Twistors, Self-Duality, and Spin^c Structures

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Abstract. The fact that every compact oriented 4-manifold admits spin^c structures was proved long ago by Hirzebruch and Hopf. However, the usual proof is neither direct nor transparent. This article gives a new proof using twistor spaces that is simpler and more geometric. After using these ideas to clarify various aspects of four-dimensional geometry, we then explain how related ideas can be used to understand both spin and spin^c structures in any dimension.

Key words: 4-manifold; spin^c structure; twistor space; self-dual 2-form

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1   Twistor spaces and spin^c structures

Every compact oriented 4-manifold admits spin^c structures. The standard proof of this fact is due to Hirzebruch and Hopf [3], although the result had previously been hinted at by Whitney [9]. For readable modernized English-language versions of the Hirzebruch–Hopf proof, see Killingback and Rees [5] or Gompf and Stipsicz [2, Section 5.7].

However, the Hirzebruch–Hopf proof is so indirect that it does not really involve the notion of a spin^c structure at all, and it proceeds by so completely isolating the topological issues from the geometric motivation as to make it seem rather formal and unenlightening. The main purpose of this article is to give a self-contained proof of this important fact that is based on ideas from twistor theory. In the process, we will also see how this result is inextricably related to other fundamental aspects of 4-dimensional geometry. The article then concludes by putting this 4-dimensional story in the context of a twistor approach to spin and spin^c structures in other dimensions.

Let us begin by recalling that dimension four is profoundly exceptional for both differential topology and differential geometry. This idiosyncrasy is largely attributable to a fluke of Lie-group theory: the rotation group $\mathrm{SO}(4)$ is not a simple Lie group. Instead, its Lie algebra splits as a direct sum

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3),$$

as a consequence of the fact that left- and right-multiplication by the unit quaternions $\mathrm{Sp}(1)$ belong to different subgroups of the rotation group. On an oriented Riemannian 4-manifold (M^4, g), this gives rise to an invariant direct-sum decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

of the bundle of 2-forms, because the action of $\mathrm{SO}(4)$ on 2-forms is isomorphic, via index raising, to its adjoint representation on the Lie algebra $\mathfrak{so}(4)$ of skew 4×4 matrices. This decomposition

in fact coincides with the decomposition of the 2-forms into the $(\pm)$-eigenspaces of the Hodge star operator

$$\star \colon \Lambda^2 \to \Lambda^2.$$

We will now emphasize our choice of an orientation by focusing on the bundle Λ^+ of *self-dual 2-forms* φ, which are characterized by the condition $\star\varphi = \varphi$.

While the rank-3 oriented vector bundle $\Lambda^+ \to M$ depends on the conformal class $[g]$ of the Riemannian metric g, the bundles Λ_g^+ and $\Lambda_{g'}^+$ associated with two different metrics are nonetheless canonically bundle-isomorphic via the natural identification $\Lambda_g^+ = \Lambda^2/\Lambda_g^-$, because we always have $\Lambda_{g'}^+ \cap \Lambda_g^- = 0$. This algorithm for producing an isomorphism suffers from some defects, though. First of all, interchanging g and g' does not produce the inverse isomorphism. Second, the isomorphism produced by this algorithm does not preserve the relevant inner products. Fortunately, however, the latter can be corrected by applying a unique positive, self-adjoint endomorphism to Λ^+, and this then allows us to identify the oriented bundles-with-inner-product Λ^+ for two different metrics in a manner that is unique up to isotopy. This will suffice to give a metric-independent meaning to the notions that are the main focus of our discussion.

We now fix a Riemannian metric g on our oriented 4-manifold M, and notice that, since $\mathrm{SO}(4)/\mathbb{Z}_2 \cong \mathrm{SO}(3) \times \mathrm{SO}(3)$, the 4-dimensional rotation group acts transitively on the unit sphere in Λ^+. For this reason, any $\omega \in \Lambda_x^+$, $x \in M$, with $|\omega| = \sqrt{2}$ can be expressed as

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4$$

in some oriented orthonormal basis for T_xM, and hence corresponds, via index raising, to the endomorphism $\jmath \colon T_xM \to T_xM$ represented by the matrix

$$\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}.$$

In other words, any such ω defines an almost-complex structure $\jmath$ at x that is compatible with the metric g and determines the given orientation. Conversely, if $\jmath \colon T_xM \to T_xM$ satisfies $\jmath^2 = -I$ and $\jmath^*g = g$, and also determines the given orientation of M, then $\jmath$ arises, via index raising, from a unique $\omega \in \Lambda_x^+$ with $|\omega| = \sqrt{2}$.

We now define the *twistor space* of our oriented Riemannian 4-manifold (M,g) to be the total space

$$Z := S_{\sqrt{2}}(\Lambda^+) = \left\{ \omega \in \Lambda^+ \,|\, |\omega| = \sqrt{2} \right\}$$

of the 2-sphere bundle $\wp \colon Z \to M$ associated with the oriented rank-3 vector bundle $\Lambda^+ \to M$ of self-dual 2-forms. We may then give Z an almost-complex structure $J \colon TZ \to TZ$, $J^2 = -I$, by the following construction, which is essentially due to Atiyah–Hitchin–Singer [1], and which provides a general Riemannian context for Penrose's non-linear graviton construction [7]. We begin by decomposing TZ into vertical and horizontal components,

$$TZ = \mathsf{V} \oplus \mathsf{H}, \tag{1.1}$$

where $\mathsf{V} := \ker \mathrm{d}\wp$ and H is induced by parallel transport in Λ^2 with respect to the Riemannian connection of g. Now notice that index raising gives us an alternative, conformally invariant description

$$Z = \left\{ \jmath \colon T_xM \to T_xM,\ x \in M \,|\, \jmath^2 = -I,\ \jmath^*g = g,\ \jmath > 0 \right\}$$

