - b_5\kappa_x^*(y_i - y_j, y_k) - b_7\kappa_x^*(y_i - y_j, y_{[n]}) \\ &\quad - a_2\kappa_x^*(x_i - x_j, x_k) - a_6(\kappa_x^*(x_i, y_j) - \kappa_x^*(x_j, y_i)) \\ &\quad + (b_2 + a_5)\kappa_x^*(x_k, y_i - y_j) + a_7\kappa_x^*(x_{[n]}, y_i - y_j). \end{aligned}$$

In particular,

$$\phi_{1,1}^C(y_i, y_j, x_k) = 0 \quad \text{and} \quad \phi_{1,1}^L(y_i, y_j, x_k) = \gamma_4(x_i - x_j) + \gamma_5(y_i - y_j),$$

where

$$\begin{aligned} \gamma_4 &= -a_2(b_6 + b_5 + nb_7) - a_2a_1 - a_2b_6 + a_6(a_6 + b_2 + na_7) + a_7(a_3 - a_6 - a_5), \\ \gamma_5 &= -b_2(b_6 + b_5 + nb_7) - a_2b_1 - a_6b_5 + b_6(a_6 + a_5 + na_7) + a_7(b_3 - b_6 - b_5). \end{aligned}$$

When $x = g$ is a transposition, we have

$$\begin{aligned} \phi_{g,1}^*(y_i, y_j, x_k) &= (b_2 + a_5)\kappa_g^*(x_k, y_i - y_j) \\ &= \begin{cases} \pm(b_2 + a_5)\kappa_g^*(x_k, y_l) & \text{if } g = (lk) \text{ with } l = i \text{ or } l = j \text{ respectively,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and we define

$$\gamma_6 = -(b_2 + a_5)\kappa_g^C(x_k, y_j) = c(b_2 + a_5).$$

4.3 Two basis vectors in W or W^* and two distinct indices

For any distinct indices i, j with $1 \leq i, j \leq n$, we have

$$\begin{aligned} \phi_{x,1}^*(x_i, x_j, y_j) &= (a_3 + a_5)\kappa_x^*(x_i, x_j) - b_1\kappa_x^*(y_i, y_j) + a_4\kappa_x^*(x_i, x_{[n]}) - a_7\kappa_x^*(x_j, x_{[n]}) \\ &\quad + (-a_1 + b_3)\kappa_x^*(x_i, y_j) - b_5\kappa_x^*(x_j, y_i) + (a_1 - b_6)\kappa_x^*(x_j, y_j) \\ &\quad + b_4\kappa_x^*(x_i, y_{[n]}) - b_7\kappa_x^*(x_j, y_{[n]}). \end{aligned}$$

In particular if $x = 1$ and $* = C$ then we set

$$\gamma_7 = \phi_{1,1}^C(x_i, x_j, y_j) = \alpha(a_1 - b_6 + b_4 - b_7) - \beta(a_1 - b_3 + b_5 - (n-1)(b_4 - b_7)),$$

and if $x = 1$ and $* = L$ then

$$\phi_{1,1}^L(x_i, x_j, y_j) = \gamma_8x_{[n]} + \gamma_9x_i + \gamma_{10}x_j + \gamma_{11}y_{[n]} + \gamma_{12}y_i - \gamma_{13}y_j,$$

where

$$\begin{aligned} \gamma_8 &= a_7(b_3 - b_5) - a_4b_6 + (b_4 - b_7)(a_4 + (n-1)a_7 + a_6), \\ \gamma_9 &= a_1(a_3 + na_4) + a_5(b_3 + nb_4) + b_4(a_3 - a_5 - a_6) - a_2b_1 - b_5a_6, \\ \gamma_{10} &= -a_1(a_5 + a_6 + na_7) - a_5(b_5 + nb_7) - b_7(a_3 - a_5 - a_6) + a_2b_1 + b_3a_6 - a_3b_6, \\ \gamma_{11} &= b_7(b_3 - b_5) - b_4b_6 + (b_4 - b_7)(a_1 + b_4 + (n-1)b_7 + b_6) - b_1(a_4 - a_7), \\ \gamma_{12} &= b_1(a_5 - b_2 + a_3 + na_4) - b_5(a_1 + b_6 - b_3 - nb_4) + b_4(b_3 - b_5 - b_6), \\ \gamma_{13} &= b_1(-b_2 + a_5 + a_3 + na_7) + b_5(b_5 + nb_7) + b_7(b_3 - b_5 - b_6) - a_1(b_3 - b_6). \end{aligned}$$

For $x = g$ a transposition, we have that

$$\begin{aligned}\phi_{g,1}^*(x_i, x_j, y_j) &= (-a_1 + b_3)\kappa_g^*(x_i, y_j) - b_5\kappa_g^*(x_j, y_i) + (a_1 - b_6)\kappa_g^*(x_j, y_j) \\ &= \begin{cases} (2a_1 - b_3 + b_5 - b_6)\kappa_g^*(x_j, y_j), & \text{if } g = (ij), \\ 0, & \text{otherwise,} \end{cases}\end{aligned}$$

and we define

$$\gamma_{14} = (2a_1 - b_3 + b_5 - b_6)\kappa_g^C(x_j, y_j) = c(2a_1 - b_3 + b_5 - b_6).$$

Interchanging the roles of x and y and recomputing yields that for any distinct indices i and j with $1 \leq i, j \leq n$,

$$\begin{aligned}\phi_{x,1}^*(y_i, y_j, x_j) &= -(b_3 + b_6)\kappa_x^*(y_i, y_j) - a_2\kappa_x^*(x_i, x_j) - b_4\kappa_x^*(y_i, y_{[n]}) + b_7\kappa_x^*(y_j, y_{[n]}) \\ &\quad + (a_3 + b_2)\kappa_x^*(x_j, y_i) - a_6\kappa_x^*(x_i, y_j) - (a_5 + b_2)\kappa_x^*(x_j, y_j) \\ &\quad + a_4\kappa_x^*(x_{[n]}, y_i) - a_7\kappa_x^*(x_{[n]}, y_j).\end{aligned}$$

In particular, we set

$$\gamma_{15} = \phi_{1,1}^C(y_i, y_j, x_j) = \alpha(-b_2 - a_5 + a_4 - a_7) + \beta(b_2 + a_3 - a_6 + (n-1)(a_4 - a_7)),$$

and

$$\phi_{1,1}^L(y_i, y_j, x_j) = \gamma_{16}x_{[n]} + \gamma_{17}x_i + \gamma_{18}x_j + \gamma_{19}y_{[n]} + \gamma_{20}y_i + \gamma_{21}y_j,$$

where

$$\begin{aligned}\gamma_{16} &= a_7(a_3 - a_6) - a_4a_5 + (a_4 - a_7)(-b_2 + a_4 + (n-1)a_7 + a_5) + a_2(b_4 - b_7), \\ \gamma_{17} &= -a_2(a_1 + b_6 + b_3 + nb_4) + a_6(b_2 - a_5 + a_3 + na_4) + a_4(a_3 - a_5 - a_6), \\ \gamma_{18} &= a_2(a_1 + b_6 + b_3 + nb_7) - a_6(a_6 + na_7) - a_7(a_3 - a_5 - a_6) - b_2(a_3 - a_5), \\ \gamma_{19} &= b_7(a_3 - a_6) - b_4a_5 + (a_4 - a_7)(b_4 + (n-1)b_7 + b_5), \\ \gamma_{20} &= -b_2(b_3 + nb_4) + b_6(a_3 + na_4) + a_4(b_3 - b_5 - b_6) - a_2b_1 - a_6b_5, \\ \gamma_{21} &= b_2(b_5 + b_6 + nb_7) - b_6(a_6 + na_7) - a_7(b_3 - b_5 - b_6) + a_2b_1 + a_3b_5 - b_3a_5.\end{aligned}$$

Lastly, when $x = g$ is a transposition, we have

$$\begin{aligned}\phi_{g,1}^*(x_i, x_j, y_j) &= (a_3 + b_2)\kappa_g^*(x_j, y_i) - a_6\kappa_g^*(x_i, y_j) - (a_5 + b_2)\kappa_g^*(x_j, y_j) \\ &= \begin{cases} -(2b_2 + a_3 + a_5 - a_6)\kappa_g^*(x_j, y_j), & \text{if } g = (ij), \\ 0, & \text{otherwise,} \end{cases}\end{aligned}$$

and we define

$$\gamma_{22} = (2b_2 + a_3 + a_5 - a_6)\kappa_g^*(x_j, y_j) = c(2b_2 + a_3 + a_5 - a_6).$$

4.4 Lie orbifold algebra maps

We now use the calculations in Sections 4.1–4.3 to describe all Drinfeld orbifold algebra maps with linear part supported only on the identity, i.e., all Lie orbifold algebra maps $\kappa_1^L + \kappa^C$. The corresponding Lie orbifold algebras are described in Theorem 7.1.

Theorem 4.1. *Let S_n ($n \geq 3$) act on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ by the doubled permutation representation, and let κ_1^L and κ_1^C be as described in Definition 3.7 and κ_{ref}^C be as in Definition 3.8 with complex parameters $a_1, \dots, a_7, b_1, \dots, b_7, \alpha, \beta$, and c . The Lie orbifold algebra maps are precisely the maps of the form $\kappa = \kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ satisfying the conditions $\gamma_i = 0$ for $1 \leq i \leq 22$.*

Proof. Let κ^L be a pre-Drinfeld orbifold algebra map supported only on the identity. By Corollary 3.5 we know $\kappa^L = \kappa_1^L$ is as given in terms of a_i and b_i for $1 \leq i \leq 7$ in Definition 3.7.

In considering property (2.4) when $g = 1$, note that $\psi_1 \equiv 0$ for any κ^C , so we must have $\phi_{1,1} \equiv 0$ as well. The result of computing $\phi(\kappa_1^L, \kappa_1^L)$ is given in Sections 4.1–4.3 as the values of the various $\phi_{1,1}^L(u, v, w)$. These show that $\phi_{1,1} \equiv 0$ precisely when the parameters a_i, b_i for $i = 1, \dots, 7$ satisfy the conditions

$$\gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = 0 \quad \text{and} \quad \gamma_i = 0 \quad \text{for} \quad 8 \leq i \leq 13, \quad \text{and} \quad 16 \leq i \leq 21. \quad (4.1)$$

Since κ^L is supported only on the identity we must also consider property (2.4) for $g \neq 1$ and find all G -invariant constant 2-cochains such that $\psi(\kappa^C) = 0$ (i.e., satisfying the mixed Jacobi identity). By Corollary 3.6, these are supported on the identity and transpositions and are given by $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C$ with κ_1^C defined using α and β as in Definition 3.7 and κ_{ref}^C defined using c as in Definition 3.8.

Now assume $\kappa_1^C + \kappa_{\text{ref}}^C$ clears the first obstruction, i.e., that the conditions in (4.1) do hold for κ_1^L , and consider property (2.5) for each of κ_1^C and κ_{ref}^C . Using the values of $\phi_{1,1}^C(u, v, w)$ and $\phi_{g,1}^C(u, v, w)$ in Sections 4.1–4.3 we see that κ_1^C clears the second obstruction for κ_1^L , i.e., $\phi(\kappa_1^C, \kappa_1^L) = 0$, precisely when in addition

$$\gamma_7 = \gamma_{15} = 0. \quad (4.2)$$

We also see that κ_{ref}^C clears the second obstruction for κ_1^L , i.e., that $\phi(\kappa_{\text{ref}}^C, \kappa_1^L) = 0$, precisely when in addition,

$$\gamma_3 = \gamma_6 = \gamma_{14} = \gamma_{22} = 0. \quad (4.3)$$

Thus κ_1^L lifts to $\kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ if and only if $\gamma_i = 0$ for $1 \leq i \leq 22$. ■

Corollary 4.2. *Theorem 4.1 includes precisely the following special cases:*

- (1) $\kappa = \kappa_1^L$ satisfying conditions (4.1),
- (2) $\kappa = \kappa_1^L + \kappa_1^C$ satisfying conditions (4.1) and (4.2), and
- (3) $\kappa = \kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ with $\kappa_{\text{ref}}^C \neq 0$ satisfying conditions (4.4)–(4.17).

Proof.

Cases (1) and (2). These are immediate by the forms of (4.2) and (4.3) since $\kappa_1^C \equiv 0$ if and only if $\alpha = \beta = 0$ and $\kappa_{\text{ref}}^C \equiv 0$ if and only if $c = 0$.

