

Twistor Theory of Dancing Paths

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Abstract. Given a path geometry on a surface \mathcal{U} , we construct a causal structure on a four-manifold which is the configuration space of non-incident pairs (point, path) on \mathcal{U} . This causal structure corresponds to a conformal structure if and only if \mathcal{U} is a real projective plane, and the paths are lines. We give the example of the causal structure given by a symmetric sextic, which corresponds on an $SL(2, \mathbb{R})$ -invariant projective structure where the paths are ellipses of area π centred at the origin. We shall also discuss a causal structure on a seven-dimensional manifold corresponding to non-incident pairs (point, conic) on a projective plane.

Key words: path geometry; twistor theory; causal structures

2020 Mathematics Subject Classification: 32L25; 53A20

*Dedicated to Roger Penrose
on the occasion of his 90th birthday*

1 Introduction

It is a great honor and pleasure to be able to contribute to the celebrations of Roger's 90th birthday. Roger is now twice as old as his nonlinear graviton construction [17]. When I arrived in Oxford as a graduate student, it was already known that the integrability of the nonlinear solitonic systems has its roots in the Ward transform for the anti-self-dual Yang–Mills equations [16, 20]. I spent three years of my PhD showing, under Lionel Mason's supervision, that the curved space twistor theory behind the nonlinear graviton construction underlies the integrability of the complementary class of the, so called, dispersionless integrable systems [8]. Roger made several comments to me about my work, but on average it took me three months to understand his geometric insight, and another three to realise that he was right. At that time Roger had already moved to the notoriously difficult googly problem, as well as the interplay between gravity and quantum mechanics. The photograph on Figure 1 was taken at the GRG meeting in Pune, India in 1997, where Roger gave a public lecture about the latter. The lecture proved to be so popular among the citizens of Pune that the lecture hall had to be sealed off by riot police, to prevent overcrowding.

The nonlinear graviton construction describes a link between algebraic geometry of holomorphic curves in a complex three-fold (the twistor space), and differential geometry on the moduli space M of these curves. Two curves in three dimensions generically do not intersect (this is backed by a geometric intuition based on curves in \mathbb{R}^3 which carries over to the twistor space, as the curves are holomorphic). The intersection condition defines a notion of null separation:

This paper is a contribution to the Special Issue on Twistors from Geometry to Physics in honor of Roger Penrose. The full collection is available at <https://www.emis.de/journals/SIGMA/Penrose.html>



Figure 1. Pune, December 1997. From the left: Devendra Kapadia, George Sparling, Maciej Dunajski, Roger Penrose, Jerzy Lewandowski.

two points on M are null separated iff the corresponding curves in the twistor space intersect. All curves intersecting a fixed curve C form a hyper-surface in M , consisting of points null separated from C . The assumptions made about the normal bundle of twistor curves in the nonlinear graviton construction ensure that this null separation condition leads to quadratic light cones on TM , and therefore defines a conformal structure. Dropping some of these assumptions leads to *causal structures*, where the associated null cones are no-longer quadratic, but instead of higher order [11, 12, 13, 15]. In this paper I give examples of such structures which result from path geometries on two-dimensional surfaces. These geometries, as well as their duals, generalise the Klein duality between \mathbb{RP}^2 and $(\mathbb{RP}^2)^*$.

2 Summary of the results

2.1 Path geometry

A path geometry \mathbf{u} on a two-dimensional surface \mathcal{U} is a family of unparametrised smooth curves, from now on called paths, one through each point of \mathcal{U} in each direction. If the paths of $(\mathcal{U}, \mathbf{u})$ are unparametrised geodesics of a torsion-free affine connection ∇ , then the path geometry is called *projective*. The corresponding projective structure is an equivalence class $[\nabla]$ of torsion-free affine connections sharing their unparametrised geodesics with ∇ . If at least one connection in $[\nabla]$ is the Levi-Civita connection of some (pseudo) Riemannian metric g , then the corresponding path geometry is called *metrisable*. The projective path geometries were invariantly characterised more than a century ago [6, 14], but the necessary and sufficient conditions for a path geometry to be metrisable were only given relatively recently [5].