of the twistor space. Since the derivative $\mathrm{d}\wp\colon TZ \to TM$ of the bundle projection $\wp$ induces a tautological isomorphism $\mathsf{H} \cong \wp^*TM$, we may therefore define an endomorphism $J_{\mathsf{H}}\colon \mathsf{H} \to \mathsf{H}$ whose action at $\jmath \in \wp^{-1}(x)$ is the horizontal lift of $\jmath\colon T_xM \to T_xM$. Meanwhile, since each fiber $\wp^{-1}(x)$ of $\wp\colon Z \to M$ is a round 2-sphere in an oriented 3-dimensional inner-product space Λ_x^+, we can therefore declare $J_{\mathsf{V}}\colon \mathsf{V} \to \mathsf{V}$ to be $+90°$ rotation in the tangent space of each fiber 2-sphere with respect to the outward-pointing orientation. Since these recipes guarantee that $J_{\mathsf{H}}^2 = -I_{\mathsf{H}}$ and $J_{\mathsf{V}}^2 = -I_{\mathsf{V}}$, setting

$$J := J_{\mathsf{H}} \oplus J_{\mathsf{V}}$$

now produces an almost-complex structure J on Z, as illustrated by Figure 1.

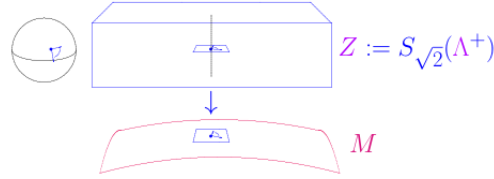**Figure 1.** The twistor space Z of an oriented Riemannian 4-manifold (M, g) is the sphere bundle of the oriented rank-3 bundle $\Lambda^+ \to M$ of self-dual 2-forms. This 6-manifold can be given a canonical almost-complex structure J that is compatible with the decomposition of TZ into vertical and horizontal subspaces.

The almost-complex structure J is actually conformally invariant, even though the decomposition (1.1) is not; however, J is only integrable [1, 7] if the Weyl curvature of (M, g) is anti-self-dual. Nonetheless, some useful remnants of integrability persist, even in the general case; in particular, each fiber $\wp^{-1}(x) \cong \mathbb{CP}_1$ is a J-holomorphic curve, and the normal bundle $\nu = T^{1,0}Z/\mathsf{V}^{1,0}$ of each fiber is a holomorphic bundle $\nu \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ over this $\mathbb{CP}_1$.

Since this emphasizes the fact that $\wp\colon Z \to M$ may be thought of as a $\mathbb{CP}_1$-bundle, it seems natural to ask whether this $\mathbb{CP}_1$-bundle can always be expressed as the projectivization $\mathbb{P}(\mathbb{V}_+)$ of a rank-2 complex vector bundle $\mathbb{V}_+ \to M$. As we will see, the answer always turns out to be, "Yes!" This assertion exactly amounts to the fact that any oriented 4-manifold admits spin^c structures. The road that will bring us to this realization begins with the following definition:

Geometric Definition. A *spin^c structure* on a connected oriented Riemannian 4-manifold (M, g) is a complex line bundle $\mathscr{L} \to Z$ on the twistor space that has degree 1 on any S^2 fiber of $Z \to M$.

Here, two isomorphic complex line bundles on Z are considered to define the same spin^c structure. However, the first Chern class c_1 defines a bijection between equivalence classes of complex line bundles on Z and $H^2(Z, \mathbb{Z})$, in a manner that converts the tensor-product of line bundles into the addition of cohomology classes. This means that the above definition can be reformulated as saying that a spin^c structure on M is a cohomology class $\mathbf{a} \in H^2(Z, \mathbb{Z})$ with $\langle \mathbf{a}, F \rangle = 1$, where $F \in H_2(Z, \mathbb{Z})$ is the homology class of a fiber $\wp^{-1}(x) \approx S^2$.

The above should be compared and contrasted with the following:

Standard Definition. A *spin^c structure* on an oriented Riemannian 4-manifold (M, g) is a circle bundle $\widehat{\mathfrak{F}} \to \mathfrak{F}$ over the oriented orthonormal frame bundle that is also compatibly endowed with the structure of a principal Spin^c(4)-bundle, where

$$\mathrm{Spin}^c(4) := [\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{U}(1)]/\langle(-1, -1, -1)\rangle.$$

Fortunately, these competing definitions are exactly equivalent. For example, one can pass from the geometric definition to the standard definition by first expressing Z as $\mathfrak{F}/\mathrm{U}(2)$. Thus, if $\mathscr{L} \to Z$ is a complex line bundle of fiber degree 1, we can pull the circle bundle $S(\mathscr{L})$ back to obtain a circle bundle over $\mathfrak{F}$, and this pull-back can then be made into principal $\mathrm{Spin}^c(4)$-bundle over M using the fact that $H^1(\mathrm{SO}(4), \mathcal{E}^\times) = H^1(\mathrm{SO}(4), \mathbb{C}^\times) = H^2(\mathrm{SO}(4), \mathbb{Z}) = \mathbb{Z}_2$.

In the opposite direction, given a principal $\mathrm{Spin}^c(4)$-bundle over M that is also a circle bundle $\widehat{\mathfrak{F}} \to \mathfrak{F}$, we first construct a vector bundle $\mathbb{V}_+ \to M$ by applying the associated bundle construction to the representation $\mathrm{Spin}^c(4) \to \mathrm{U}(2) \cong [\mathrm{Sp}(1) \times \mathrm{U}(1)]/\langle(-1,-1)\rangle$ obtained by dropping the second $\mathrm{Sp}(1)$. The map $\widehat{\mathfrak{F}} \to \mathfrak{F}$ then allows us to identify Z with $\mathbb{P}(\mathbb{V}_+)$, and the $\mathcal{O}(1)$-bundle dual to the tautological line bundle of $\mathbb{P}(\mathbb{V}_+)$ then provides the fiber-degree-1 line bundle featured in the geometric definition.