Case (3). Suppose instead that κ_1^L lifts to $\kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ with $\kappa_{\text{ref}}^C \neq 0$. Then in addition to (4.1) and (4.2), we have $c \neq 0$. This reduces the conditions in (4.3) to

$$a_1 = b_6 \quad \text{and} \quad a_1 - b_3 + b_5 = 0, \quad \text{or equivalently} \quad b_3 - b_5 - b_6 = 0, \quad (4.4)$$

$$b_2 = -a_5 \quad \text{and} \quad b_2 + a_3 - a_6 = 0, \quad \text{or equivalently} \quad a_3 - a_5 - a_6 = 0. \quad (4.5)$$

These in turn allow simplification of the conditions for $\kappa_1^C + \kappa_{\text{ref}}^C$ to clear the first obstruction if $\kappa_{\text{ref}}^C \neq 0$ clears the second obstruction for κ_1^L by reducing (4.1) to

$$a_1(a_4 - a_7) - (b_4 - b_7)(a_4 + a_6 + (n-1)a_7) = 0, \quad (4.6)$$

$$b_1(a_4 - a_7) - (b_4 - b_7)(b_4 + b_6 + (n-1)b_7) = 0, \quad (4.7)$$

$$a_2(b_4 - b_7) - (a_4 - a_7)(a_4 + a_5 + (n-1)a_7) = 0, \quad (4.8)$$

$$b_2(b_4 - b_7) - (a_4 - a_7)(b_4 + b_5 + (n-1)b_7) = 0, \quad (4.9)$$

$$a_1(a_3 + na_4) + a_5(b_3 + nb_4) - b_1a_2 - b_5a_6 = 0, \quad (4.10)$$

$$a_1(a_3 + na_7) + a_5(b_3 + nb_7) - b_1a_2 - b_5a_6 = 0, \quad (4.11)$$

$$b_1(a_3 + na_4) + b_5(b_3 + nb_4) - 2b_1b_2 - 2b_5b_6 = 0, \quad (4.12)$$

$$b_1(a_3 + na_7) + b_5(b_3 + nb_7) - 2b_1b_2 - 2b_5b_6 = 0, \quad (4.13)$$

$$-a_2(b_3 + nb_4) + a_6(a_3 + na_4) - 2a_1a_2 - 2a_5a_6 = 0, \quad (4.14)$$

$$-a_2(b_3 + nb_7) + a_6(a_3 + na_7) - 2a_1a_2 - 2a_5a_6 = 0. \quad (4.15)$$

Conditions (4.4) and (4.5) also reduce the conditions in (4.2) for κ_1^C to clear the second obstruction for κ_1^L assuming $\kappa_{\text{ref}}^C \neq 0$ also clears the second obstruction for κ_1^L to

$$(\alpha + (n-1)\beta)(b_4 - b_7) = 0, \quad (4.16)$$

$$(\alpha + (n-1)\beta)(a_4 - a_7) = 0. \quad (4.17)$$

Thus κ_1^L lifts to $\kappa_1^L + \kappa_1^C + \kappa_{\text{ref}}^C$ with $\kappa_{\text{ref}}^C \neq 0$ when (4.4)–(4.17) hold. \blacksquare

Remark 4.3. Note in Case (3) of Corollary 4.2 with $c \neq 0$ that if $\alpha \neq -(n-1)\beta$, then $a_4 = a_7$ and $b_4 = b_7$. These combined with $a_3 = a_5 + a_6$ and $b_3 = b_5 + b_6$ in (4.4)–(4.5) mean that the definitions of $\kappa_1^L(x_i, y_j)$ in (3.6) (but with j allowed to be i) and the definition of $\kappa_1^L(x_i, y_i)$ in (3.5) agree. Furthermore, conditions (4.6)–(4.9) then hold and conditions (4.10)–(4.15), simplify further to

$$a_1(a_3 + na_4) + a_5(b_3 + nb_4) - b_1a_2 - b_5a_6 = 0,$$

$$b_1(a_3 + na_4) + b_5(b_3 + nb_4) - 2b_1b_2 - 2b_5b_6 = 0,$$

$$-a_2(b_3 + nb_4) + a_6(a_3 + na_4) - 2a_1a_2 - 2a_5a_6 = 0.$$

4.5 Algebraic varieties corresponding to image constraints

The homogeneous quadratic PBW conditions $\gamma_i = 0$ for $1 \leq i \leq 22$ give rise to a projective variety that controls the parameter space for the family of maps in Theorem 4.1 and corresponding Lie orbifold algebras in Theorem 7.1. Based on computations done for a few specific values of n in Macaulay2 [14] with the graded reverse lexicographic monomial ordering and the parameter order $a_1, \dots, a_7, b_1, \dots, b_7, \alpha, \beta, c$ we conjecture that this projective variety has dimension seven. It contains the lattice of subvarieties described in Table 1 arising from the PBW conditions and additional constraints on how $\text{im } \kappa_1^L$ relates to subrepresentations of the doubled permutation representation. While not needed for Theorem 4.1, these are included as being of potential independent interest.

Remark 4.4. Observe that $x_i = \bar{x}_i + \frac{1}{n}x_{[n]}$ and $y_i = \bar{y}_i + \frac{1}{n}y_{[n]}$ can be used in (3.3)–(3.6) to

decompose the results according to $V \cong \mathfrak{h}^* \oplus \iota^* \oplus \mathfrak{h} \oplus \iota$:

$$\kappa_1^L(x_i, x_j) = a_1(\bar{x}_i - \bar{x}_j) + b_1(\bar{y}_i - \bar{y}_j), \quad (4.18)$$

$$\kappa_1^L(y_i, y_j) = a_2(\bar{x}_i - \bar{x}_j) + b_2(\bar{y}_i - \bar{y}_j), \quad (4.19)$$

$$\kappa_1^L(x_i, y_i) = a_3\bar{x}_i + \frac{1}{n}(a_3 + na_4)x_{[n]} + b_3\bar{y}_i + \frac{1}{n}(b_3 + nb_4)y_{[n]}, \quad (4.20)$$

$$\kappa_1^L(x_i, y_j) = a_5\bar{x}_i + a_6\bar{x}_j + \frac{1}{n}(a_5 + a_6 + na_7)x_{[n]} + b_5\bar{y}_i + b_6\bar{y}_j + \frac{1}{n}(b_5 + b_6 + nb_7)y_{[n]}. \quad (4.21)$$

Using Remark 4.4 we see that if the image of κ_1^L is contained within $\mathfrak{h}^* \oplus \mathfrak{h}$ then the coefficients of $x_{[n]}$ and $y_{[n]}$ are zero; i.e.,

$$a_3 + na_4 = 0, \quad b_3 + nb_4 = 0, \quad (4.22)$$

$$a_5 + a_6 + na_7 = 0, \quad b_5 + b_6 + nb_7 = 0. \quad (4.23)$$

If the image of κ_1^L contains at most one copy of the trivial representation, i.e., some linear combination of $x_{[n]}$ and $y_{[n]}$, then the coefficients of $x_{[n]}$ and $y_{[n]}$ satisfy this weaker condition:

$$(a_3 + na_4)(b_5 + b_6 + nb_7) = (b_3 + nb_4)(a_5 + a_6 + na_7). \quad (4.24)$$

Similarly, if the image of κ_1^L contains at most one copy of the standard representation then the coefficients of \bar{x}_i and \bar{y}_i satisfy these conditions:

$$a_i b_j = b_i a_j \quad \text{for } 1 \leq i < j \leq 6 \quad \text{and} \quad i, j \neq 4. \quad (4.25)$$

Lastly, if the image of κ_1^L is contained within $\iota^* \oplus \iota$ then the coefficients of \bar{x}_i and \bar{y}_i are zero:

$$a_i = b_i = 0 \quad \text{for } 1 \leq i \leq 6 \quad \text{and} \quad i \neq 4. \quad (4.26)$$

In Table 1 we list constraints on $\text{im } \kappa_1^L$, resulting conditions, in addition to (4.1)–(4.3), needed to describe the subvariety of maps subject to each constraint, and the conjectured dimension and degree based on computations. Dropping condition (4.24) in the first three rows, i.e., allowing an additional summand of the trivial representation in $\text{im } \kappa_1^L$ in those cases, did not change the computed dimension or degree of the variety.

Table 1. Conjectured dimension and degree of projective varieties for Lie orbifold algebras obtained from constraints on $\text{im } \kappa_1^L$. When $\text{im } \kappa_1^L$ is contained in the trivial representation, the resulting seven defining polynomials, the dimension and degree, and related simple κ maps can be found by hand.

Constraint on $\text{im } \kappa_1^L$	Conditions	Dimension	Degree
contained in $\text{perm} \oplus \text{std}$	(4.24)	7	8
contained in perm	(4.24), (4.25)	6	30
contained in triv	(4.24), (4.25), (4.26)	4	1
contained in $\text{std} \oplus \text{std}$	(4.22), (4.23), (4.24)	6	6
contained in std	(4.22), (4.23), (4.24), (4.25)	5	4

5 Drinfeld orbifold algebra maps that deform $S(W^* \oplus W) \# S_n$

In Section 3 we defined a pre-Drinfeld orbifold algebra map κ_{ref}^L . Here we aim to lift κ_{ref}^L to a Drinfeld orbifold algebra map. We evaluate $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ and clear the first obstruction by

defining a G -invariant 2-cochain $\kappa_{3\text{-cyc}}^C$ such that $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C)$. We then clear the second obstruction by showing $\phi(\kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L) = 0$ and giving conditions on the parameters α and β for κ_1^C and a, a^\perp, b, b^\perp for κ_{ref}^L such that $\phi(\kappa_1^C, \kappa_{\text{ref}}^L) = 0$. It follows in Theorem 5.9 that $\kappa_{\text{ref}}^L + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ is always a Drinfeld orbifold algebra map and $\kappa_{\text{ref}}^L + \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ is a Drinfeld orbifold algebra map precisely when conditions (5.3) and (5.4) hold.

Characterizing in general when $\kappa_1^L + \kappa_{\text{ref}}^L$ lifts is straightforward but rather more involved. Instead, in Theorem 5.10 we provide a nontrivial choice of parameters and verify in that case that it is possible to lift simultaneously to $\kappa_1^L + \kappa_{\text{ref}}^L + \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$. There could very well be other parameter choices for successfully lifting $\kappa_1^L + \kappa_{\text{ref}}^L$. This is indicated by the question marks in Table 2 summarizing the results in this and the previous section.

5.1 Clearing obstructions to deformations of $S(W^* \oplus W) \# S_n$

We begin by recalling a lemma from [8] that allows for a reduction in the computations necessary to remove obstructions and lift κ maps.

Invariance relations. Recall that a cochain $\mu = \sum_{g \in G} \mu_g g$ with components $\mu_g: \wedge^k V \rightarrow S(V)$ is G -invariant if and only if ${}^h \mu_g = \mu_{hgh^{-1}}$ for all $g, h \in G$. Equivalently,

$${}^h(\mu_g(v_1, \dots, v_k)) = \mu_{hgh^{-1}}({}^h v_1, \dots, {}^h v_k)$$

for all $g, h \in G$ and $v_1, \dots, v_k \in V$. Thus a G -invariant cochain is determined by its components for a set of conjugacy class representatives.

In the following lemma that applies to any finite group, one can let $\mu = \kappa^L$ or $\mu = \kappa^C$ and let $\nu = \kappa^L$ to see that if κ^L and κ^C are G -invariant, then $\phi(\kappa^*, \kappa^L)$ and $\psi(\kappa^*)$ are also G -invariant. This is helpful because, for instance, if $\phi_g = 2\psi_g$ for some $g \in G$, then acting by $h \in G$ on both sides shows $\phi_{hgh^{-1}} = 2\psi_{hgh^{-1}}$ also. Thus if $\phi_g = 2\psi_g$ for all g in a set of conjugacy class representatives, then $\phi(\kappa^L, \kappa^L) = 2\psi(\kappa^C)$. Similar reasoning applies to properties (2.3) and (2.5) of a Drinfeld orbifold algebra map.