A *dual path geometry* $(\mathcal{Z}, \mathfrak{z})$ is a surface \mathcal{Z} whose points correspond to the paths of \mathcal{U} . The paths of \mathfrak{z} correspond to points of \mathcal{U} . This gives rise to a double-fibration picture [4, 6]

$$\mathcal{U} \xleftarrow{\mu} \mathcal{I} \xrightarrow{\nu} \mathcal{Z}, \quad (2.1)$$

where $\mathcal{I} = \{(u \in \mathcal{U}, z \in \mathcal{Z}), u \in z\} \subset \mathcal{U} \times \mathcal{Z}$ is the three-dimensional incidence space. Both projections in (2.1) are submersions, and for all incident pairs $(u, z) \in \mathcal{I}$ the curves $\mu^{-1}(u)$ and $\nu^{-1}(z)$ meet transversally, and their tangents span a two-plane in $T_{(u,z)}\mathcal{I}$ which gives rise to a contact structure on \mathcal{I} .

2.2 Dancing pairs

Let $M = (\mathcal{U} \times \mathcal{Z}) \setminus \mathcal{I}$ be the four-dimensional manifold of non-incident pairs. A point in $m \in M$ consists of a point u in \mathcal{U} and a path z of \mathfrak{u} which does not contain u .

Definition 2.1. Two points (u_1, z_1) and (u_2, z_2) in M are dancing if there exists a path in $(\mathcal{U}, \mathfrak{u})$ which contains three points: u_1, u_2 , and the intersection $z_1 \cap z_2$.

The terminology is taken from [3], where the term *dancing* is introduced (and motivated) at an infinitesimal level and directly leads to the Definition 2.2. The Definition 2.1 makes sense both locally, and globally on M . In the latter case the paths z_1 and z_2 may intersect at more than one point (e.g., if \mathcal{U} is the round sphere, and the paths are the great circles). In this case we require that there exists a path containing u_1, u_2 and at least one point of intersection of z_1, z_2 . The dancing condition could of course equivalently have been defined in terms of the dual path geometry $(\mathcal{Z}, \mathfrak{z})$.

The simplest example is provided by the classical projective duality, where $\mathcal{U} = \mathbb{RP}^2$ and $\mathcal{Z} = (\mathbb{RP}^2)^*$. Both path geometries \mathfrak{u} and \mathfrak{z} are projective, and metrisable (in six different ways) by metrics of constant curvature. In this case the dancing condition gives rise to a neutral signature conformal structure on $M = \mathrm{SL}(3, \mathbb{R})/\mathrm{GL}(2, \mathbb{R})$ which contains an anti-self-dual Einstein metric with non-zero Ricci scalar, and the isometry group $\mathrm{SL}(3, \mathbb{R})$ [3, 7, 9]. We shall review this example in Section 3. In general it is convenient to work with the infinitesimal dancing condition [3], where $u_1 = u, u_2 = u + \epsilon \dot{u} + \dots$ and $z_1 = z, z_2 = z + \epsilon \dot{z} + \dots$. The tangent vector $(\dot{u}, \dot{z}) \in T_{(u,z)}M$ describes the direction of a path through u , and the turning point of the path z (i.e., the point of intersection of z with a nearby path).

Definition 2.2. A tangent vector $(\dot{u}, \dot{z}) \in T_{(u,z)}M$ is called *null* if the path through u in the direction of \dot{u} contains the turning point of z .

In Section 4 we shall show how the infinitesimal dancing condition can put in the form

$$\mathcal{L}(u, z, \dot{u}, \dot{z}) = 0, \quad (2.2)$$

where $\mathcal{L}: TM \rightarrow \mathbb{R}$. In the projectively flat case where $\mathcal{U} = \mathbb{RP}^2$ and $M = \mathrm{SL}(3)/\mathrm{GL}(2)$ described above the function \mathcal{L} is homogeneous of degree two in (\dot{u}, \dot{z}) , and gives a null-geodesic Lagrangian for a conformal structure on M . One may wonder whether other examples of path geometries $(\mathcal{U}, \mathfrak{u})$ give rise to conformal structures. The answer to this is negative, and is provided by the following local (and therefore also global) rigidity theorem

Theorem 2.3. *If the dancing condition on a path geometry \mathcal{U} defines a conformal structure on $M = (\mathcal{U} \times \mathcal{Z}) \setminus \mathcal{I}$, then $\mathcal{U} = \mathbb{RP}^2$ with its flat projective structure, and the conformal structure is represented by the $\mathrm{SL}(3)$ -invariant anti-self-dual Einstein metric on $M = \mathrm{SL}(3)/\mathrm{GL}(2)$.*