While every oriented 4-manifold will turn out to admit spin^c structures, the situation is entirely different for *spin structures*, which are much more restrictive:

Geometric Definition. A *spin structure* on an oriented Riemannian 4-manifold (M, g) is a square-root $\mathscr{V}^{1/2}$ of the vertical complex line bundle $\mathscr{V} := \mathsf{V}^{1,0}$ of the twistor space $Z \to M$.

Here, a square-root $\mathscr{V}^{1/2}$ of $\mathscr{V}$ means a line-bundle $\mathscr{L} \to Z$ that is equipped with a specific isomorphism $\Phi \colon \mathscr{L} \otimes \mathscr{L} \to \mathscr{V}$. Two such square-roots $(\mathscr{L}, \Phi)$ and $(\mathscr{L}', \Phi')$ are considered to be the same if there is an isomorphism $\Psi \colon \mathscr{L} \to \mathscr{L}'$ of complex line bundles that induces a commutative diagram

$$
\begin{array}{ccc}
\mathscr{L} \otimes \mathscr{L} & \xrightarrow{\ \Phi\ } & \\
{\scriptstyle \Psi \otimes \Psi}\big\downarrow & & \mathscr{V} \\
\mathscr{L}' \otimes \mathscr{L}' & \xrightarrow{\ \Phi'\ } &
\end{array}
$$

Standard Definition. A *spin structure* on an oriented Riemannian 4-manifold (M, g) is a double cover $\widetilde{\mathfrak{F}} \to \mathfrak{F}$ of the principal $\mathrm{SO}(4)$-bundle of oriented orthonormal frames by a principal $\mathrm{Spin}(4)$-bundle, where

$$\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1).$$

Once again, these competing definitions are exactly equivalent. Indeed, the $\mathrm{SO}(3)$-bundle $\mathsf{F} = \mathfrak{F}/\mathrm{Sp}(1)$ may be viewed as the circle bundle $S(\mathscr{V})$ over $Z = \mathsf{F}/\mathrm{SO}(2)$, so a square-root of $\mathscr{V}$ amounts to a double-cover of F. On the other hand, the quotient map $\mathfrak{F} \to \mathsf{F}$ induces an isomorphism of fundamental groups, so double covers of F are in bijective correspondence with double covers of $\mathfrak{F}$.

2   Oriented rank-3 vector bundles

We now focus on the case where our smooth oriented 4-manifold is *compact*. The *Euler class* $\mathsf{e}(\Lambda^+)$ will then play the starring role in our proofs. However, our approach will be clarified by initially working in a more general context. Let us therefore first consider the Euler class $\mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ of any oriented rank-3 bundle $\mathsf{E} \to M$.

Now, since the rank of E is odd, multiplication by -1 induces an orientation-reversing self-isomorphism $\mathsf{E} \to \mathsf{E}$. Hence the Euler class $\mathsf{e} = \mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ satisfies $\mathsf{e} = -\mathsf{e}$, and so is a 2-torsion element. This Euler class is Poincaré dual to the homology class $\in H_1(M, \mathbb{Z})$ of the oriented curve cut out by the zero set of any section of $\mathsf{E} \to M$ that is transverse to the zero section $0_M \subset \mathsf{E}$. Of course, if E had a nowhere-zero section, its Euler class $\mathsf{e}(\mathsf{E})$ would

consequently vanish. However, since rank $\mathsf{E} < \dim M$, the converse is definitely false! This common mistake seems to arise from over-familiarity with the Poincaré–Hopf theorem, which concerns the case where the rank of the bundle equals the dimension of the base; in that context, generic zeroes are just isolated points, and generic zeroes of opposite sign can then be eliminated in pairs. In our case, the zero locus of a generic section is instead a union of oriented circles, and the vanishing of the Euler class just means that this curve bounds an oriented surface.

Example. Since $H^3\big(S^4, \mathbb{Z}\big) = 0$, the oriented rank-3 bundle $\Lambda^+ \to S^4$ certainly must have $\mathsf{e}(\Lambda^+) = 0$. However, non-zero global sections of this bundle certainly *do not exist*. Otherwise, the twistor projection $Z \to S^4$ would admit a smooth global section, and we could then interpret this as an orientation-compatible almost complex structure J on S^4. However, S^4 does not admit an almost-complex structure! For example, if it did, the index of the spin^c Dirac operator generalizing $\bar{\partial} + \bar{\partial}^*$ would be $\mathrm{Td}\left(S^4\right) = \frac{\chi+\tau}{4}\left(S^4\right) = \frac{1}{2} \notin \mathbb{Z}$; for further discussion, see the commentary following (4.1) below. This contradiction shows that every smooth section of $\Lambda^+ \to S^4$ must have non-empty zero locus, even through $\mathsf{e}(\Lambda^+) = 0$.

We will show in Section 3 that $\mathsf{e}(\Lambda^+) = 0$ on any smooth oriented 4-manifold M, even if $H^3(M, \mathbb{Z}) \neq 0$. In order to do this, however, it will help to first put the question into the broader context of rank-3 oriented vector bundles on compact oriented 4-manifolds. Here, our approach will crucially depend on the following technical result:

Proposition 2.1. *Let M be a smooth connected compact oriented 4-manifold, let $\mathsf{E} \to M$ be a real oriented rank-3 vector bundle, let $\varpi \colon \mathcal{Z} \to M$ be the unit 2-sphere bundle $\mathcal{Z} = S(\mathsf{E})$ with respect to some positive-definite inner product, and let $F \in H_2((\mathcal{Z}), \mathbb{Z})$ denote the homology class of an S^2-fiber of ϖ. Then the following conditions are all equivalent:*

(i) The Euler class $\mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ vanishes;

(ii) There is a cohomology class $\mathbf{a} \in H^2(\mathcal{Z}, \mathbb{Z})$ with $\langle \mathbf{a}, F \rangle = 1$;

(iii) $H_2(\mathcal{Z}, \mathbb{Z}) \cong H_2(M, \mathbb{Z}) \oplus \mathbb{Z}$; and

(iv) $|\mathfrak{T}_2(\mathcal{Z})| = |\mathfrak{T}_2(M)|$, where $\mathfrak{T}_k$ is the torsion subgroup of $H_k(\_\_, \mathbb{Z})$.