Lemma 5.1 ([8, Lemma 5.1]). *Let G be a finite group acting linearly on $V \cong \mathbb{C}^n$. If μ and ν are G -invariant 2-cochains with ν linear and μ linear or constant, then $\phi(\mu, \nu)$ and $\psi(\mu)$ are G -invariant. Specifically, at the component level, for all $x, y, h \in G$ and $v_1, v_2, v_3 \in V$ we have*

$${}^h(\phi_{x,y}(v_1, v_2, v_3)) = \phi_{h x h^{-1}, h y h^{-1}}({}^h v_1, {}^h v_2, {}^h v_3).$$

5.2 Clearing the first obstruction

We begin by recording simplifications of a summand, $\phi_{\sigma, \tau}^*$, of the component ϕ_g^* of $\phi(\kappa_{\text{ref}}^*, \kappa_{\text{ref}}^L)$, where $*$ stands for L or C . Simplification of $\phi_{\sigma, \tau}^*(u, v, w)$ depends on the location of the basis vectors relative to W and W^* and relative to the fixed spaces V^σ and V^τ , so recall that v^* is the vector dual to v and define the following indicator function. For $g \in S_n$ and $v \in V$, let

$$\delta_g(v) = \begin{cases} 1 & \text{if } v \in V^g, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $g \in G$ and $v \in V$,

$$\delta_g(v^*) = \delta_g(v).$$

Remark. Let $\phi_{\sigma, \tau}^*$ be as in Lemma 5.2. Then for all $u, v, w \in V$ we have

$$\phi_{\sigma, \tau}^*(\tau u, \tau v, \tau w) = \phi_{\sigma, \tau}^*(u, v, w). \quad (5.1)$$

This follows from the definition of $\phi_{\sigma, \tau}^*$ and that κ_{ref}^L is τ -invariant.

Lemma 5.2. *Let κ_{ref}^* with $*$ = L or $*$ = C be as in Definition 3.8, with common parameters $a, a^\perp, b, b^\perp \in \mathbb{C}$. Denote a term of the component ϕ_g^* of $\phi(\kappa_{\text{ref}}^*, \kappa_{\text{ref}}^L)$ by $\phi_{\sigma, \tau}^*$, where σ and τ are transpositions such that $\sigma\tau = g$.*

- (1) *If $u, v, w \in W$, $u, v, w \in W^*$, $u, v \in V^\tau$, or $u \in V^\tau \cap V^\sigma$, then $\phi_{\sigma, \tau}^*(u, v, w) = 0$.*
- (2) *If $u \in V^\tau \setminus V^\sigma$ and $v \notin V^\tau$, then the basis vectors moved by τ are of the form $v, {}^\tau v, v^*, {}^\tau v^*$. We have $\phi_{\sigma, \tau}^*(v, {}^\tau v, u^*) = 0$,*

$$\phi_{\sigma, \tau}^*(u, v, v^*) = -\phi_{\sigma, \tau}^*(u, v, {}^\tau v^*) = \begin{cases} 2(a^\perp - a)[1 - \delta_\tau(\sigma u)]\kappa_\sigma^*(u^*, u) & \text{if } u \in W, \\ 2(b^\perp - b)[1 - \delta_\tau(\sigma u)]\kappa_\sigma^*(u, u^*) & \text{if } u \in W^*, \end{cases}$$

and by (5.1),

$$\phi_{\sigma, \tau}^*(u, {}^\tau v, {}^\tau v^*) = -\phi_{\sigma, \tau}^*(u, {}^\tau v, v^*) = \phi_{\sigma, \tau}^*(u, v, v^*).$$

- (3) *If $v \notin V^\tau$, then*

$$\phi_{\sigma, \tau}^*(v, {}^\tau v, v^*) = \begin{cases} 2(a^\perp - a)[\delta_\tau(\sigma v)\kappa_\sigma^*(v^*, v) + \delta_\tau(\sigma^\tau v)\kappa_\sigma^*({}^\tau v^*, {}^\tau v)] & \text{if } v \in W, \\ 2(b^\perp - b)[\delta_\tau(\sigma v)\kappa_\sigma^*(v, v^*) + \delta_\tau(\sigma^\tau v)\kappa_\sigma^*({}^\tau v, {}^\tau v^*)] & \text{if } v \in W^*, \end{cases}$$

and by (5.1),

$$\phi_{\sigma, \tau}^*(v, {}^\tau v, {}^\tau v^*) = -\phi_{\sigma, \tau}^*(v, {}^\tau v, v^*).$$

Proof.

Case (1). When $u, v, w \in W$ or $u, v, w \in W^*$, then $\phi_{\sigma, \tau}^*(u, v, w) = 0$ because

$$\phi_{\sigma, \tau}^*(u, v, w) = \kappa_\sigma^*(u + {}^\tau u, \kappa_\tau^L(v, w)) + \kappa_\sigma^*(v + {}^\tau v, \kappa_\tau^L(w, u)) + \kappa_\sigma^*(w + {}^\tau w, \kappa_\tau^L(u, v)), \quad (5.2)$$

and κ_τ^L is zero whenever both input vectors are in W^* or both are in W . As in [8], $\phi_{\sigma, \tau}^*(u, v, w) = 0$ when $u, v \in V^\tau$ follows from (5.2) and that $V^\tau \subseteq \ker \kappa_\tau^*$, while $\phi_{\sigma, \tau}^*(u, v, w) = 0$ when $u \in V^\tau \cap V^\sigma$ uses also $V^\sigma \subseteq \ker \kappa_\sigma^*$.

Case (2). Assume $u \in V^\tau \setminus V^\sigma$ and $v \notin V^\tau$. First note that

$$\phi_{\sigma, \tau}^*(v, {}^\tau v, u^*) = 0$$

by using (5.1) and the alternating property to see that $\phi_{\sigma, \tau}^*(v, {}^\tau v, u^*) = -\phi_{\sigma, \tau}^*(v, {}^\tau v, u^*)$.

We can reduce to $\phi_{\sigma, \tau}^*(u, v, v^*) = 2\kappa_\sigma^*(u, \kappa_\tau^L(v, v^*))$ by using $u \in V^\tau \subseteq \ker \kappa_\tau^*$ in (5.2). Using bilinearity and $V^\sigma \subseteq \ker \kappa_\sigma^*$, the right hand side is a linear combination of expressions $\kappa_\sigma^*(u, {}^h u)$ and $\kappa_\sigma^*(u, {}^h u^*)$ for $h \in \langle \sigma \rangle$. The appropriate coefficients in terms of a, a^\perp, b, b^\perp can be described in terms of the indicator function for the fixed space of τ and depend on whether $u \in W^*$ or $u \in W$. Also note that $\sum_{h \in \langle \sigma \rangle} {}^h u \in V^\sigma \subseteq \ker \kappa_\sigma^*$. Thus, for $u \in W^*$ we have

$$\begin{aligned} & \kappa_\sigma^*(u, \kappa_\tau^L(v, v^*)) \\ &= \sum_{h \in \langle \sigma \rangle} [a(1 - \delta_\tau({}^h u)) + a^\perp \delta_\tau({}^h u)]\kappa_\sigma^*(u, {}^h u) + [b(1 - \delta_\tau({}^h u^*)) + b^\perp \delta_\tau({}^h u^*)]\kappa_\sigma^*(u, {}^h u^*) \\ &= (a^\perp - a) \sum_{h \in \langle \sigma \rangle} \delta_\tau({}^h u)\kappa_\sigma^*(u, {}^h u) + (b^\perp - b) \sum_{h \in \langle \sigma \rangle} \delta_\tau({}^h u^*)\kappa_\sigma^*(u, {}^h u^*) \\ &= (b^\perp - b) [\delta_\tau(u)\kappa_\sigma^*(u, u^*) + \delta_\tau(\sigma u)\kappa_\sigma^*(u, \sigma u^*)], \end{aligned}$$

where the term with coefficient $a^\perp - a$ is zero because $\kappa_\sigma^*(u, u) = \kappa_\sigma^*(u, \sigma u) = 0$. Since $u \in V^\tau$ and $\kappa_\sigma^*(u, \sigma u^*) = -\kappa_\sigma^*(u, u^*)$, this yields

$$\phi_{\sigma, \tau}^*(u, v, v^*) = 2\kappa_\sigma^*(u, \kappa_\tau^L(v, v^*)) = 2(b^\perp - b)[1 - \delta_\tau(\sigma u)]\kappa_\sigma^*(u, u^*).$$

When $u \in W$, the calculation of $\kappa_\sigma^*(u, \kappa_\tau^L(v, v^*))$ involves a sign difference, and the coefficients on the two sums are reversed, so the one with coefficient $a^\perp - a$ survives and yields $\phi_{\sigma, \tau}^*(u, v, v^*) = 2(a^\perp - a)[1 - \delta_\tau(\sigma u)]\kappa_\sigma^*(u^*, u)$.

Similar calculations show

$$\phi_{\sigma, \tau}^*(u, v, \tau v^*) = -\phi_{\sigma, \tau}^*(u, v, v^*).$$

Case (3). Assume $v \notin V^\tau$. Then $\kappa_\tau^L(v, \tau v) = 0$ and hence

$$\phi_{\sigma, \tau}^*(v, \tau v, v^*) = \kappa_\sigma^*(v + \tau v, \kappa_\tau^L(\tau v, v^*)) + \kappa_\sigma^*(\tau v + v, \kappa_\tau^L(v^*, v)) = -2\kappa_\sigma^*(v + \tau v, \kappa_\tau^L(v, v^*)).$$

A calculation as in case (2), using $\kappa_\tau^L(\tau v, \tau v^*) = \kappa_\tau^L(v, v^*)$ and that $\delta_\tau(v) = \delta_\tau(\tau v) = 0$ yields

$$\begin{aligned} \phi_{\sigma, \tau}^*(v, \tau v, v^*) &= -2[\kappa_\sigma^*(v, \kappa_\tau^L(v, v^*)) + \kappa_\sigma^*(\tau v, \kappa_\tau^L(\tau v, \tau v^*))] \\ &= \begin{cases} 2(a^\perp - a)[\delta_\tau(\sigma v)\kappa_\sigma^*(v^*, v) + \delta_\tau(\sigma \tau v)\kappa_\sigma^*(\tau v^*, \tau v)] & \text{if } v \in W, \\ 2(b^\perp - b)[\delta_\tau(\sigma v)\kappa_\sigma^*(v, v^*) + \delta_\tau(\sigma \tau v)\kappa_\sigma^*(\tau v, \tau v^*)] & \text{if } v \in W^*. \end{cases} \quad \blacksquare \end{aligned}$$

As mentioned in the outline of the proof of Theorem 5.9, the next two propositions are used to evaluate both $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ and $\phi(\kappa_{\text{ref}}^C, \kappa_{\text{ref}}^L)$.

Proposition 5.3. *Let κ_{ref}^* with $*$ = L or $*$ = C be as in Definition 3.8. For $g \in S_n$ where $n \geq 3$, let ϕ_g^* be the g -component of $\phi(\kappa_{\text{ref}}^*, \kappa_{\text{ref}}^L)$. If g is not a 3-cycle then $\phi_g^* \equiv 0$.*

Proof. Since κ_{ref}^* is supported only on transpositions and the only cycle types that arise as a product of two transpositions are the identity, double transpositions, and 3-cycles, it suffices to consider only the components ϕ_1^* and ϕ_g^* where g is a double transposition, and in fact only ϕ_1^* when $n = 3$. Since ${}^h\sigma^h\tau = {}^h(\sigma\tau)$, it suffices to use only representatives of orbits of factor pairs under the action of S_n by diagonal conjugation.

Case 1 ($g = 1$). The identity component ϕ_1^* of $\phi(\kappa_{\text{ref}}^*, \kappa_{\text{ref}}^L)$ is a sum of terms $\phi_{\sigma, \sigma^{-1}}^*$, where σ ranges over the set of transpositions in S_n . For each transposition σ , since $\sigma^{-1} = \sigma$ we have $\text{im } \kappa_{\sigma^{-1}}^L = \text{im } \kappa_\sigma^L \subseteq V^\sigma \subseteq \ker \kappa_\sigma^*$, and thus $\kappa_\sigma^*(u, \kappa_{\sigma^{-1}}^L(v, w)) = 0$ for all $u, v, w \in V$. It follows that $\phi_{\sigma, \sigma^{-1}}^* \equiv 0$ for each transposition $\sigma \in S_n$, and hence, $\phi_1^* \equiv 0$.