We shall prove this theorem in Section 4 by twistor methods. In Section 5 we shall consider a path geometry arising from an $\mathrm{SL}(2, \mathbb{R})$ invariant projective structure. We shall prove

Theorem 2.4. *Let $\mathcal{U} = \mathbb{R}^2$ and let \mathcal{Z} be the set of ellipses in \mathcal{U} centred at the origin and of area π . The dancing condition on $M = (\mathcal{U} \times \mathcal{Z}) \setminus \mathcal{I}$ defines an $\mathrm{SL}(2, \mathbb{R})$ -invariant sextic (2.2) on TM which is quartic in \dot{u} and quadratic in \dot{z} .*

The path geometry $(\mathcal{U}, \mathbf{u})$ from Theorem 2.4 is projective and metrisable. The dual path geometry is not projective.

Finally in Section 6 we shall move away from path geometries, and instead consider a seven-dimensional manifold $N \subset \mathbb{P}^2 \times \mathbb{P}^5$ consisting of non-incident pairs (a, A) , where $a \in \mathbb{P}^2$ is a point, and $A \subset \mathbb{P}^2$ is an irreducible conic not containing a . We shall say that two pairs (a, A) and (b, B) of points-conics are dancing if there exists an irreducible conic containing (a, b) and the four points $A \cap B$ of intersections between A and B . We shall prove

Theorem 2.5. *Let A, B be 3 by 3 matrices representing non-singular conics in \mathbb{RP}^2 , and let a, b be vectors in $\mathbb{R}^3 \setminus \{0\}$ representing points in \mathbb{RP}^2 . Two pairs (a, A) and (b, B) are dancing iff*

$$(a^T A a)(b^T B b) - (a^T B a)(b^T A b) = 0.$$

The infinitesimal dancing condition gives rise to a conformal structure on TN of signature $(++--000)$ which is degenerate along the fibres of a projection $N \rightarrow (\mathbb{P}^2 \times (\mathbb{P}^2)^) \setminus \mathcal{I}$ sending the conic A to the polar line of (a, A) .*

3 Dancing on \mathbb{RP}^2

Let $(\mathcal{U} = \mathbb{RP}^2, \mathbf{u})$ be a path geometry corresponding to a flat projective structure on the projective plane. The dual path geometry $(\mathcal{Z} = (\mathbb{RP}^2)^*, \mathfrak{z})$ is also flat, and the double fibration (2.1) is just the classical Klein projective duality between lines in \mathbb{RP}^2 and points in $(\mathbb{RP}^2)^*$. In this section we shall represent a point $u \in \mathcal{U}$ by an equivalence class of vectors $P \in \mathbb{R}^3$, the components of a vector being the homogeneous coordinates. Similarly a point $z \in \mathcal{Z}$ will be represented by a vector L corresponding to a line in \mathbb{RP}^2 .

Let $M \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$ be set of non-incident pairs (P, L) , where $P \in \mathbb{P}^2$, and $L \subset \mathbb{P}^2$ is a line. Two pairs (P, L) and (\tilde{P}, \tilde{L}) are null-separated in the sense of the Definition 2.1 if there exists a line which contains the three points $(P, \tilde{P}, L \cap \tilde{L})$. This null condition defines a co-dimension one cone in TM : generically there is no line through three given points. The action of $\mathrm{SL}(3, \mathbb{R})$ given by $(P, L) \rightarrow (\mathbf{A}P, L\mathbf{A}^{-1})$, where $\mathbf{A} \in \mathrm{SL}(3, \mathbb{R})$ is transitive on M , and a group stabilising a non-incident pair (point, line) is $\mathrm{GL}(2, \mathbb{R})$ which sits in $\mathrm{SL}(3, \mathbb{R})$ as a lower-diagonal block. Therefore M can be identified with $\mathrm{SL}(3, \mathbb{R})/\mathrm{GL}(2, \mathbb{R})$.

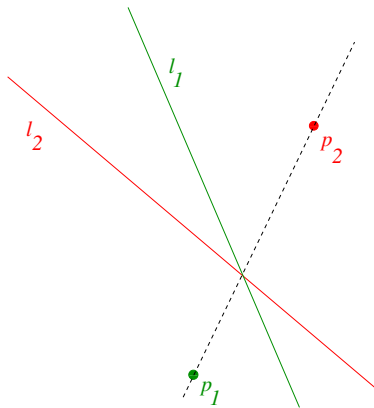


Figure 2. Dancing pair.