Proof. Let us first recall that the cohomology of $\mathcal{Z}$ is related to that of M by the Gysin exact sequence [6, Section 12]

$$\cdots H^{k-3}(M) \overset{\cup\mathsf{e}}{\to} H^k(M) \overset{\varpi^*}{\to} H^k(\mathcal{Z}) \overset{\varpi_*}{\to} H^{k-2}(M) \overset{\cup\mathsf{e}}{\to} H^{k+1}(M) \cdots, \tag{2.1}$$

where $\mathsf{e} = \mathsf{e}(\mathsf{E})$ is the Euler class of the oriented rank-3 bundle E. This is really just a disguised form of the long exact sequence

$$\cdots \to H^k(\mathsf{E}, \mathsf{E} - M) \to H^k(\mathsf{E}) \to H^k(\mathsf{E} - M) \to H^{k+1}(\mathsf{E}, \mathsf{E} - M) \to \cdots$$

of the pair $(\mathsf{E}, \mathsf{E} - M)$, because the zero section $M \hookrightarrow \mathsf{E}$ is a deformation retract of E, and $\mathcal{Z}$ is a deformation retract of $\mathsf{E} - M$; the Thom isomorphism $H^{k-3}(M) \to H^k(\mathsf{E}, \mathsf{E} - M)$, given by cupping with the Thom class, therefore converts one exact sequence into the other. While this works perfectly well with coefficients in any ring, we will actually only use it for $\mathbb{Z}$, $\mathbb{Z}_2$, and $\mathbb{R}$ coefficients in this article.

First observe that the exactness of the Gysin sequence

$$\cdots \to H^2(Z, \mathbb{Z}) \overset{\varpi_*}{\to} H^0(M, \mathbb{Z}) \overset{\mathsf{e};}{\to} H^3(M, \mathbb{Z}) \to \cdots$$

implies that (i) $\implies$ (ii), because the vanishing of $\mathsf{e} = \mathsf{e}(\mathsf{E})$ implies the surjectivity of the map $\varpi_* \colon H^2(\mathcal{Z}, \mathbb{Z}) \to H^0(M, \mathbb{Z}) = \mathbb{Z}$ given by $\mathbf{a} \mapsto \langle \mathbf{a}, F \rangle$.

Next, since $H^4(M, \mathbb{Z}) = \mathbb{Z}$ is free, while $\mathsf{e}(\mathsf{E})$ is a torsion class, observe that the terminal segment of the Gysin sequence (2.1) breaks off as the short exact sequence

$$0 \to H^4(M, \mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z}, \mathbb{Z}) \xrightarrow{\varpi_*} H^2(M, \mathbb{Z}) \to 0,$$

where the image of $H^4(M, \mathbb{Z})$ in $H^4(\mathcal{Z}, \mathbb{Z})$ is generated by the Poincaré dual of the fiber class F. Applying Poincaré duality in both $\mathcal{Z}$ and M therefore converts this into an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot F} H_2(\mathcal{Z}, \mathbb{Z}) \to H_2(M, \mathbb{Z}) \to 0. \tag{2.2}$$

If statement (ii) holds, then there is an $\mathbf{a} \in H^2(\mathcal{Z}, \mathbb{Z})$ with $\langle \mathbf{a}, F \rangle = 1$, and pairing with $\mathbf{a}$ then provides a left inverse of $\mathbb{Z} \hookrightarrow H_2(\mathcal{Z}, \mathbb{Z})$. Thus (ii) implies that the exact sequence (2.2) *splits*, and so yields an isomorphism

$$H_2(\mathcal{Z}, \mathbb{Z}) \cong \mathbb{Z} \oplus H_2(M, \mathbb{Z}). \tag{2.3}$$

This shows that (ii) $\Longrightarrow$ (iii).

Next, observe that whenever (2.3) holds, one also has

$$\mathfrak{T}_2(\mathcal{Z}) \cong \mathfrak{T}_2(M),$$

where $\mathfrak{T}_k$ denotes the torsion subgroup of the integer homology H_k. Since this in particular implies that these torsion subgroups have the same order, it follows that (iii) $\Longrightarrow$ (iv).

On the other hand, the universal coefficients theorem tells us that

$$\mathfrak{T}^3(\mathcal{Z}) \cong \mathfrak{T}_2(\mathcal{Z}) \qquad \text{and} \qquad \mathfrak{T}^3(M) \cong \mathfrak{T}_2(M), \tag{2.4}$$

where $\mathfrak{T}^k = \text{Ext}(\mathfrak{T}_{k-1}, \mathbb{Z}) \cong \mathfrak{T}_{k-1}$ denotes the torsion subgroup of the integer cohomology H^k. Thus, if $|\mathfrak{T}_2(\mathcal{Z})| = |\mathfrak{T}_2(M)|$, it then follows that $|\mathfrak{T}^3(\mathcal{Z})| = |\mathfrak{T}^3(M)|$. However, the central portion of the Gysin sequence reads

$$\cdots \to H^0(M, \mathbb{Z}) \xrightarrow{\mathsf{e};} H^3(M, \mathbb{Z}) \to H^3(\mathcal{Z}, \mathbb{Z}) \to H^1(M, \mathbb{Z}) \to \cdots,$$

where $\mathsf{e} = \mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ is a 2-torsion class. Since $H^1(M, \mathbb{Z})$ is torsion-free, this means, in particular, that

$$\mathfrak{T}^3(\mathcal{Z}) \cong \begin{cases} \mathfrak{T}^3(M) & \text{if } \mathsf{e}(\mathsf{E}) = 0, \\ \mathfrak{T}^3(M)/\mathbb{Z}_2 & \text{otherwise.} \end{cases} \tag{2.5}$$

Thus, $|\mathfrak{T}^3(\mathcal{Z})| = |\mathfrak{T}^3(M)|$ only when $\mathsf{e}(\mathsf{E}) = 0$. This proves that (iv) $\Longrightarrow$ (i). We have thus shown that