Case 2 ($g = (12)(34)$). Note that $\phi_g^* = \phi_{(12), (34)}^* + \phi_{(34), (12)}^*$. Since we know that both $\text{im } \kappa_{(34)}^L \subseteq V^{(12)} \subseteq \ker \kappa_{(12)}^*$ and $\text{im } \kappa_{(12)}^L \subseteq V^{(34)} \subseteq \ker \kappa_{(34)}^*$, we see that $\phi_g^*(u, v, w) \equiv 0$. \blacksquare

Proposition 5.4. *Let $\kappa_{\text{ref}} = \kappa_{\text{ref}}^L + \kappa_{\text{ref}}^C$ be as in Definition 3.8 with parameters $a, b, c \in \mathbb{C}$, and let ϕ_g^* denote the g -component of $\phi(\kappa_{\text{ref}}^*, \kappa_{\text{ref}}^L)$, where $*$ = L or $*$ = C and g is a 3-cycle. Then $\phi_g^C \equiv 0$. For ϕ_g^L we have $\phi_g^L(u, v, w) = 0$ if $u \in V^g$, and for $v \notin V^g$, $\phi_g^L(v, {}^g v, {}^g v) = 0$, and*

$$\begin{aligned} \phi_g^L(v, {}^g v, v^*) &= \phi_g^L({}^g v, {}^g v, v^*) = \phi_g^L({}^g v, v, v^*) \\ &= \begin{cases} 2(a^\perp - a)(b^\perp - b)({}^g v - v) + 2(a^\perp - a)^2({}^g v^* - v^*) & \text{if } v \in W, \\ 2(a^\perp - a)(b^\perp - b)({}^g v - v) + 2(b^\perp - b)^2({}^g v^* - v^*) & \text{if } v \in W^*, \end{cases} \end{aligned}$$

with values on triples involving a third basis vector of the form ${}^g v^*$ or ${}^g v^*$ obtained by acting by g or g^2 respectively.

Proof. By the orbit property in Lemma 5.1, it suffices to evaluate ϕ_g^* for the conjugacy class representative $g = (123)$. Note that $Z(g) = \langle (123) \rangle \times \text{Sym}_{\{4, \dots, n\}}$, and the factorizations of g as a product of transpositions are all in the same $Z(g)$ -orbit under diagonal conjugation, so

$$\phi_g^* = \phi_{(12),(23)}^* + \phi_{(23),(31)}^* + \phi_{(31),(12)}^*.$$

For each pair of transpositions σ and τ with $\sigma\tau = g = (123)$, we have $V^g \subseteq V^\sigma \cap V^\tau$ and $V^\sigma \cap V^\tau \subseteq \ker \phi_{\sigma,\tau}^*$ by Lemma 5.2(1). So if any vector in a basis triple is in V^g , then ϕ_g^* evaluates to zero. It remains only to consider triples $\{u, v, w\} \subseteq \{x_1, x_2, x_3, y_1, y_2, y_3\}$.

Lemma 5.2(1) yields $\phi_g^*(x_1, x_2, x_3) = \phi_g^*(y_1, y_2, y_3) = 0$. There are, up to permutation, $\frac{1}{2} \binom{6}{3} - 1 = 9$ basis triples with two elements in W^* and one element in W :

$$\begin{array}{lll} x_1, x_2, y_1, & x_2, x_3, y_1, & x_3, x_1, y_1, \\ x_2, x_3, y_2, & x_3, x_1, y_2, & x_1, x_2, y_2, \\ x_3, x_1, y_3, & x_1, x_2, y_3, & x_2, x_3, y_3. \end{array}$$

The G -invariance in Lemma 5.1 yields

$$\phi_g^*({}^g u, {}^g v, {}^g w) = {}^g \phi_g^*(u, v, w),$$

and thus since each of the three columns of basis triples is a g -orbit, it suffices to compute ϕ_g^* on just the basis triples

$$x_1, x_2, y_1, \quad x_2, x_3, y_1, \quad \text{and} \quad x_3, x_1, y_1.$$

For each of these three basis triples, u, v, w , the result $\phi_g^*(u, v, w)$ will be the same by Lemma 5.2 (although the reason varies for a given term on different triples), namely

$$\begin{aligned} \phi_g^*(u, v, w) &= \phi_{(12),(23)}^*(u, v, w) + \phi_{(23),(31)}^*(u, v, w) + \phi_{(31),(12)}^*(u, v, w) \\ &= 0 - 2(b^\perp - b)\kappa_{(23)}^*(x_3, y_3) + 2(b^\perp - b)\kappa_{(31)}^*(x_1, y_1) \\ &= \begin{cases} 0 & \text{if } * = C, \\ -2(b^\perp - b)[ax_{2,3} + a^\perp x_{2,3}^\perp + by_{2,3} + b^\perp y_{2,3}^\perp] \\ \quad + 2(b^\perp - b)[ax_{1,3} + a^\perp x_{1,3}^\perp + by_{1,3} + b^\perp y_{1,3}^\perp] & \text{if } * = L \end{cases} \\ &= \begin{cases} 0 & \text{if } * = C, \\ 2(b^\perp - b)[(a^\perp - a)(x_2 - x_1) + (b^\perp - b)(y_2 - y_1)] & \text{if } * = L. \end{cases} \end{aligned}$$

A similar reduction and computation applies to the nine basis triples with two elements in W and one element in W^* , and that combined with the orbiting properties in Lemma 5.1 lead to the conclusion in the statement. \blacksquare

Using the form of $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ in Propositions 5.3 and 5.4 we define the cochain $\kappa_{3\text{-cyc}}^C$ in Definition 3.9 to ensure $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C)$ as in the next proposition.

Proposition 5.5. *Let κ_{ref}^L and $\kappa_{3\text{-cyc}}^C$ be as in Definitions 3.8 and 3.9, with common parameters $a, a^\perp, b, b^\perp \in \mathbb{C}$. Then $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C)$.*

Proof. We compare the component ϕ_g of $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ with the component $2\psi_g$ of $2\psi(\kappa_{3\text{-cyc}}^C)$. If g is not a 3-cycle, then $\phi_g \equiv 0$ by Proposition 5.3; and $\kappa_{3\text{-cyc}}^C$ is not supported on g , so $\psi_g \equiv 0$ as well. If g is a 3-cycle, then it suffices to compare components ϕ_g and $2\psi_g$ on basis triples of the form in the statement of Proposition 5.4.

Case 1. If $u \in V^g$, then $\phi_g(u, v, w) = 0$ by Proposition 5.4 and $\psi_g(u, v, w) = 0$ because $g u - u = 0$ and $V^g \subseteq \ker \kappa_g^C$.

Case 2. If $v \notin V^g$, then $\phi_g(v, g v, g^2 v) = 0$ by Proposition 5.4 and $\psi_g(v, g v, g^2 v) = 0$ by g -invariance of κ_g^C .

Case 3. For triples of the form $(v, g v, v^*)$, $(g v, g^2 v, v^*)$, and $(g^2 v, v, v^*)$ with $v \in V^g$, use Proposition 5.4 to find $\phi_g(v, g v, v^*) = \phi_g(g v, g^2 v, v^*) = \phi_g(g^2 v, v, v^*)$ and Definition 3.9 to confirm that $\phi_g = 2\psi_g$ on each such triple, using that

$$\begin{aligned}\psi_g(v, g v, v^*) &= \kappa_g^C(v, g v)(g v^* - v^*) + \kappa_g^C(g v, v^*)(g v - v) + \kappa_g^C(v^*, v)(g^2 v - g v), \\ \psi_g(g v, g^2 v, v^*) &= \kappa_g^C(g v, g^2 v)(g v^* - v^*) + \kappa_g^C(g^2 v, v^*)(g^2 v - g v) + \kappa_g^C(v^*, g v)(v - g^2 v), \\ \psi_g(g^2 v, v, v^*) &= \kappa_g^C(g^2 v, v)(g v^* - v^*) + \kappa_g^C(v, v^*)(v - g^2 v) + \kappa_g^C(v^*, g^2 v)(g v - v).\end{aligned}\quad \blacksquare$$

5.3 Clearing the second obstruction

The final step in determining when the cochain $\kappa = \kappa_{\text{ref}}^L + \kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C + \kappa_1^C$ is a Drinfeld orbifold algebra map is to understand when $\phi(\kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C + \kappa_1^C, \kappa_{\text{ref}}^L) = 0$, which is stated as Corollary 5.8 and follows immediately from Propositions 5.3 and 5.4 and Lemmas 5.6 and 5.7. This clears the second obstruction and completes the proof of Theorem 5.9.

Lemma 5.6. *Let κ_{ref}^L and $\kappa_{3\text{-cyc}}^C$ be as in Definitions 3.8 and 3.9, with common parameters $a, a^\perp, b, b^\perp \in \mathbb{C}$. Denote a term of the component ϕ_g of $\phi(\kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L)$ by $\phi_{\sigma, \tau}$, where σ is a 3-cycle and τ is a transposition such that $\sigma\tau = g$. Then $\phi_{\sigma, \tau} \equiv 0$.*

Proof. The proof proceeds by considering the same exhaustive cases as in Lemma 5.2, but using the definition of $\kappa_{3\text{-cyc}}^C$ to show in fact $\phi_{\sigma, \tau} \equiv 0$ in cases (2) and (3). Showing that $\phi_{\sigma, \tau}^*(u, v, w) = 0$ when $u, v, w \in W$, $u, v, w \in W^*$, $u, v \in V^\tau$, or $u \in V^\tau \cap V^\sigma$ proceeds exactly as in the proof of Lemma 5.2 since the methods did not depend on anything about κ_σ^* other than $V^\sigma \subseteq \ker \kappa_\sigma^*$.

As in the proof of case (2) in Lemma 5.2, assume $u \in V^\tau \setminus V^\sigma$ and $v \notin V^\tau$ and note

$$\phi_{\sigma, \tau}(v, \tau v, u^*) = 0,$$

and

$$\phi_{\sigma, \tau}(u, v, v^*) = 2\kappa_\sigma^C(u, \kappa_\tau^L(v, v^*)).$$

Using bilinearity and $V^\sigma \subseteq \ker \kappa_\sigma^*$, the right hand side is a linear combination of expressions $\kappa_\sigma^*(u, h u)$ for $h \in \langle \sigma \rangle$. The appropriate coefficients in terms of a, a^\perp, b, b^\perp can be described in terms of the indicator function for the fixed space of τ and depend on whether $u \in W^*$ or $u \in W$. Also, $\sum_{h \in \langle \sigma \rangle} h u \in V^\sigma \subseteq \ker \kappa_\sigma^*$. Thus for $u \in W^*$,

$$\begin{aligned}\kappa_\sigma^C(u, \kappa_\tau^L(v, v^*)) &= (a^\perp - a) \sum_{h \in \langle \sigma \rangle} \delta_\tau(h u) \kappa_\sigma^C(u, h u) + (b^\perp - b) \sum_{h \in \langle \sigma \rangle} \delta_\tau(h u^*) \kappa_\sigma^C(u, h u^*) \\ &= (a^\perp - a) [\delta_\tau(\sigma u) \kappa_\sigma^C(u, \sigma u) + \delta_\tau(\sigma^2 u) \kappa_\sigma^C(u, \sigma^2 u)] \\ &\quad + (b^\perp - b) [\delta_\tau(\sigma u) \kappa_\sigma^C(u, \sigma u^*) + \delta_\tau(\sigma^2 u) \kappa_\sigma^C(u, \sigma^2 u^*)] \\ &= [\delta_\tau(\sigma u) - \delta_\tau(\sigma^2 u)] [(a^\perp - a)(b^\perp - b)^2 - (b^\perp - b)(a^\perp - a)(b^\perp - b)] = 0,\end{aligned}$$

and hence $\phi_{\sigma, \tau}(u, v, v^*) = 0$. A similar calculation shows that

$$\phi_{\sigma, \tau}(u, v, \tau v^*) = -\phi_{\sigma, \tau}(u, v, v^*) = 0$$

and applying G -invariance leads to the same conclusions when $u \in W$. Then (5.1) also implies

$$\phi_{\sigma,\tau}(u, {}^\tau v, {}^\tau v^*) = -\phi_{\sigma,\tau}(u, {}^\tau v, v^*) = \phi_{\sigma,\tau}(u, v, v^*) = 0.$$

For case (3) assume $v \notin V^\tau$ and note that as in the proof of Lemma 5.2,

$$\phi_{\sigma,\tau}(v, {}^\tau v, v^*) = -2\kappa_\sigma^C(v + {}^\tau v, \kappa_\tau^L(v, v^*)).$$

Since $\kappa_\sigma^C({}^\tau v, \kappa_\tau^L(v, v^*)) = \kappa_\sigma^C({}^\tau v, \kappa_\tau^L({}^\tau v, {}^\tau v^*)) = \kappa_\sigma^C(v, \kappa_\tau^L(v, v^*)) = 0$ by the same calculation as in case (2) except with ${}^\tau v$ and v in place of u , it follows that

$$\phi_{\sigma,\tau}(v, {}^\tau v, v^*) = -2[\kappa_\sigma^C(v, \kappa_\tau^L(v, v^*)) + \kappa_\sigma^C({}^\tau v, \kappa_\tau^L(v, v^*))] = 0.$$

Lastly, (5.1) yields

$$\phi_{\sigma,\tau}(v, {}^\tau v, {}^\tau v^*) = -\phi_{\sigma,\tau}(v, {}^\tau v, v^*) = 0. \quad \blacksquare$$

The case where $* = L$ is included in the preliminary calculations of the following lemma because it will be useful as a starting point in the proof of Theorem 5.10. We note that when $n = 2$ conditions (5.3) and (5.4) need to be modified to $a(\alpha + \beta) = b(\alpha + \beta) = 0$.