Proposition 3.1. *There exist affine coordinates $(x^0, x^1, \zeta_0, \zeta_1)$ on $M = \mathrm{SL}(3)/\mathrm{GL}(2)$ such that two points are dancing if and only if they are null separated with respect to the anti-self-dual Einstein metric*

$$g = d\zeta_0 dx^0 + d\zeta_1 dx^1 + (\zeta \cdot dx)^2, \quad \text{where } \zeta \cdot dx \equiv \zeta_0 dx^0 + \zeta_1 dx^1. \quad (3.1)$$

Proof. To find an analytic expression for the conformal structure resulting from the dancing condition consider two pairs (P, L) and (\tilde{P}, \tilde{L}) of non-incident points and lines. Let $L + t\tilde{L}$ be a pencil of lines. There exists t such that

$$P \cdot (L + t\tilde{L}) = 0, \quad \tilde{P} \cdot (L + t\tilde{L}) = 0. \quad (3.2)$$

Eliminating t from (3.2) gives

$$(P \cdot L)(\tilde{P} \cdot \tilde{L}) - (\tilde{P} \cdot L)(P \cdot \tilde{L}) = 0.$$

Setting $\tilde{P} = P + dP$, $\tilde{L} = L + dL$ yields a metric g representing the conformal structure

$$g = \frac{dP \cdot dL}{P \cdot L} - \frac{1}{(P \cdot L)^2} (L \cdot dP)(P \cdot dL).$$

We can use the normalisation $P \cdot L = 1$, so that $P \cdot dL = -L \cdot dP$, and

$$g = dP \cdot dL + (L \cdot dP)^2.$$

In [9] the affine coordinates $(x^0, x^1, \zeta_0, \zeta_1)$ were introduced on M such that

$$P = [x^0, x^1, 1], \quad L = [\zeta_0, \zeta_1, 1 - x^0 \zeta_0 - x^1 \zeta_1]$$

with a normalisation $P \cdot L = 1$. The resulting metric is (3.1). ■

In [9] the metric (3.1) was shown to be Einstein and anti-self-dual. Its isometry group is $\mathrm{SL}(3, \mathbb{R})$, and the underlying embedding of $\mathfrak{sl}(3)$ into $\mathfrak{sl}(4)$ was constructed in [10]. The metric was analysed (albeit in different coordinates) in the dancing context in [3], and in connection with path geometries in [7].

Let the flag manifold $\mathcal{I} \equiv F_{12}(\mathbb{C}^3) \in \mathbb{P}^2 \times \mathbb{P}^{2*}$ be the set of incident pairs (p, l) , such that $p \cdot l = 0$. This is the twistor space [2, 19] of (M, g) . A \mathbb{P}^1 of $(p, l) \in \mathcal{I}$ corresponding to a point (P, L) consists of all lines l through P , and all points $p = l \cap L$:

$$P \cdot l = 0, \quad p \cdot L = 0, \quad p \cdot l = 0.$$

Let (P, L) and (\tilde{P}, \tilde{L}) be null separated. The corresponding lines in F_{12} intersect at a point (p, l) given by $p = L \wedge \tilde{L}$, $l = P \wedge \tilde{P}$, where $[L \wedge \tilde{L}]^\gamma = \epsilon^{\alpha\beta\gamma} L_\alpha \tilde{L}_\beta$ etc. The incidence condition $p \cdot l = 0$ now gives the conformal structure (3.2). This is an illustration of Penrose's nonlinear graviton construction [17], adapted to the non-zero cosmological constant [19].

4 Dancing condition for general path geometries

The incidence relation encoding the double fibration (2.1) is given by a map $\psi: \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$, where \mathcal{I} is the inverse image $\psi^{-1}(0) \subset \mathcal{U} \times \mathcal{Z}$. Let $(u, z) \in M$, i.e., $\psi(u, z) \neq 0$, and let $(\dot{u}, \dot{z}) \in T_{(u,z)}M$. The infinitesimal dancing condition is equivalent to the existence of a pair (u^*, z^*) such that

$$\psi(u, z^*) = 0, \quad \psi(u^*, z) = 0, \quad \psi(u^*, z^*) = 0, \quad \dot{u} \frac{\partial \psi}{\partial u}(u, z^*) = 0, \quad \dot{z} \frac{\partial \psi}{\partial z}(u^*, z) = 0,$$

where z^* is the path in the direction \dot{u} containing u and the turning point u^* of the path z . Here the notation $\dot{u}\partial/\partial u$ denotes a differentiation along a vector field.