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(i)},$$

and our proof of the proposition is therefore complete.   ∎

3   Existence of spin^c structures

Proposition 2.1 and some twistor geometry now imply the following:

Theorem 3.1. *Any smooth compact oriented* 4*-manifold admits spin^c structures.*

Proof. For any Riemannian metric g on M, our geometric definition of a spin^c structure restates the claim as asserting the existence of a cohomology class $\mathbf{a} \in H^2(Z, \mathbb{Z})$ with $\langle \mathbf{a}, F \rangle = 1$, where $Z = S(\Lambda^+)$ is the twistor space, and $F = [S^2]$ is the fiber homology class. The equivalence (ii) $\Longleftrightarrow$ (iv) in Proposition 2.1 therefore says that it suffices to show that $|\mathfrak{T}_2(Z)| = |\mathfrak{T}_2(M)|$, where $\mathfrak{T}_2$ denotes the torsion subgroup of the integer homology H_2. Since (2.4) and (2.5) also guarantee that $|\mathfrak{T}_2(Z)| \leq |\mathfrak{T}_2(M)|$, it therefore suffices to show that the homomorphism $\wp_* \colon \mathfrak{T}_2(Z) \to \mathfrak{T}_2(M)$ is surjective.

To see this, we begin by recalling that any element of $H_2(M, \mathbb{Z})$ can be represented by a smooth compact connected embedded oriented surface $\Sigma^2 \subset M^4$. Indeed, any homology class $[\Sigma] \in H_2(M, \mathbb{Z})$ is Poincaré dual to an element of $H^2(M, \mathbb{Z})$, which can then be realized as the first Chern class of a complex line bundle on M. A generic smooth section of this line bundle then provides a smooth compact oriented embedded surface Σ_0 representing $[\Sigma]$. This representative might still not be connected, but we can then correct this by connecting the various components of Σ_0 by narrow tubes $I \times S^1$ that approximate a collection of disjoint arcs between the different connected components.

Now suppose that $\Sigma \subset M$ is a smooth compact connected oriented surface representing a torsion class $[\Sigma] \in \mathfrak{T}_2(M) \subset H_2(M, \mathbb{Z})$. Since the homological self-intersection $[\Sigma] \bullet [\Sigma]$ of a torsion class must vanish, it therefore follows that the normal bundle $\mathfrak{N}$ of $\Sigma \subset M$ has trivial Euler class, and is therefore trivial. We now define an orientation- and g-compatible almost-complex structure $\jmath$ on $TM|_\Sigma = \mathfrak{N} \oplus T\Sigma$ by declaring it to be given by $90°$ rotation in both $\mathfrak{N}$ and $T\Sigma$. Since this $\jmath$ defines a section of $Z|_\Sigma$, its image defines an embedded surface $\hat{\Sigma} \subset Z$ that projects diffeomorphically to Σ via the twistor projection. However, the twistor almost-complex structure J_{H} on the horizontal bundle $\mathsf{H} \subset TZ$ then restricts to $\hat{\Sigma}$ as an exact copy of the action of $\jmath$ on $TM|_\Sigma$. Since the normal bundle $\mathfrak{N}$ of Σ is trivial, we thus have

$$\left\langle c_1(\mathscr{H}), [\hat{\Sigma}] \right\rangle = \left\langle c_1(\mathbb{C} \oplus T^{1,0}\Sigma), [\Sigma] \right\rangle = \chi(\Sigma) = 2(1 - \mathfrak{g}),$$

where $\mathfrak{g}$ denotes the genus of Σ and where, once again, $\mathscr{H} = \mathsf{H}^{1,0}$.

Now consider the homology class $A = [\hat{\Sigma}] + (\mathfrak{g} - 1)F \in H_2(Z, \mathbb{Z})$, where F once again denotes the homology class of a fiber of the twistor projection $\wp \colon Z \to M$. Since $\wp$ induces an oriented diffeomorphism $\hat{\Sigma} \to \Sigma$, and since $\wp$ collapses F to a point, $\wp_* \colon H_2(Z, \mathbb{Z}) \to H_2(M, \mathbb{Z})$ therefore sends $A \mapsto [\Sigma]$. We will now prove that A is a torsion class. To show this, it suffices to check that $\langle \mathbf{a}, A \rangle = 0$ for every $\mathbf{a} \in H^2(Z, \mathbb{R})$. However, the $\mathbb{R}$-coefficient version of the Gysin sequence (2.1) implies that $H^2(Z, \mathbb{R}) = \wp^* H^2(M, \mathbb{R}) \oplus \mathbb{R} c_1(\mathscr{H})$, since $\langle c_1(\mathscr{H}), F \rangle = 2 \neq 0$. But pairing A with an element of $\wp^* H^2(M, \mathbb{R})$ amounts to pairing $\wp_*(A) = [\Sigma]$ with an element of $H^2(M, \mathbb{R})$, which yields zero because $[\Sigma]$ is a torsion class by hypothesis. But because the restriction of $\mathscr{H}$ to an S^2 fiber has degree $+2$, we also have

$$\langle c_1(\mathscr{H}), A \rangle = \left\langle c_1(\mathscr{H}), [\hat{\Sigma}] \right\rangle + (\mathfrak{g} - 1)\langle c_1(\mathscr{H}), F \rangle = 2(1 - \mathfrak{g}) + (\mathfrak{g} - 1)2 = 0.$$

This shows that A is a torsion class in $H_2(Z, \mathbb{Z})$ with $\wp_*(A) = [\Sigma]$, and therefore proves that $\wp_* \colon \mathfrak{T}_2(Z) \to \mathfrak{T}_2(M)$ is surjective, as claimed. $\blacksquare$

Applying Proposition 2.1 to the Gysin sequence, we thus have:

Corollary 3.2. *Any compact oriented Riemannian 4-manifold (M, g) satisfies $\mathsf{e}(\Lambda^+) = 0$. Consequently, with either $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients,*

$$0 \to H^k(M) \xrightarrow{\wp^*} H^k(Z) \xrightarrow{\wp_*} H^{k-2}(M) \to 0$$

is exact for every k, where $\wp \colon Z \to M$ is the twistor projection.