Lemma 5.7. *Let κ_1^C and κ_{ref}^L be as in Definitions 3.7 and 3.8, with parameters $\alpha, \beta \in \mathbb{C}$ and $a, a^\perp, b, b^\perp \in \mathbb{C}$ respectively. Denote a term of the component ϕ_g of $\phi(\kappa_1^*, \kappa_{\text{ref}}^L)$ by $\phi_{1,g}^*$, where $* = C$ or $* = L$ and g is a transposition. Then $\phi_{1,g}^C = 0$ if and only if the following conditions hold*

$$\alpha a + \beta(a + (n-2)a^\perp) = 0, \quad \alpha a^\perp + \beta(2a + (n-3)a^\perp) = 0, \quad (5.3)$$

$$\alpha b + \beta(b + (n-2)b^\perp) = 0, \quad \alpha b^\perp + \beta(2b + (n-3)b^\perp) = 0. \quad (5.4)$$

Proof. As in Section 4, it suffices to compute $\phi_{1,g}^*$ on basis triples of the following forms for $1 \leq i, j, k \leq n$.

1. All basis vectors in W or in W^* and i, j, k distinct: $(x_i, x_j, x_k), (y_i, y_j, y_k)$.
2. Two basis vectors in W or in W^* and i, j, k distinct: $(x_i, x_j, y_k), (y_i, y_j, x_k)$.
3. Two basis vectors in W or W^* and i, j distinct: $(x_i, x_j, y_j), (y_i, y_j, x_j)$.

Case 1. For any distinct indices i, j, k with $1 \leq i, j, k \leq n$, using (2.6) and Definition 3.8 of κ_{ref}^L it is immediate that $\phi_{1,g}^*(x_i, x_j, x_k) = 0$, and in similar fashion $\phi_{1,g}^*(y_i, y_j, y_k) = 0$, for any (distinct i, j, k with) $1 \leq i, j, k \leq n$. Thus this case imposes no conditions on any parameters.

Case 2. For any distinct indices i, j, k with $1 \leq i, j, k \leq n$, using the definitions of κ_{ref}^L and κ_1^C , bilinearity, and skew-symmetry yields

$$\begin{aligned} \phi_{1,g}^*(x_i, x_j, y_k) &= \begin{cases} 2\kappa_1^*(x_j, ax_{ik} + a^\perp x_{ik}^\perp + by_{ik} + b^\perp y_{ik}^\perp) & \text{if } g = (ik), \\ -2\kappa_1^*(x_i, ax_{jk} + a^\perp x_{jk}^\perp + by_{jk} + b^\perp y_{jk}^\perp) & \text{if } g = (jk), \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 2[\alpha b^\perp + \beta(2b + (n-3)b^\perp)] & \text{if } g = (ik) \text{ and } * = C, \\ -2[\alpha b^\perp + \beta(2b + (n-3)b^\perp)] & \text{if } g = (jk) \text{ and } * = C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Interchanging the roles of x and y and recomputing yields that for any distinct indices i, j, k with $1 \leq i, j, k \leq n$,

$$\phi_{1,g}^*(y_i, y_j, x_k) = \begin{cases} 2[\alpha a^\perp + \beta(2a + (n-3)a^\perp)] & \text{if } g = (ik) \text{ and } * = C, \\ -2[\alpha a^\perp + \beta(2a + (n-3)a^\perp)] & \text{if } g = (jk) \text{ and } * = C, \\ 0 & \text{otherwise.} \end{cases}$$

Case 3. For any distinct indices i, j with $1 \leq i, j \leq n$, using the definitions of κ_{ref}^L and κ_1^C , bilinearity, and skew-symmetry yields

$$\begin{aligned} \phi_{1,g}^*(x_i, x_j, y_j) &= \begin{cases} 2\kappa_1^*(x_i + x_j, ax_{ij} + a^\perp x_{ij}^\perp + by_{ij} + b^\perp y_{ij}^\perp) & \text{if } g = (ij), \\ 2\kappa_1^*(x_i, ax_{jk} + a^\perp x_{jk}^\perp + by_{jk} + b^\perp y_{jk}^\perp) & \text{if } g = (jk), \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 4[\alpha b + \beta(b + (n-2)b^\perp)] & \text{if } g = (ij) \text{ and } * = C, \\ 2[\alpha b^\perp + \beta(2b + (n-3)b^\perp)] & \text{if } g = (jk) \text{ and } * = C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Interchanging the roles of x and y and recomputing yields that for any distinct indices i, j with $1 \leq i, j \leq n$,

$$\phi_{1,g}^*(y_i, y_j, x_j) = \begin{cases} 4[\alpha a + \beta(a + (n-2)a^\perp)] & \text{if } g = (ij) \text{ and } * = C, \\ 2[\alpha a^\perp + \beta(2a + (n-3)a^\perp)] & \text{if } g = (jk) \text{ and } * = C, \\ 0 & \text{otherwise.} \end{cases}$$

Setting the results in cases 2 and 3 equal to zero yields conditions (5.3) and (5.4). \blacksquare

Corollary 5.8. *Let κ_{ref}^L and $\kappa_{3\text{-cyc}}^C$ be as in Definitions 3.8 and 3.9, with common parameters $a, a^\perp, b, b^\perp \in \mathbb{C}$. For every $g \in S_n$, the component ϕ_g of $\phi(\kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L)$ is identically zero. The component ϕ_g of $\phi(\kappa_1^C, \kappa_{\text{ref}}^L)$ is zero for all $g \in S_n$ if and only if conditions (5.3) and (5.4) given in Lemma 5.7 hold.*

Proof. In fact, each component of $\phi(\kappa_{\text{ref}}^C, \kappa_{\text{ref}}^L)$ is identically zero by Propositions 5.3 and 5.4, and each component of $\phi(\kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L)$ is identically zero by Lemma 5.6. By Lemma 5.7 we have $\phi(\kappa_1^C, \kappa_{\text{ref}}^L) = 0$ if and only if conditions (5.3) and (5.4) are satisfied because the component ϕ_g is identically zero for g not a transposition by the definitions of κ_1^C and κ_{ref}^L . \blacksquare

Conditions (5.3) and (5.4) given in Lemma 5.7 give rise to a variety that controls the parameter space for the family of maps in Theorem 5.9 and Drinfeld orbifold algebras in Theorem 7.2. We conjecture that this projective variety has dimension four based on computations done for a few specific values of n in Macaulay2 [14] with the graded reverse lexicographic monomial ordering and the parameter order $a, a^\perp, b, b^\perp, \alpha, \beta, c$.

5.4 Drinfeld orbifold algebra maps

Now we use the details of clearing the obstructions from Section 5.1 to describe all Drinfeld orbifold algebra maps with linear part supported only off the identity. The corresponding Drinfeld orbifold algebras are given in Theorem 7.2.

Theorem 5.9. *For S_n ($n \geq 3$) acting on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ by the doubled permutation representation, the Drinfeld orbifold algebra maps supported only off the identity are precisely the maps of the form $\kappa = \kappa_{\text{ref}}^L + \kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C + \kappa_1^C$, with κ_{ref}^L and κ_{ref}^C as in Definition 3.8, $\kappa_{3\text{-cyc}}^C$ as in Definition 3.9, κ_1^C as in Definition 3.7, and with the parameters $a, a^\perp, b, b^\perp, c, \alpha$, and β satisfying these conditions derived in Lemma 5.7:*

$$(5.3) \quad \alpha a + \beta(a + (n-2)a^\perp) = 0, \quad \alpha a^\perp + \beta(2a + (n-3)a^\perp) = 0,$$

$$(5.4) \quad \alpha b + \beta(b + (n-2)b^\perp) = 0, \quad \alpha b^\perp + \beta(2b + (n-3)b^\perp) = 0.$$

In particular, $\kappa = \kappa_{\text{ref}}^L + \kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C$ is always a Drinfeld orbifold algebra map.

Proof. Suppose κ^L is a pre-Drinfeld orbifold algebra map supported only off the identity. By Corollary 3.5 we must have $\kappa^L = \kappa_{\text{ref}}^L$ for some parameters $a, a^\perp, b, b^\perp \in \mathbb{C}$ as in Definition 3.8. It remains to find all G -invariant maps κ^C such that properties (2.4) and (2.5) of a Drinfeld orbifold algebra map also hold.

First we find a particular lift.

- *First obstruction.* Propositions 5.3 and 5.4 give the value of $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$. These values suggest how to construct the S_n -invariant map $\kappa_{3\text{-cyc}}^C$ such that property (2.4) holds, as given in Definition 3.9. Proposition 5.5 then verifies that $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L)$ and $2\psi(\kappa_{3\text{-cyc}}^C)$ are indeed equal.
- *Second obstruction.* By Lemma 5.6, we have that $\phi(\kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L) = 0$.

Thus $\kappa = \kappa_{\text{ref}}^L + \kappa_{3\text{-cyc}}^C$ is a Drinfeld orbifold algebra map.

Next, we modify this particular lift to obtain all possible lifts. Let κ^C be any G -invariant constant 2-cochain.

- *First obstruction.* Since $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C)$, it follows that $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa^C)$ if and only if $\psi(\kappa^C - \kappa_{3\text{-cyc}}^C) = 0$. By Corollary 3.6 this occurs if and only if $\kappa^C - \kappa_{3\text{-cyc}}^C = \kappa_{\text{ref}}^C + \kappa_1^C$, with κ_{ref}^C as in Definition 3.8 for some parameter $c \in \mathbb{C}$ and κ_1^C as in Definition 3.7 for some parameters $\alpha, \beta \in \mathbb{C}$.
- *Second obstruction.* By Corollary 5.8, $\phi(\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L) = 0$ if and only if (5.3) and (5.4) are satisfied. This occurs in particular when $\kappa_1^C \equiv 0$ in which case $\kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ clears the second obstruction and κ_{ref}^L lifts with no restrictions on the parameters a, a^\perp, b, b^\perp , or c .

Thus the lifts of κ_{ref}^L to a Drinfeld orbifold algebra map are precisely the maps of the form $\kappa = \kappa_{\text{ref}}^L + \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ satisfying conditions (5.3) and (5.4). ■

Lastly, by specifying parameters we obtain some Drinfeld orbifold algebra maps that are supported both on and off the identity. The proof uses results related to clearing obstructions that appeared in Section 4 and Section 5.1.

Theorem 5.10. *For S_n ($n \geq 3$) acting on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ by the doubled permutation representation, there are Drinfeld orbifold algebra maps of the form $\kappa = \kappa^L + \kappa^C$ with linear part $\kappa^L = \kappa_1^L + \kappa_{\text{ref}}^L$ and constant part $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$, where*

- (1) κ_1^L is as described in Definition 3.7 with $a_i = b_i = 0$ for $i = 1, 2, 3, 5, 6$ and $a_4 = a_7$ and $b_4 = b_7$ are not both zero,
- (2) κ_1^C is as described in Definition 3.7 with $\alpha = \beta$,
- (3) κ_{ref}^L and κ_{ref}^C are as in Definition 3.8 and $\kappa_{3\text{-cyc}}^C$ is as in Definition 3.9 with $2a + (n-2)a^\perp = 2b + (n-2)b^\perp = 0$, but a, a^\perp, b , and b^\perp not all zero.

Proof. As in the proof of Theorem 5.9, even without the given parameter choices we have $\phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C) = 2\psi(\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C)$. Setting $a_i = b_i = 0$ for $i = 1, 2, 3, 5, 6$, $a_4 = a_7$, and $b_4 = b_7$ in (4.1) and in the values of $\phi_{g,1}^L(u, v, w)$ in Sections 4.1–4.3 shows $\phi(\kappa_1^L, \kappa_1^L) = 0$ and $\phi(\kappa_{\text{ref}}^L, \kappa_1^L) = 0$ respectively. Using the forms of $\phi_{1,g}^L$ given in Lemma 5.7 and

the assumptions that $a_4 = a_7$ and $b_4 = b_7$ yield that

$$\phi_{1,g}^L(x_i, x_j, y_k) = \begin{cases} 2(2b + (n-2)b^\perp)(a_4x_{[n]} + b_4y_{[n]}) & \text{if } g = (ik), \\ -2(2b + (n-2)b^\perp)(a_4x_{[n]} + b_4y_{[n]}) & \text{if } g = (jk), \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{1,g}^L(x_i, x_j, y_j) = \begin{cases} 4(2b + (n-2)b^\perp)(a_4x_{[n]} + b_4y_{[n]}) & \text{if } g = (ij), \\ 2(2b + (n-2)b^\perp)(a_4x_{[n]} + b_4y_{[n]}) & \text{if } g = (jk), \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for $\phi_{1,g}^L(y_i, y_j, x_k)$ and $\phi_{1,g}^L(y_i, y_j, x_j)$, but replacing b with a and b^\perp with a^\perp . By the hypothesis on a , a^\perp , b , and b^\perp , all of these are zero, so $\phi(\kappa_1^L, \kappa_{\text{ref}}^L) = 0$ and hence

$$\phi(\kappa_1^L + \kappa_{\text{ref}}^L, \kappa_1^L + \kappa_{\text{ref}}^L) = \phi(\kappa_{\text{ref}}^L, \kappa_{\text{ref}}^L) = 2\psi(\kappa_{3\text{-cyc}}^C) = 2\psi(\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C).$$

Thus with the given parameter choices $\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ clears the first obstruction for $\kappa_1^L + \kappa_{\text{ref}}^L$.