We now eliminate z^* between the 1st and the 4th condition, and eliminate u^* between the 2nd and the 5th condition. Substituting the resulting expressions to the 3rd condition gives one relation between (u, z, \dot{u}, \dot{z}) which is of the form (2.2).

We shall say that a two-dimensional surface $\sigma \subset M$ is totally null if all tangent vector fields in $T_m\sigma, m \in M$ are null in the sense of Definition 2.2.

Lemma 4.1. *There exists a three-parameter family of totally null surfaces (call them α -surfaces) in M which correspond to points in \mathcal{I} . For any $(l, p) \in \mathcal{I}$ the corresponding α -surface consists of $(L, P) \in M$ such that*

$$P \subset l, \quad p \subset L, \quad P \neq p, \quad L \neq l. \quad (4.1)$$

Proof. The points in \mathcal{I} clearly correspond to surfaces in M : there is a one-dimensional family of P s satisfying (4.1), and once P has been fixed, there is a one dimensional family of L s. To show that these surfaces are null note that given (L, P) and (\tilde{L}, \tilde{P}) on the α -surface (l, p) , the points $(P, \tilde{P}, L \cap \tilde{L})$ all belong to the line l in agreement with the Definition 2.1. ■

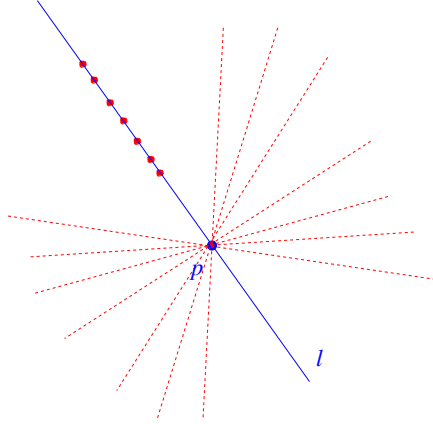


Figure 3. α -surface.

In addition to the α surfaces M also admits two two-parameter families of null surfaces (call them β -surfaces) defined as follows. The first family β_1 consist of a fixed point $P \in \mathcal{U}$ (hence two parameters), and all lines L in \mathcal{U} not containing P . The second family β_2 consist of a fixed line $L \subset \mathcal{U}$ and all points P not on L . If we have instead chosen to model M on the points and paths in $(\mathcal{Z}, \mathfrak{z})$, then the roles of β_1 and β_2 would swap, but the notion of α surfaces would stay the same. The result of Lemma 4.1 fits into the framework of *half-flat causal structures* [13, 15], where half-flatness is defined by the existence of a three-parameter family of surfaces whose tangent planes at each point give the ruling for the (not necessarily quadratic) null cone at that point. This terminology is motivated by the paradigm example of Penrose [17], where the cones are quadratic, and half-flatness condition is equivalent to the anti-self-duality of the Weyl tensor.

We are now ready to prove Theorem 2.3, and thus establish the uniqueness of the dancing metric.

Proof of Theorem 2.3. If there exists a conformal structure on M , then, by Lemma 4.1 it is necessarily anti-self-dual (with the right choice of the orientation). Indeed, there exists a three-parameter family of null surfaces. All null surfaces in four dimensions are self-dual (SD) or

anti-self-dual (ASD), and we choose an orientation of M such that the α -surfaces are SD. Thus the Weyl tensor of any metric g in the conformal class $[g]$ is ASD by Penrose's nonlinear graviton theorem [17] adapted to neutral signature.

Now we shall argue that the conformal structure $[g]$ defined by the α -surfaces of Lemma 4.1 must contain an Einstein metric. To show it, it is enough to demonstrate that the twistor space \mathcal{I} of incident pairs (α -surfaces) admits a contact structure. The existence of an Einstein metric will then be a consequence of the theorem of Ward [19]. The contact structure is intrinsically defined by the double-fibration picture (2.1). The space \mathcal{I} is equipped with two one-dimensional distributions defining the quotients to \mathcal{U} and \mathcal{Z} . The resulting two-dimensional distribution is non-integrable, and thus it defines a one-form θ on \mathcal{I} such that $\theta \wedge d\theta \neq 0$.