As an application, we therefore have the following:

Proposition 3.3. *One has $w_2(TM) = w_2(\Lambda^+)$ for every compact oriented Riemannian four-manifold (M, g). Moreover, $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ is the mod-2 reduction of some integer cohomology class $\mathbf{c} \in H^2(M, \mathbb{Z})$.*

Proof. We have $\wp^* w_2(TM) = w_2(\wp^* TM) = w_2(\mathscr{H}) = \rho[c_1(\mathscr{H})] = \rho[c_1(\wedge^2 \mathscr{H})] = w_2(\wedge^2 \mathscr{H})$ $= w_2(\mathbb{R} \oplus \wedge^2 \mathscr{H}^*) = w_2(\wp^* \Lambda^+) = \wp^* w_2(\Lambda^+)$. Since $\wp^* \colon H^2(M, \mathbb{Z}_2) \to H^2(Z, \mathbb{Z}_2)$ is injective by Corollary 3.2, this implies that $w_2(TM) = w_2(\Lambda^+)$.

Now consider the third integer Stiefel–Whitney class defined by $W_3 := \beta(w_2)$, where β is the Bockstein homomorphism of the long exact sequence

$$\cdots \to H^2(\_\_, \mathbb{Z}) \xrightarrow{2:} H^2(\_\_, \mathbb{Z}) \xrightarrow{\rho} H^2(\_\_, \mathbb{Z}_2) \xrightarrow{\beta} H^3(\_\_, \mathbb{Z}) \to \cdots \tag{3.1}$$

induced by

$$0 \to \mathbb{Z} \xrightarrow{2:} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}_2 \to 0,$$

with ρ denoting reduction mod 2. Since $w_2(\wp^* TM) = \rho[c_1(\mathscr{H})]$, it follows that $\wp^* W_3(TM) = W_3(\wp^* TM) = \beta[w_2(\wp^* TM)] = 0$. The injectivity of $\wp^* \colon H^3(M, \mathbb{Z}) \to H^3(Z, \mathbb{Z})$ guaranteed by Corollary 3.2 thus implies that $W_3(TM) = \beta[w_2(TM)]$ vanishes, and the exactness of (3.1) then tells us that $w_2(TM)$ belongs to the image of $\rho \colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$. ∎

It is now easy to deduce an analogous result for the simplest non-compact 4-manifolds:

Corollary 3.4. *Let M be the interior of a smooth compact oriented 4-manifold-with-boundary. Then M admits spin^c structures.*

Proof. If M is displayed as the interior of a compact oriented 4-manifold-with-boundary X, we first construct the double $N = X \cup_{\partial X} \overline{X}$ of X, where $\overline{X}$ denotes X equipped with the opposite orientation. Each component of N then admits a spin^c structure by Theorem 3.1, so taking a union over components gives us a complex line bundle $\mathscr{L} \to S(\Lambda^+)$, where $\Lambda^+ \to N$ is defined with respect to some Riemannian metric on N. Since M is an open subset of N, the twistor space of M is an open subset of the twistor space of N, and restricting $\mathscr{L} \to S(\Lambda^+)$ to this subset now gives us a spin^c structure on M. Of course, this construction is carried out with respect to a Riemannian metric on M that happens to arises by restriction from N, but the conclusion does not depend on a choice of metric, since the bundle-isomorphism class of $\Lambda^+ \to M$ is actually metric-independent. ∎

There seems to be a widespread consensus [2, 5] that this result should also hold for general non-compact 4-manifolds, including those that are not homotopy-equivalent to finite cell complexes. Unfortunately, however, a watertight proof of this assertion is currently lacking. For example, it does not suffice to exhaust M by precompact regions $M_1 \subset M_2 \subset \cdots \subset M_j \subset \cdots$ with smooth boundary, and then observe that $\mathsf{e}(\Lambda^+) = 0$ vanishes on each M_j by Corollary 3.4, because $H^*(M, \mathbb{Z}) \neq \varprojlim H^*(M_j, \mathbb{Z})$ in general; cf. [6, p. 109].

4   Spin and spin^c geometry

While every compact oriented 4-manifold admits spin^c structures, such structures are typically far from unique:

Theorem 4.1. *On any smooth compact oriented 4-manifold M, the cohomology group $H^2(M, \mathbb{Z})$ acts freely and transitively on the set of spin^c structures on M.*

Proof. Assuming, for simplicity, that M is connected, Corollary 3.2 tells us that there is an exact sequence

$$0 \to H^2(M, \mathbb{Z}) \xrightarrow{\wp^*} H^2(Z, \mathbb{Z}) \xrightarrow{\wp_*} \mathbb{Z} \to 0,$$

while our geometric definition tells us that {spin^c structures on M} is exactly $\wp_*^{-1}(1) \subset H^2(Z, \mathbb{Z})$. This corresponds to tensoring $\mathscr{L} \to Z$ with pull-backs of line bundles on M. ■

By contrast, many 4-manifolds do not admit spin structures:

Theorem 4.2. *A smooth oriented 4-manifold M admits a spin structure iff $w_2(TM) = 0$. When this happens, $H^1(M, \mathbb{Z}_2)$ acts freely and transitively on the spin structures of M.*

Proof. By our geometric definition, a spin structure exists iff the vertical line bundle $\mathscr{V} \to Z$ has a square root. However, as pointed out by Hitchin [4], there is a canonical isomorphism $\mathscr{V} \cong \wedge^2 \mathscr{H}$, because these two bundles have tautological identifications with the very same line sub-bundle of $\mathbb{C} \otimes \wp^*(\Lambda^+)$. Thus, M admits a spin structure iff $c_1(\mathscr{H}) = c_1(\wedge^2 \mathscr{H}) = c_1(\mathscr{V})$ is divisible by 2 in $H^2(Z, \mathbb{Z})$. But because

$$\cdots \to H^2(Z, \mathbb{Z}) \xrightarrow{2\cdot} H^2(Z, \mathbb{Z}) \xrightarrow{\rho} H^2(Z, \mathbb{Z}_2) \to \cdots$$

is exact, this happens iff $\rho[c_1(\mathscr{H})] = w_2(\mathscr{H}) = \wp^* w_2(TM)$ vanishes. Since $\wp^* \colon H^2(M, \mathbb{Z}_2) \to H^2(Z, \mathbb{Z}_2)$ is injective by Corollary 3.2, it therefore follows that M admits a spin structure iff $w_2(TM) = 0$.