We claim $\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ also clears the second obstruction for $\kappa_1^L + \kappa_{\text{ref}}^L$ because

$$\phi(\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C, \kappa_1^L + \kappa_{\text{ref}}^L) = 0.$$

By the assumptions on a_i and b_i for $1 \leq i \leq 7$ conditions (4.2) and (4.3) are satisfied and thus $\phi(\kappa_1^C + \kappa_{\text{ref}}^C, \kappa_1^L) = 0$. Also $\phi(\kappa_{3\text{-cyc}}^C, \kappa_1^L) = 0$. This is because for any $1 \leq i, j, k \leq n$ and any 3-cycle g , by $a_1 = a_2 = b_1 = b_2 = 0$ we have

$$\phi_{g,1}^C(x_i, x_j, x_k) = \phi_{g,1}^C(y_i, y_j, y_k) = 0,$$

and by $a_4 = a_7$ and $b_4 = b_7$ we have

$$\phi_{g,1}^C(x_i, x_j, y_k) = \kappa_g^C(x_i, \kappa_1^L(x_j, y_k)) + \kappa_g^C(x_j, \kappa_1^L(y_k, x_i)) = \kappa_g^C(x_i - x_j, a_4x_{[n]} + b_4y_{[n]}) = 0$$

because $\kappa_g^C(x_i, {}^g x_i + {}^{g^{-1}} x_i) = 0$ regardless of whether $i \in V^g$ by Definition 3.9, and similarly $\phi_{g,1}^C(y_i, y_j, x_k) = 0$. By Corollary 5.8 we know that $\phi(\kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C, \kappa_{\text{ref}}^L) = 0$ in general, and that $\phi(\kappa_1^C, \kappa_{\text{ref}}^L) = 0$ because conditions (5.3) and (5.4) are satisfied when $\alpha = \beta$ and $2a + (n-2)a^\perp = 2b + (n-2)b^\perp = 0$.

Thus with the given choices of parameters, $\kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ clears the second obstruction as well and lifts $\kappa_1^L + \kappa_{\text{ref}}^L$ to a Drinfeld orbifold algebra map. \blacksquare

6 Drinfeld orbifold algebra maps that deform $S(\mathfrak{h}^* \oplus \mathfrak{h}) \# S_n$

We now use the results in Sections 4 and 5 on Lie and Drinfeld orbifold algebra maps that produce deformations of the skew group algebra $S(W^* \oplus W) \# S_n$ in order to understand which maps produce deformations of $S(\mathfrak{h}^* \oplus \mathfrak{h}) \# S_n$.

In contrast to the complicated families of Lie orbifold algebras and maps in Theorems 4.1 and 7.1, when S_n instead acts on its doubled standard subrepresentation $\mathfrak{h}^* \oplus \mathfrak{h}$ there are no Lie orbifold algebra maps with nonzero linear part (Theorem 6.3). However, Theorem 6.4 describes a three-parameter family of Drinfeld orbifold algebra maps that do provide polynomial degree one deformations generalizing the \mathfrak{sl}_n -type rational Cherednik algebras $H_{0,c}$ (see also Theorem 7.3). We begin with a result that applies to any finite group and provides conditions under which we can combine Drinfeld orbifold algebra maps for subrepresentations into a map for their direct sum.

Proposition 6.1. *Let G be a finite group acting linearly on finite-dimensional vector spaces U_1, U_2, \dots, U_r . Given Drinfeld orbifold algebra maps $\kappa|_{U_i}$ for G acting on U_i ($i = 1, \dots, r$), define κ on $\bigwedge^2(\bigoplus_{i=1}^r U_i)$ so that κ agrees with $\kappa|_{U_i}$ for pairs of vectors from the same U_i and is zero on mixed pairs, i.e., $\kappa(U_i, U_j) = 0$ for $i \neq j$. Then κ is a Drinfeld orbifold algebra map for G acting on $\bigoplus_{i=1}^r U_i$ if and only if whenever $i \neq j$ all group elements in the support of $\kappa|_{U_i}$ act trivially on U_j .*

Proof. For $i = 1, \dots, r$, suppose $\kappa|_{U_i}$ is a Drinfeld orbifold algebra map for G acting on U_i and define κ on $\bigwedge^2(\bigoplus_{i=1}^r U_i)$ as above. Conditions (2.1) and (2.2) of the definition of a Drinfeld orbifold algebra map are straightforward to verify. For conditions (2.3)–(2.5), consider a triple of vectors from $\bigoplus_{i=1}^r U_i$. If all three vectors are from the same U_i , then equations (2.3)–(2.5) hold by virtue of $\kappa|_{U_i}$ being a Drinfeld orbifold algebra map. If the three vectors are from U_i, U_j , and U_k with i, j, k distinct, then conditions (2.3)–(2.5) are easily seen to hold because κ is defined to be zero on pairs of vectors from different summands.

By multilinearity and skew-symmetry, all that remains is to examine the case where two vectors, say u and v , are from the same U_i and the third vector, say w , is from some U_j with $j \neq i$. Recall that

$$\phi_{x,y}^*(u, v, w) = \kappa_x^*(u + {}^y u, \kappa_y^L(v, w)) + \kappa_x^*(v + {}^y v, \kappa_y^L(w, u)) + \kappa_x^*(w + {}^y w, \kappa_y^L(u, v)).$$

In the present case, the first two terms are zero because $\kappa_y^L(U_i, U_j) = 0$, and the last term is zero because $\kappa_y^L(U_i, U_i) \subseteq U_i$ and $\kappa_x^*(U_j, U_i) = 0$. Thus $\phi_{x,y}^*(u, v, w) = 0$, which implies equation (2.5) is satisfied for all g in G . We also see (2.3) and (2.4) will be satisfied if and only if for all $g \in G$, we have $\psi_g^*(u, v, w) = 0$ for $* = L$ and $* = C$, respectively.

To this end, recall that

$$\psi_g^*(u, v, w) = \kappa_g^*(u, v)({}^g w - w) + \kappa_g^*(v, w)({}^g u - u) + \kappa_g^*(w, u)({}^g v - v).$$

Continuing with $u, v \in U_i$ and $w \in U_j$, we see that $\psi_g^*(u, v, w) = \kappa_g^*(u, v)({}^g w - w)$ because $\kappa_g^*(U_i, U_j) = 0$. Thus for conditions (2.3) and (2.4) to hold for all g in G and all triples of this type, it is both necessary and sufficient for group elements in the support of $\kappa|_{U_i}$ to act trivially on U_j whenever $i \neq j$. \blacksquare

As a corollary, in the case of two summands U_1 and U_2 with G acting trivially on U_1 and $\kappa|_{U_1} \equiv 0$ (so that the support of $\kappa|_{U_1}$ is empty), we have:

Corollary 6.2. *Let G be a finite group acting linearly on a vector space $V = U_1 \oplus U_2$, where each U_i is a subrepresentation. If G acts trivially on U_1 , then every Drinfeld orbifold algebra map $\kappa|_{U_2}$ on $\bigwedge^2 U_2$ extends to a Drinfeld orbifold algebra map κ on $\bigwedge^2 V$ with $\text{im } \kappa^L \subseteq U_2$ and such that $U_1 \subseteq \ker \kappa$.*

We now use Corollary 6.2 to show there are no Drinfeld orbifold algebra maps with linear part supported only on the identity for S_n acting on $\mathfrak{h}^* \oplus \mathfrak{h}$.

Theorem 6.3. *For S_n ($n \geq 3$) acting on $\mathfrak{h}^* \oplus \mathfrak{h} \cong \mathbb{C}^{2n-2}$ by the doubled standard representation there are no degree-one Lie orbifold algebra maps.*

Proof. If there were a Lie orbifold algebra map $\kappa^L + \kappa^C$ for the doubled standard representation with $\kappa^C: \bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}) \rightarrow \mathbb{C}S_n$ and nonzero $\kappa^L: \bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}) \rightarrow \mathfrak{h}^* \oplus \mathfrak{h}$, then it could be extended as in Corollary 6.2 to yield a Lie orbifold algebra map for S_n acting on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ via the doubled permutation representation. The possible forms of Lie orbifold algebra maps κ for the doubled permutation representation are controlled by Theorem 4.1, which includes the PBW conditions $\gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = 0$ in (4.1). We will use these to show that in fact $\kappa^L \equiv 0$

by first imposing the image constraint $\text{im } \kappa^L \subseteq \mathfrak{h}^* \oplus \mathfrak{h}$ and the kernel constraint $\iota^* \oplus \iota \subseteq \ker \kappa$ from Corollary 6.2.

First, use $x_i = \bar{x}_i + \frac{1}{n}x_{[n]}$ and $y_i = \bar{y}_i + \frac{1}{n}y_{[n]}$ in (3.3)–(3.6) to write the values of κ_1^L according to the decomposition $V \cong \mathfrak{h}^* \oplus \iota^* \oplus \mathfrak{h} \oplus \iota$:

$$\begin{aligned}\kappa_1^L(x_i, x_j) &= a_1(\bar{x}_i - \bar{x}_j) + b_1(\bar{y}_i - \bar{y}_j), \\ \kappa_1^L(y_i, y_j) &= a_2(\bar{x}_i - \bar{x}_j) + b_2(\bar{y}_i - \bar{y}_j), \\ \kappa_1^L(x_i, y_i) &= a_3\bar{x}_i + \frac{1}{n}(a_3 + na_4)x_{[n]} + b_3\bar{y}_i + \frac{1}{n}(b_3 + nb_4)y_{[n]}, \\ \kappa_1^L(x_i, y_j) &= a_5\bar{x}_i + a_6\bar{x}_j + \frac{1}{n}(a_5 + a_6 + na_7)x_{[n]} + b_5\bar{y}_i + b_6\bar{y}_j + \frac{1}{n}(b_5 + b_6 + nb_7)y_{[n]}.\end{aligned}$$

Thus $\text{im } \kappa^L \subseteq \mathfrak{h}^* \oplus \mathfrak{h}$ implies

$$\begin{aligned}a_3 + na_4 &= 0, & b_3 + nb_4 &= 0, \\ a_5 + a_6 + na_7 &= 0, & b_5 + b_6 + nb_7 &= 0.\end{aligned}$$

Second, impose the extension conditions $\kappa^L(w_i, v_j) = \kappa^L(v_i, v_j) = 0$ for w_i in the basis $\{x_{i+1} - x_i, y_{i+1} - y_i \mid 1 \leq i \leq n-1\}$ of $\mathfrak{h}^* \oplus \mathfrak{h}$ and v_i, v_j in the basis $\{x_{[n]}, y_{[n]}\}$ of $\iota^* \oplus \iota = (\mathfrak{h}^* \oplus \mathfrak{h})^\perp$. From

$$\begin{aligned}\kappa^L(x_{i+1} - x_i, x_{[n]}) &= na_1(x_{i+1} - x_i) + nb_1(y_{i+1} - y_i) = 0, \\ \kappa^L(y_{i+1} - y_i, y_{[n]}) &= na_2(x_{i+1} - x_i) + nb_2(y_{i+1} - y_i) = 0\end{aligned}$$

we obtain $a_1 = b_1 = a_2 = b_2 = 0$. We also require that

$$\kappa^L(x_{i+1} - x_i, y_{[n]}) = (a_3 + na_5 - (a_5 + a_6))(x_{i+1} - x_i) + (b_3 + nb_5 - (b_5 + b_6))(y_{i+1} - y_i) = 0$$

and

$$\kappa^L(x_{[n]}, y_{i+1} - y_i) = (a_3 + na_6 - (a_5 + a_6))(x_{i+1} - x_i) + (b_3 + nb_6 - (b_5 + b_6))(y_{i+1} - y_i) = 0,$$

and thus all four coefficients are zero. Using the results of the image constraint to simplify those coefficients yields

$$\begin{aligned}a_4 - a_5 - a_7 &= 0, & b_4 - b_5 - b_7 &= 0, \\ a_4 - a_6 - a_7 &= 0, & b_4 - b_6 - b_7 &= 0,\end{aligned}$$

from which it follows that $a_5 = a_6$ and $b_5 = b_6$. The remaining extension requirement imposes no further constraints on the parameters because one verifies $\kappa^L(x_{[n]}, y_{[n]}) = 0$ using $a_5 + a_6 + na_7 = b_5 + b_6 + nb_7 = 0$.