We have shown (after the proof of Lemma 4.1) that on top of the SD α -surfaces there exist the two, two-parameter family of β -surfaces. Let (\mathbb{S}, ϵ) , and (\mathbb{S}, ϵ') be two symplectic real vector bundles over M (see [1, 2]). We shall make use of the two isomorphisms: $TM = \mathbb{S} \otimes \mathbb{S}'$, and $\Lambda^2_- = \mathbb{S} \odot \mathbb{S}$, where Λ^2_- is the rank-3 vector bundle of ASD two-forms. Let the two families of β surfaces correspond to ASD two-forms Σ_1 and Σ_2 in $\Gamma(\Lambda^2_-)$, or equivalently to two sections $\iota, \tilde{\iota}$ of \mathbb{S} .

We can rescale ι so that the corresponding two-form is closed, and proportional to $dx^1 \wedge dx^2$ for some functions (x^1, x^2) on M which are constant on each β -surface in the first two parameter family. The Frobenius integrability conditions now imply the local existence of two more coordinates such that ζ_0 and ζ_1 on M such that $\text{Ker}(\Sigma_1) = \text{span}\{\partial/\partial\zeta_0, \partial/\partial\zeta_1\}$. The functions (ζ_0, ζ_1) are then the coordinates on the β -surface. The corresponding metric takes the form

$$g = d\zeta_A \odot dx^A + \Theta_{AB}(x, \zeta) dx^A \odot dx^B, \quad A, B = 0, 1 \quad (4.2)$$

for some symmetric two-by-two matrix Θ . The anti-self-duality condition on the Weyl tensor forces the components of Θ to be at most cubic in (ζ_0, ζ_1) , with additional algebraic relations between the components. Imposing the Einstein condition gives

$$\Theta_{AB} = \zeta_A \zeta_B + P_{(AB)} - \Gamma_{AB}^C \zeta_C, \quad (4.3)$$

where the coefficients Γ_{AB}^C only depend on (x^0, x^1) , and P_{AB} is the projective Schouten tensor of affine connection ∇ with connection components Γ_{AB}^C . The β -distribution defined by Σ_1 is parallel, and

$$\nabla^g \Sigma_1 = 6\mathcal{A} \otimes \Sigma_1,$$

where ∇^g is the Levi-Civita connection of g , and the one-form \mathcal{A} is a symplectic connection for the symplectic structure [9] given by

$$\Omega = d\mathcal{A} = d\zeta_A \wedge dx^A + P_{AB} dx^A \wedge dx^B. \quad (4.4)$$

Up to this point our computation has followed [9]. Now consider the second family of β surfaces given by the ASD two-form Σ_2 , which in our chosen coordinate system is given by

$$\Sigma_2 = \epsilon^{AB} (d\zeta_A + \Theta_{AC} dx^C) \wedge (d\zeta_B + \Theta_{BD} dx^D).$$

The normalisation condition $\epsilon(\iota, \tilde{\iota}) = 1$, and the condition that the distribution defined by Σ_2 is parallel together imply that this two-form satisfies

$$\nabla^g \Sigma_2 = -6\mathcal{A} \otimes \Sigma_2,$$

where \mathcal{A} is given by (4.4). We will need only the skew-symmetric part of this equation, which gives

$$0 = d\Sigma_2 + 6\mathcal{A} \wedge \Sigma_2 = \epsilon^{CD} \nabla_{[A} P_{B]C} dx^A \wedge dx^B \wedge d\zeta_D,$$

where the second equality follows from a rather complicated computation which we have performed on MAPLE. Therefore this second family of β -surfaces exists iff $\nabla_{[A}P_{B]C} = 0$, which (for two-dimensional projective structure) is precisely the condition (see, e.g., [5]) that the connection ∇ is projectively equivalent to a flat connection, where we can choose $\Gamma_{AB}^C = 0$ and $P_{AB} = 0$. It was shown in [9] that the metric (4.2) with Θ_{AB} given by (4.3) is invariant under projective changes of connection complemented by an affine translation of ξ_A . Therefore the resulting metric is of the form (3.1), which completes the proof. ■