When $w_2(TM) = 0$, the spin structures are exactly those double covers of the principal $\mathbb{C}^\times$-bundle $\mathscr{V}^\times = \mathscr{V} - 0_Z$ that also doubly cover the fiber; equivalently, they correspond to elements of $H^1(S(\mathscr{V}), \mathbb{Z}_2)$ that are non-zero on the fiber circle. Since $w_2(TM) = 0$ implies that $w_2(\mathscr{V}) = 0$, the Gysin sequence of $\mathscr{V} \to Z$ then simplifies to yield

$$0 \to H^1(Z, \mathbb{Z}_2) \to H^1(S(\mathscr{V}), \mathbb{Z}_2) \to H^0(Z, \mathbb{Z}_2) \to 0$$

and it follows that $H^1(Z, \mathbb{Z}_2)$ acts freely and transitively on such elements of $H^1(S(\mathscr{V}), \mathbb{Z}_2)$. Since we also have $H^1(M, \mathbb{Z}_2) = H^1(Z, \mathbb{Z}_2)$ by Corollary 3.2, $H^1(M, \mathbb{Z}_2)$ therefore acts freely and transitively on spin structures by tensoring $\mathscr{V}^{1/2}$ with pull-backs of real line bundles. ■

We now give a direct twistorial construction of the spinor bundles of a spin structure, and twisted-spinor bundles of a spin^c structure. This provides yet another way of seeing that our geometric definitions of such structures are equivalent to the standard definitions.

Given a square-root $\mathscr{V}^{1/2}$ of the vertical line bundle $\mathscr{V} \to Z$ on the twistor space, we begin by noticing that $\mathscr{V}$ is canonically isomorphic to the tangent bundle $T^{1,0} \mathbb{CP}_1(x)$ on any twistor fiber $\mathbb{CP}_1(x) := \wp^{-1}(x)$, x. This gives $\mathscr{V}^{1/2}$ a natural fiber-wise holomorphic structure, and we may therefore define two 2-dimensional vector spaces at each $x \in M$ by

$$\mathbb{S}_{+x} = H^0\big(\mathbb{CP}_1(x), \mathcal{O}(\mathscr{V}^{1/2})\big), \qquad \mathbb{S}_{-x} = H^0\big(\mathbb{CP}_1(x), \mathcal{O}(\nu \otimes \mathscr{V}^{-1/2})\big),$$

where the normal bundle $\nu = \mathscr{H}|_{\mathbb{CP}_1(x)}$ of $\mathbb{CP}_1(x) \subset Z$ is thought of as a holomorphic bundle $\cong \mathcal{O}(1) \oplus \mathcal{O}(1)$. These naturally define smooth vector bundles $\mathbb{S}_\pm$, because we may define the smooth sections of $\mathbb{S}_+ \to M$ (respectively, $\mathbb{S}_- \to M$) to be the smooth sections of $\mathscr{V}^{1/2} \to Z$ (respectively, $\nu \otimes \mathscr{V}^{-1/2} \to Z$) that are holomorphic on each fiber of $\wp$. These bundles naturally reduce to the structure group $\mathrm{SU}(2) = \mathrm{Sp}(1)$, and the principal $\mathrm{Spin}(4)$-bundle $\widetilde{\mathfrak{F}} \to M$ then arises as a bundle of adapted frames for $\mathbb{S}_+ \oplus \mathbb{S}_-$.

If we instead start with a degree-1 complex line-bundle $\mathscr{L} \to Z$, the Gysin sequence allows us to choose an isomorphism $\mathscr{L}^2 = \mathscr{V} \otimes \wp^* \mathsf{L}$ for a unique complex line-bundle $\mathsf{L} \to M$. Since $\wp^* \mathsf{L}$

has a natural flat connection on each twistor fiber $\mathbb{CP}_1(x)$, this gives $\mathscr{L} = (\mathscr{V} \otimes \wp^* \mathsf{L})^{1/2}$ a natural fiber-wise holomorphic structure. We thus obtain a pair of vector bundles $\mathbb{V}_\pm \to M$ by setting

$$\mathbb{V}_{+x} = H^0\big(\mathbb{CP}_1(x), \mathcal{O}(\mathscr{L})\big), \qquad \mathbb{V}_{-x} = H^0\big(\mathbb{CP}_1(x), \mathcal{O}(\nu \otimes \mathscr{L} \otimes \mathscr{V}^*)\big).$$

On any spin subset of M, these bundles can then be re-expressed as $\mathbb{V}_\pm = \mathbb{S}_\pm \otimes \mathsf{L}^{1/2}$. Moreover, $\wedge^2 \mathbb{V}_+ = \wedge^2 \mathbb{V}_- = \mathsf{L}$, and the principal Spin^c(4)-bundle $\widehat{\mathfrak{F}} \to M$ arises as a bundle of adapted frames for $\mathbb{V}_+ \oplus \mathbb{V}_-$.