To analyze the PBW conditions $\gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = 0$ in (4.1) it will help to first observe that the above constraints $a_5 = a_6$ and $a_5 + a_6 + na_7 = 0$ yield that

$$a_5 = a_6 = -\frac{n}{2}a_7,$$

and hence that

$$a_4 = a_5 + a_7 = -\frac{n-2}{2}a_7 \quad \text{and} \quad a_3 = -na_4 = \frac{n(n-2)}{2}a_7,$$

with corresponding expressions in terms of b_7 for $b_3, b_4, b_5,$ and b_6 . These allow the simplification

$$\begin{aligned}\phi_{1,1}(x_i, x_j, y_k) &= [-b_5a_6 + b_7(a_3 - a_5 - a_6)](x_i - x_j) + [b_5(b_5 + nb_7) + b_7(b_3 - b_5 - b_6)](y_i - y_j) \\ &= \frac{n^2}{4}a_7b_7(x_i - x_j) + \frac{n^2}{4}b_7^2(y_i - y_j).\end{aligned}$$

Similarly,

$$\phi_{1,1}(y_i, y_j, x_k) = \frac{n^2}{4} a_7^2 (x_i - x_j) + \frac{n^2}{4} a_7 b_7 (y_i - y_j).$$

Requiring each of these to be zero forces $a_7 = b_7 = 0$, and thus $a_i = b_i = 0$ for $3 \leq i \leq 6$ as well. Since we already have $a_1 = b_1 = a_2 = b_2 = 0$, this proves there are no Lie orbifold algebra maps for S_n acting on the doubled standard subrepresentation $\mathfrak{h}^* \oplus \mathfrak{h}$ with $\kappa^L \neq 0$. ■

For maps with linear part supported only off the identity there is instead a three-parameter family of Drinfeld orbifold algebra maps that generalize the commutator relations for the rational Cherednik algebra $H_{0,c}$.

Theorem 6.4. *Let S_n ($n \geq 3$) act via the doubled standard representation on the space $\mathfrak{h}^* \oplus \mathfrak{h} \cong \mathbb{C}^{2n-2}$ spanned by $\bar{x}_i = x_i + \frac{1}{n}x_{[n]}$ and $\bar{y}_i = y_i + \frac{1}{n}y_{[n]}$ with $1 \leq i \leq n$. Let $a^\perp, b^\perp, c \in \mathbb{C}$. All Drinfeld orbifold algebra maps with nonzero linear part supported only off the identity have the form $\kappa^L + \kappa^C$ defined for $1 \leq i \neq j \leq n$ by*

$$\begin{aligned} \kappa^L(\bar{x}_i, \bar{x}_j) &= \kappa^L(\bar{y}_i, \bar{y}_j) = 0, \\ \kappa^L(\bar{x}_i, \bar{y}_i) &= -\frac{n}{2} \sum_{k \neq i} (a^\perp(\bar{x}_i + \bar{x}_k) + b^\perp(\bar{y}_i + \bar{y}_k)) \otimes (ik), \\ \kappa^L(\bar{x}_i, \bar{y}_j) &= \frac{n}{2} (a^\perp(\bar{x}_i + \bar{x}_j) + b^\perp(\bar{y}_i + \bar{y}_j)) \otimes (ij) \end{aligned}$$

and

$$\begin{aligned} \kappa^C(\bar{x}_i, \bar{x}_j) &= \frac{n^2}{4} (b^\perp)^2 \sum_{k \neq i, j} (ijk) - (kji), & \kappa^C(\bar{y}_i, \bar{y}_j) &= \frac{n^2}{4} (a^\perp)^2 \sum_{k \neq i, j} (ijk) - (kji), \\ \kappa^C(\bar{x}_i, \bar{y}_i) &= c \sum_{k \neq i} (ik), & \kappa^C(\bar{x}_i, \bar{y}_j) &= -c(ij) - \frac{n^2}{4} a^\perp b^\perp \sum_{k \neq i, j} (ijk) - (kji). \end{aligned}$$

Proof. Suppose $\kappa = \kappa^L + \kappa^C$ is a Drinfeld orbifold algebra map for the doubled standard representation with $\kappa^C: \Lambda^2(\mathfrak{h}^* \oplus \mathfrak{h}) \rightarrow \mathbb{C}S_n$ and nonzero $\kappa^L: \Lambda^2(\mathfrak{h}^* \oplus \mathfrak{h}) \rightarrow (\mathfrak{h}^* \oplus \mathfrak{h}) \otimes \mathbb{C}S_n$ supported only off the identity. Extend κ as described in Corollary 6.2 to yield a Drinfeld orbifold algebra map for S_n acting on $V = W^* \oplus W \cong \mathbb{C}^{2n}$ via the doubled permutation representation. By Theorem 5.9 the possible forms of such extensions are $\kappa_{\text{ref}}^L + \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ satisfying the PBW conditions (5.3) and (5.4).

We start by imposing the condition $\text{im } \kappa^L \subseteq \mathfrak{h}^* \oplus \mathfrak{h}$. Use $\bar{x}_i = x_i - \frac{1}{n}x_{[n]}$, $\bar{y}_i = y_i - \frac{1}{n}y_{[n]}$, $\bar{x}_{ij} := \bar{x}_i + \bar{x}_j$, and $\bar{y}_{ij} := \bar{y}_i + \bar{y}_j$ to rewrite the nonzero values of $\kappa_{(ij)}^L$ as

$$\begin{aligned} \kappa_{(ij)}^L(x_i, y_i) &= \kappa_{(ij)}^L(x_j, y_j) = -\kappa_{(ij)}^L(x_i, y_j) = -\kappa_{(ij)}^L(x_j, y_i) \\ &= ax_{ij} + a^\perp x_{ij}^\perp + by_{ij} + b^\perp y_{ij}^\perp = (a - a^\perp)x_{ij} + a^\perp x_{[n]} + (b - b^\perp)y_{ij} + b^\perp y_{[n]} \\ &= (a - a^\perp)\bar{x}_{ij} + \frac{1}{n}(2a + (n-2)a^\perp)x_{[n]} + (b - b^\perp)\bar{y}_{ij} + \frac{1}{n}(2b + (n-2)b^\perp)y_{[n]}. \end{aligned}$$

This shows $2a + (n-2)a^\perp = 2b + (n-2)b^\perp = 0$ or

$$a - a^\perp = -\frac{n}{2}a^\perp \quad \text{and} \quad b - b^\perp = -\frac{n}{2}b^\perp \tag{6.1}$$

and substituting these into (5.3) and (5.4) yields that

$$a(\alpha - \beta) = a^\perp(\alpha - \beta) = b(\alpha - \beta) = b^\perp(\alpha - \beta) = 0.$$

Thus since $\kappa^L \neq 0$ we must also have $\alpha = \beta$ in κ_1^C .

Next consider conditions arising from $\iota^* \oplus \iota \subseteq \ker \kappa$ in Corollary 6.2, i.e., $\kappa^C(w, v) = \kappa^C(u, v) = 0$ for all w in the basis $\{x_{i+1} - x_i, y_{i+1} - y_i \mid 1 \leq i \leq n-1\}$ of $\mathfrak{h}^* \oplus \mathfrak{h}$ and u, v in the basis $\{x_{[n]}, y_{[n]}\}$ of $\iota^* \oplus \iota = (\mathfrak{h}^* \oplus \mathfrak{h})^\perp$.

For κ_1^C , since $\kappa_1^C(x_i, x_j) = \kappa_1^C(y_i, y_j) = 0$ and $\kappa_1^C(x_i, y_{[n]}) = \kappa_1^C(x_{[n]}, y_i) = \alpha + (n-1)\beta$ for any $1 \leq i, j \leq n$, the only extension condition that is not automatically satisfied is $\kappa_1^C(x_{[n]}, y_{[n]}) = n(\alpha + (n-1)\beta) = 0$. Together with $\alpha = \beta$ this forces $\alpha = \beta = 0$ and thus $\kappa_1^C \equiv 0$.

For κ_{ref}^* and $\kappa_{3\text{-cyc}}^C$, the definitions of κ_g^L and κ_g^C when g is a transposition and of κ_g^C when g is a 3-cycle yield that

$$\kappa_g^* \left(v, \sum_{h \in \langle g \rangle} h v \right) = \kappa_g^* \left(v, \sum_{h \in \langle g \rangle} h v^* \right) = 0 \quad (6.2)$$

for all $v \in \{x_1, \dots, x_n, y_1, \dots, y_n\}$. This in turn implies that all of the extension conditions hold for κ_{ref}^* and $\kappa_{3\text{-cyc}}^C$, yielding no further constraints on parameters.

We now evaluate κ_{ref}^L , κ_{ref}^C , and $\kappa_{3\text{-cyc}}^C$ at pairs of vectors from $\{\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n\}$. For g a transposition, i an index moved by g , \bar{v} a vector in $\{\bar{x}_i, {}^g \bar{x}_i, \bar{y}_i, {}^g \bar{y}_i\}$, and $* = L$ or $* = C$, we have $\kappa_g^*(\bar{v}, {}^g \bar{v}) = 0$ and we use (6.2) to observe that for $1 \leq i \leq n$,

$$\begin{aligned} \kappa_g^*(\bar{x}_i, \bar{y}_i) &= -\kappa_g^*(\bar{x}_i, {}^g \bar{y}_i) = \kappa_g^*(x_i, y_i) - \frac{1}{n} \kappa_g^*(x_i, y_{[n]}) - \frac{1}{n} \kappa_g^*(x_{[n]}, y_j) + \frac{1}{n^2} \kappa_g^*(x_{[n]}, y_{[n]}) \\ &= \kappa_g^*(x_i, y_i). \end{aligned}$$

It then follows from (6.1) that

$$\begin{aligned} \kappa_g^L(\bar{x}_i, \bar{y}_i) &= -\kappa_g^L(\bar{x}_i, {}^g \bar{y}_i) = -\frac{n}{2} a^\perp \bar{x}_{ij} - \frac{n}{2} b^\perp \bar{y}_{ij}, \\ \kappa_g^C(\bar{x}_i, \bar{y}_i) &= -\kappa_g^C(\bar{x}_i, {}^g \bar{y}_i) = c. \end{aligned}$$

For g a 3-cycle, by the orbit property in (6.2) and by (6.1) we see that

$$\begin{aligned} \kappa_g^C(\bar{v}, \bar{v}^*) &= 0, \\ \kappa_g^C(\bar{v}, {}^g \bar{v}) &= \begin{cases} \frac{n^2}{4} (a^\perp)^2 & \text{if } v \in W, \\ \frac{n^2}{4} (b^\perp)^2 & \text{if } v \in W^*, \end{cases} \\ \kappa_g^C({}^g \bar{v}, \bar{v}^*) &= -\kappa_g^C(\bar{v}, {}^g \bar{v}^*) = \frac{n^2}{4} a^\perp b^\perp. \end{aligned}$$

These components produce the given definition of $\kappa^L + \kappa^C$. ■

Remark 6.5. In the case $\kappa_{\text{ref}}^L = \kappa_{\text{ref}}^C \equiv 0$ then also $\kappa_{3\text{-cyc}}^C \equiv 0$, $\alpha = -(n-1)\beta$, and the restriction of the constant 2-cochain $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C$ to $\wedge^2(\mathfrak{h}^* \oplus \mathfrak{h})$ is given by

$$\begin{aligned} \kappa_1^C(\bar{x}_i, \bar{x}_j) &= \kappa_1^C(\bar{y}_i, \bar{y}_j) = 0, & \kappa_1^C(\bar{x}_i, \bar{y}_i) &= -(n-1)\beta, & \kappa_1^C(\bar{x}_i, \bar{y}_j) &= \beta, \\ \kappa_g^C(\bar{x}_i, \bar{x}_j) &= \kappa_g^C(\bar{y}_i, \bar{y}_j) = 0, & \kappa_g^C(\bar{x}_i, \bar{y}_i) &= c, & \kappa_g^C(\bar{x}_i, \bar{y}_j) &= -c, \end{aligned}$$

where $\beta, c \in \mathbb{C}$, g is a transposition, and $1 \leq i \neq j \leq n$. This corresponds to the rational Cherednik algebra $H_{n\beta, c}$.