5 Dancing ellipses

5.1 A dual pair of $\mathrm{SL}(2)$ -invariant path geometries

Let \mathcal{Z} be the set of ellipses in \mathbb{R}^2 , centered at the origin and of area π . Each such ellipse is given uniquely by an equation of the form

$$Ex^2 + 2Fxy + Gy^2 = 1, \quad (5.1)$$

where $(E, F, G) \in \mathbb{R}^3$ satisfy

$$EG - F^2 = 1, \quad E > 0. \quad (5.2)$$

Thus \mathcal{Z} can be identified with a two-dimensional surface, one sheet of the two-sheeted hyperboloid in \mathbb{R}^3 given by equation (5.2).

For each $z = (E, F, G) \in \mathcal{Z}$, denote by $\widehat{z} \subset \mathcal{U} := \mathbb{R}^2 \setminus \{(0, 0)\}$ the corresponding ellipse (the solution space of (5.1) with the given E, F, G). For example, for $z = (1, 0, 1)$, \widehat{z} is the unit circle centered at the origin.

For each $u = (x, y) \in \mathcal{U}$ let $\widehat{u} \subset \mathcal{Z}$ be the set of ellipses that pass through u . Using our hyperboloid model for \mathcal{Z} , we can see from equation (5.1) that \widehat{u} is the intersection of some affine plane in \mathbb{R}^3 with \mathcal{Z} . For example, for $u = (1, 0)$, \widehat{u} is the parabola $G = F^2 + 1$ in the plane $E = 1$. In fact, the space \mathcal{Z} is the hyperboloid model of the hyperbolic plane and the curves $\widehat{u} \subset \mathcal{Z}$, $u \in \mathcal{U}$, are the *horocycles* of \mathcal{Z} (circles tangent to the real axis, in the Poincaré upper half plane model). Note that the map $u \mapsto \widehat{u}$ is 2 : 1. Namely, u and $-u$ define the same curve in \mathcal{Z} .

In this section we shall find it convenient to use the following form of the infinitesimal dancing condition (compare Definition 2.2)

Definition 5.1. Let $M \subset \mathcal{U} \times \mathcal{Z}$ be the subset of *non-incident* pairs (u, z) . A tangent vector $(\dot{u}, \dot{z}) \in T_{(u, z)}M$ satisfies the *dancing condition* if there is an *incident* pair $(u^*, z^*) \in \mathcal{U} \times \mathcal{Z}$ such that

- (1) \dot{u} is tangent to \widehat{z}^* at u ,
- (2) \dot{z} is tangent to \widehat{u}^* at z .

Therefore the vector is null in the sense of Definition 2.2 iff it satisfies the dancing condition.

The manifold M is the disjoint union of two connected components $M = M_{\mathrm{in}} \cup M_{\mathrm{out}}$, where M_{in} is the set of pairs (u, z) such that u lies inside of the ellipse \widehat{z} (or z lies inside the horocycle \widehat{u}) and M_{out} is the set of pairs (u, z) such that u lies outside \widehat{z} (or z lies outside \widehat{u}). The group $\mathrm{SL}(2, \mathbb{R})$ acts (diagonally) on M , preserving each of the two components and the dancing condition on it.

5.2 Dancing conditons and sextics

We are now ready to establish Theorem 2.4. We shall explicitly construct a sextic form

$$S \in \text{Sym}^4(T\mathcal{U}) \otimes \text{Sym}^2(T\mathcal{Z}) \subset \text{Sym}^6(TM)$$

such that a vector field $V \in \Gamma(TM)$ is null iff $S(V, V, V, V, V, V) = 0$.

Proof of Theorem 2.4. We parametrize \mathcal{Z} by the upper half plane $H = \{(a, b) \in \mathbb{R}^2 \mid b > 0\}$. Define $f: H \rightarrow \mathcal{Z}$ by

$$f(a, b) = \frac{1}{b}(a^2 + b^2, -a, 1).$$

Then the incidence condition equation (5.1) can be written as

$$\Phi(x, y, a, b) := (a^2 + b^2)x^2 - 2axy + y^2 - b = 0. \quad (5.3)$$

The dancing condition on $(\dot{u}, \dot{z}) \in T_{(u,z)}M$ amounts to the existence of a pair $(u^*, z^*) \in \mathcal{U} \times \mathcal{Z}$ satisfying

$$\Phi(u, z^*) = \Phi(u^*, z^*) = \Phi(u^*, z) = \frac{\partial \Phi}{\partial u}(u, z^*)\dot{u} = \frac{\partial \Phi}{\partial z}(u^*, z)\dot{z} = 0. \quad (5.4)$$