In particular, any spin^c structure gives us a rank-2 complex vector bundle $\mathbb{V}_+ \to M$ such that $\mathbb{P}(\mathbb{V}_+) = Z = S(\Lambda^+)$. However, because $\text{rank}_{\mathbb{R}} \mathbb{V}_+ = 4 = \dim M$, the Poincaré–Hopf paradigm applies, and can be used to predict the existence of non-zero sections of $\mathbb{V}_+$. If M is compact, we can always choose a generic section of $\mathbb{V}_+$ that vanishes at only a finite number of points; by following a suitable self-isotopy of $\mathbb{V}_+ \to M$, we can then arrange for all of these zeroes to be contained in an arbitrarily small ball $B_\varepsilon(p) \subset M$, and then use a local trivialization over this ball to alter this section so that it only vanishes at the center p of the ball (albeit typically with high multiplicity). Applying the projection $\mathbb{V}_+ - 0_M \to \mathbb{P}(\mathbb{V}_+) = Z$, we thus obtain a section of $\wp\colon Z - \wp^{-1}(p) \to M - \{p\}$, and we may then interpret this section as an almost-complex structure J on $M - \{p\}$. Moreover, the image of this J is a closed codimension-2 submanifold that is Poincaré dual to $c_1(\mathscr{L})$ for the given spin^c structure, and so completely determines the spin^c structure on $M - \{p\}$. On the other hand, removing a point from M does not change $H^2(\mathbb{Z})$, and Corollary 3.2 therefore tells us that spin^c structures on M are completely determined by their restrictions to $M - \{p\}$. In summary, we have proved:

Theorem 4.3. *Let (M, g) be a compact connected oriented Riemannian 4-manifold, and let $p \in M$ be an arbitrary base-point. Then*

- *$M - \{p\}$ admits almost-complex structures J compatible with the given metric and orientation;*
- *any such J determines a spin^c structure on M; and*
- *every spin^c structure on M arises this way.*

Similarly, on the interior of any compact oriented 4-manifold-with-nonempty-boundary, there always exist almost-complex structures compatible with the given orientation. Every such almost-complex structure moreover determines a spin^c structure, and every spin^c structure arises in this way.

However, in the compact case, the count of zeroes with multiplicity for a section of $\mathbb{V}_+$ is given by the Euler number

$$\int_M c_2(\mathbb{V}_+) = \frac{c_1^2(\mathsf{L}) - (2\chi + 3\tau)(M)}{4}, \tag{4.1}$$

where $\chi(M)$ and $\tau(M)$ respectively denote Euler characteristic and signature of M. One can therefore find a global almost-complex structure on M if and only if $c_1(\mathsf{L}) \in H^2(M, \mathbb{Z})$ can be chosen to make the right-hand side of (4.1) vanish. This therefore happens [3] if and only if there is some $\mathbf{c} \in H^2(M, \mathbb{Z})$ with $\rho(\mathbf{c}) = w_2(TM)$ and $\mathbf{c}^2 = (2\chi + 3\tau)(M)$.

5   Other dimensions

Our discussion has shown that a twistorial perspective can shed new light on spin^c geometry in dimension four. We now conclude by pointing out some partial generalizations of these ideas to other dimension.

If (M, g) is any oriented Riemannian $2m$-manifold, its *twistor space* Z is the fiber bundle $Z = \mathfrak{F}/\mathrm{U}(m)$, where $\mathfrak{F} \to M$ is the principal $\mathrm{SO}(2m)$-bundle of oriented orthonormal frames for TM. Each fiber F of Z is thus a copy of the homogeneous space $\mathrm{SO}(2m)/\mathrm{U}(m)$ of real dimension $m(m-1)$. However, F is actually a compact *Hermitian symmetric space*, and so is naturally a compact complex Fano manifold of complex dimension $\mathsf{d} = \binom{m}{2}$. The twistor fiber F_x over $x \in M$ thus parameterizes the complex structures on $T_x M \cong \mathbb{R}^{2m}$ that are compatible with the given metric and orientation; equivalently, F is just the space of α-planes [8, Appendix] in $\mathbb{C} \otimes T_x M \cong \mathbb{C}^{2m}$. Consequently, the twistor space Z again admits a tautological almost-complex structure $J = J_{\mathsf{H}} \oplus J_{\mathsf{V}}$, allowing us to view the horizontal and vertical subspaces of TZ as complex vector bundles $\mathscr{H} = \mathsf{H}^{1,0}$ and $\mathscr{V} = \mathsf{V}^{1,0}$. Because there is a natural isomorphism $\mathscr{V} = \wedge^2 \mathscr{H}$, the vertical anti-canonical line-bundle $K_\wp^{-1} = \wedge^{\mathsf{d}} \mathscr{V}$ has a natural $(m-1)^{\mathrm{st}}$ root $K_\wp^{-1/(m-1)}$ given by $\wedge^m \mathscr{H}$. On the other hand, $c_1\big(K_\wp^{-1/(m-1)}\big)$ restricts to any fiber as twice the generator of $H^2(F, \mathbb{Z}) = \mathbb{Z}$. A spin structure on M is just a square root $K_\wp^{-1/2(m-1)}$ of $K_\wp^{-1/(m-1)}$, while a spin^c structure on M is simply a line bundle on Z whose Chern class restricts to a fiber F as the generator of $H^2(F, \mathbb{Z}) = \mathbb{Z}$. The corresponding spinor and twisted-spinor vector bundles can then be manufactured by a straightforward generalization of the constructions described in Section 4.

The odd-dimensional case is similar. If (M, g) is an oriented Riemannian $(2m-1)$-manifold, with oriented orthonormal frame bundle $\mathfrak{F} \to M$, its twistor space is defined to be $\mathfrak{F}/\mathrm{U}(m-1)$, and its fibers $F = \mathrm{SO}(2m-1)/\mathrm{U}(m-1) \cong \mathrm{SO}(2m)/\mathrm{U}(m)$ are thus identical to the fibers discussed above. The vertical anti-canonical line bundle still has a natural $(m-1)^{\mathrm{st}}$ root $K_\wp^{-1/(m-1)}$, and a spin structure on M is again just a square-root $K_\wp^{-1/2(m-1)}$ of this line-bundle. A spin^c structure is once again just a line-bundle on Z of fiber-degree 1.

However, our Gysin-sequence approach to the 4-dimensional case does not generalize to higher dimensions. This makes the use of the Leray–Serre spectral sequence absolutely essential for a full understanding of the topological issues that arise in this broader context.

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