In the theory of rational Cherednik algebras, $H_{t, c}$, for the symmetric group, a natural isomorphism between $H_{t, c}$ and $H_{\lambda t, \lambda c}$ when $\lambda \in \mathbb{C}^\times$ means that only two distinct cases need be considered, $t \neq 0$ and $t = 0$. Theorems 6.3 and 6.4 show that in the first case there are no further deformations in polynomial degree one with the linear part of the parameter map supported only on the identity while there is a three-parameter family of such deformations in the second case with the linear part of the parameter map supported only off the identity.

Remark 6.6. What if κ^L were supported both on and off the identity? The specializations of parameter values for κ_1^L in part (1) and for κ_{ref}^L in part (3) of Theorem 5.10 on the combined lift of $\kappa_1^L + \kappa_{\text{ref}}^L$ means that

$$\begin{aligned}\kappa_1^L(x_i, y_i) &= \kappa_1^L(x_i, y_j) = a_4 x_{[n]} + b_4 y_{[n]}, \\ \kappa_{(ij)}^L(x_i, y_i) &= -\kappa_{(ij)}^L(x_i, y_j) = -\frac{n}{2} a^\perp(\bar{x}_i + \bar{x}_j) - \frac{n}{2} b^\perp(\bar{y}_i + \bar{y}_j).\end{aligned}$$

But then $\text{im}(\kappa_1^L + \kappa_{\text{ref}}^L) \subseteq \mathfrak{h}^* \oplus \mathfrak{h}$ would require $a_4 = b_4 = 0$ so $\kappa_1^L \equiv 0$. This combined with part (2) of Theorem 5.10 shows there is no Drinfeld orbifold algebra map for S_n on $\mathfrak{h}^* \oplus \mathfrak{h}$ with linear part supported both on and off the identity which extends to a map κ of the form in Theorem 5.10. But since Theorem 5.10 is not exhaustive, it is not clear whether there exist such maps in general.

7 Descriptions of degree-one rational Cherednik algebras

Here we present, via generators and relations, degree-one PBW deformations of the skew group algebras $S(W^* \oplus W) \# S_n$ and $S(\mathfrak{h}^* \oplus \mathfrak{h}) \# S_n$ that result from Theorems 4.1, 5.9, and 6.4 when $n \geq 3$. This facilitates comparison with degree-zero deformations (i.e., rational Cherednik algebras) and with the PBW deformations of $S(W) \# S_n$ in [8]. The classifications are summarized in Tables 2 and 3. We reiterate that the case when $n = 2$ can be analyzed in similar fashion, but involves some differences in the dimensions of spaces of pre-Drinfeld orbifold algebra maps and in the parameter relations required in order to satisfy the PBW conditions.

7.1 Algebras for the doubled permutation representation

First, the Lie orbifold algebra maps involving 17 parameters classified in Theorem 4.1 yield a variety controlling the Lie orbifold algebras that deform $S(W^* \oplus W) \# S_n$ in degree one. Based on representative calculations in Macaulay2 [14] we conjecture that this projective variety is of dimension seven. Some subvarieties of potential interest are indicated in Table 1 in Section 4. When $\kappa_1^L \equiv 0$ these Lie orbifold algebras specialize to rational Cherednik algebras corresponding to the parameter c and the general G -invariant skew-symmetric bilinear form κ_1^C involving α and β (because W is decomposable — see [6, proof of Theorem 1.3]).

Theorem 7.1 (Lie orbifold algebras for doubled permutation representation over $\mathbb{C}[t]$). *Let S_n ($n \geq 3$) act on $V = W^* \oplus W$ with basis $\mathcal{B} = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ by the doubled permutation representation. For $a_1, \dots, a_7, b_1, \dots, b_7, \alpha, \beta, c \in \mathbb{C}$ subject to conditions (4.1), (4.2), and (4.3), define $\kappa^L = \kappa_1^L$ and $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C$ to be the linear and constant cochains such that for $1 \leq i \neq j \leq n$,*

$$\begin{aligned}\kappa^L(x_i, x_j) &= (a_1(x_i - x_j) + b_1(y_i - y_j)), & \kappa^C(x_i, x_j) &= 0, \\ \kappa^L(y_i, y_j) &= (a_2(x_i - x_j) + b_2(y_i - y_j)), & \kappa^C(y_i, y_j) &= 0, \\ \kappa^L(x_i, y_i) &= (a_3 x_i + a_4 x_{[n]} + b_3 y_i + b_4 y_{[n]}), & \kappa^C(x_i, y_i) &= \alpha + c \sum_{k \neq i} (ik), \\ \kappa^L(x_i, y_j) &= (a_5 x_i + a_6 x_j + a_7 x_{[n]} + b_5 y_i + b_6 y_j + b_7 y_{[n]}), & \kappa^C(x_i, y_j) &= \beta - c(ij).\end{aligned}$$

Then the quotient $\mathcal{H}_{\kappa, t}$ of $T(V) \# S_n[t]$ by the ideal generated by

$$\{uv - vu - \kappa^L(u, v)t - \kappa^C(u, v)t^2 \mid u, v \in \mathcal{B}\}$$

is a Lie orbifold algebra over $\mathbb{C}[t]$. In fact, the algebras $\mathcal{H}_{\kappa, 1}$ are precisely the Drinfeld orbifold algebras such that κ^L is supported only on the identity.

Table 2. Classification of Drinfeld orbifold algebra maps for S_n acting on $W^* \oplus W$.

Linear part κ^L	Constant part κ^C	Parameter relations	Reference
κ_1^L	0 κ_1^C $\kappa_1^C + \kappa_{\text{ref}}^C$ with $\kappa_{\text{ref}}^C \neq 0$	(4.1) (4.1)–(4.2) (4.4)–(4.17)	Theorem 4.1
κ_{ref}^L	$\kappa_{3\text{-cyc}}^C$ $\kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C$ $\kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C + \kappa_1^C$	none none (5.3)–(5.4)	Theorem 5.9
$\kappa_1^L + \kappa_{\text{ref}}^L$	$\kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C + \kappa_1^C$?	Theorem 5.10(1)–(3) ?	Theorem 5.10
0	$\kappa_1^C + \kappa_{\text{ref}}^C$	none	

When the indicated parameter relations are satisfied, the map $\kappa = \kappa^L + \kappa^C$ is a Drinfeld orbifold algebra map. The question marks indicate there could be further maps with $\kappa^L = \kappa_1^L + \kappa_{\text{ref}}^L$.

Second, for κ^L supported only off the identity, Theorem 5.9 shows that by comparison there is only a seven-parameter family of Drinfeld orbifold algebra maps and these are controlled by a projective variety which, according to a few representative calculations in Macaulay2 [14], appears to be four-dimensional. The resulting algebras also specialize to rational Cherednik algebras parametrized by α , β , and c when $\kappa_{\text{ref}}^L = \kappa_{3\text{-cyc}}^C \equiv 0$.

Theorem 7.2 (Drinfeld orbifold algebras for doubled permutation representation over $\mathbb{C}[t]$). *Let S_n ($n \geq 3$) act on $V = W^* \oplus W$ with basis $\mathcal{B} = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ by the doubled permutation representation. Suppose $a, a^\perp, b, b^\perp, c, \alpha, \beta \in \mathbb{C}$ satisfy conditions (5.3) and (5.4). Define $\kappa^L = \kappa_1^L$ and $\kappa^C = \kappa_1^C + \kappa_{\text{ref}}^C + \kappa_{3\text{-cyc}}^C$ to be the cochains such that for $1 \leq i \neq j \leq n$,*

$$\begin{aligned} \kappa^L(x_i, x_j) &= \kappa^L(y_i, y_j) = 0, \\ \kappa^L(x_i, y_i) &= \sum_{k \neq i} ((a - a^\perp)x_{i,k} + a^\perp x_{[n]} + (b - b^\perp)y_{i,k} + b^\perp y_{[n]}) \otimes (ik), \\ \kappa^L(x_i, y_j) &= -((a - a^\perp)x_{i,j} + a^\perp x_{[n]} + (b - b^\perp)y_{i,j} + b^\perp y_{[n]}) \otimes (ij) \end{aligned}$$

and

$$\begin{aligned} \kappa^C(x_i, y_j) &= \beta - c(ij) - (a - a^\perp)(b - b^\perp) \sum_{k \neq i, j} (ijk) - (kji), \\ \kappa^C(x_i, x_j) &= (b - b^\perp)^2 \sum_{k \neq i, j} (ijk) - (kji), \\ \kappa^C(y_i, y_j) &= (a - a^\perp)^2 \sum_{k \neq i, j} (ijk) - (kji), \\ \kappa^C(x_i, y_i) &= \alpha + c \sum_{k \neq i} (ik). \end{aligned}$$

Then the quotient $\mathcal{H}_{\kappa, t}$ of $T(V) \# S_n[t]$ by the ideal generated by

$$\{uv - vu - \kappa^L(u, v)t - \kappa^C(u, v)t^2 \mid u, v \in \mathcal{B}\}$$

is a Drinfeld orbifold algebra over $\mathbb{C}[t]$. Further, the algebras $\mathcal{H}_{\kappa, 1}$ are precisely the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq V^g$ for each $g \in S_n$ and κ^L is supported only off the identity.

An analogous statement may be made for algebras constructed from the family of lifts of $\kappa_1^L + \kappa_{\text{ref}}^L$ described in Theorem 5.10 but is omitted here.

Table 3. Classification of Drinfeld orbifold algebra maps for S_n acting on $\mathfrak{h}^* \oplus \mathfrak{h}$.

Linear part κ^L	Constant part κ^C	Parameter relations	Reference
κ_{ref}^L	$\kappa_{3\text{-cyc}}^C$	$2a + (n-2)a^\perp = 0$ $2b + (n-2)b^\perp = 0$	Theorem 6.4
	$\kappa_{3\text{-cyc}}^C + \kappa_{\text{ref}}^C$	$2a + (n-2)a^\perp = 0$ $2b + (n-2)b^\perp = 0$	
0	κ_{ref}^C	none	Remark 6.5
	κ_1^C	$\alpha + (n-1)\beta = 0$	
	$\kappa_1^C + \kappa_{\text{ref}}^C$	$\alpha + (n-1)\beta = 0$	

When the parameter relations hold, the map $\kappa = \kappa^L + \kappa^C$ is a Drinfeld orbifold algebra map.

7.2 Algebras for the doubled standard representation

By Theorem 6.3 the only Lie orbifold algebras for S_n acting on $\mathfrak{h}^* \oplus \mathfrak{h}$ by the doubled standard representation are the known rational Cherednik algebras $H_{n,\beta,c}$. However, by Theorem 6.4 there is in this case a three-parameter family of Drinfeld orbifold algebras which are not graded Hecke algebras, but which specialize when $a^\perp = b^\perp = 0$ to the rational Cherednik algebras $H_{0,c}$ for S_n .

Theorem 7.3 (Drinfeld orbifold algebras for doubled standard representation over $\mathbb{C}[t]$). *Let S_n ($n \geq 3$) act on $\mathfrak{h}^* \oplus \mathfrak{h}$ by the doubled standard representation. For $a^\perp, b^\perp, c \in \mathbb{C}$ and \bar{x}_i and \bar{y}_i as in (3.1) define κ^L and κ^C as in Theorem 6.4. Then the quotient $\mathcal{H}_{\kappa,t}$ of $T(\mathfrak{h}^* \oplus \mathfrak{h})\#S_n[t]$ by the ideal generated by*

$$\{\bar{u}\bar{v} - \bar{v}\bar{u} - \kappa^L(\bar{u}, \bar{v})t - \kappa^C(\bar{u}, \bar{v})t^2 \mid \bar{u}, \bar{v} \in \bar{\mathcal{B}}\}$$

is a Drinfeld orbifold algebra of $S(\mathfrak{h}^ \oplus \mathfrak{h})\#S_n$ over $\mathbb{C}[t]$. Further, the algebras $\mathcal{H}_{\kappa,1}$ are precisely the Drinfeld orbifold algebras such that $\text{im } \kappa_g^L \subseteq (\mathfrak{h}^* \oplus \mathfrak{h})^g$ for each $g \in S_n$ and κ^L is supported only off the identity. Specializing $a^\perp = b^\perp = 0$ yields the rational Cherednik algebra $H_{0,c}$.*

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