Using the coordinates

$$u = (x, y), \quad z = (a, b), \quad u^* = (X, Y), \quad z^* = (A, B), \quad \dot{u} = (\dot{x}, \dot{y}), \quad \dot{z} = (\dot{a}, \dot{b}),$$

these equations are

$$\begin{aligned} (A^2 + B^2)x^2 - 2Axy + y^2 - B &= 0, \\ (A^2 + B^2)X^2 - 2AXY + Y^2 - B &= 0, \\ (a^2 + b^2)X^2 - 2aXY + Y^2 - b &= 0, \\ [(A^2 + B^2)x - Ay]\dot{x} + (y - Ax)\dot{y} &= 0, \\ 2X(aX - Y)\dot{a} + (2bX^2 - 1)\dot{b} &= 0. \end{aligned} \quad (5.5)$$

The dancing condition on TM is $\text{SL}(2, \mathbb{R})$ -invariant, so it is enough to study it along a *section* of the $\text{SL}(2, \mathbb{R})$ -action on M , i.e., a subset $\Sigma \subset M$ intersecting every $\text{SL}(2, \mathbb{R})$ -orbit. We take $\Sigma = \{x = 1, y = 0, a = 0\} \subset M$, with b a coordinate along Σ ($0 < b < 1$ for M_{in} and $b > 1$ for M_{out}). Along Σ , equations (5.5) reduce to

$$\begin{aligned} A^2 + B^2 &= B, \\ B(X^2 - 1) - 2AXY + Y^2 &= 0 \\ b^2(X^2 - 1) + Y^2 &= 0 \\ B\dot{x} - A\dot{y} &= 0 \\ 2XY\dot{a} + (1 - 2bX^2)\dot{b} &= 0. \end{aligned}$$

We use the 1st and 4th equation to solve for A, B and substitute in the other 3 equations, obtaining

$$\begin{aligned} Y^2\dot{x}^2 - 2XY\dot{x}\dot{y} + (X^2 + Y^2 - 1)\dot{y}^2 &= 0, \\ b^2X^2 + Y^2 - b &= 0, \\ 2XY\dot{a} + (1 - 2bX^2)\dot{b} &= 0. \end{aligned} \quad (5.6)$$

Now we substitute $Y = PX$ in equations (5.6), use the 2nd obtained equation to solve for X^2 and substitute in the other two, obtaining

$$\begin{aligned} [bx^2 + (b-1)y^2]P^2 - 2b\dot{x}yP - b(b-1)\dot{y}^2 &= 0, \\ \dot{b}P^2 + 2b\dot{a}P - b^2\dot{b} &= 0. \end{aligned} \quad (5.7)$$

To eliminate P we take the resultant¹ of the two quadratic polynomials in P in equations (5.7), obtaining (after dividing by b^2)

$$\begin{aligned} b^4\dot{b}^2\dot{x}^4 - 4b^3\dot{a}\dot{b}\dot{x}^3\dot{y} + 2b^2[(b-2)b-1]\dot{b}^2 - 2(b-1)\dot{a}^2]x^2\dot{y}^2 + 4b(1-b^2)\dot{a}\dot{b}\dot{x}\dot{y}^3 \\ + (b-1)^2[(b-1)^2\dot{b}^2 - 4b\dot{a}^2]\dot{y}^4 = 0. \end{aligned} \quad (5.8) \quad \blacksquare$$

The bi-degree of the sextic (5.8) has a geometric explanation. For fixed (u, z) and \dot{u} , one draws in \mathcal{Z} the horocycle \widehat{u} , the point z and the turning point $z^* \in \widehat{u}$ corresponding to \dot{u} . There are then exactly two horocycles \widehat{u}_1^* , \widehat{u}_2^* passing through z, z^* , giving two tangent directions \dot{z} at z .

Similarly, for fixed (u, z) and \dot{z} , there are four turning points $\pm u_1^*, \pm u_2^* \in \widehat{z}$, corresponding to \dot{z} . For each antipodal pair $\pm u_i^*$ there are two ellipses \widehat{z}^* passing through $u, \pm u_i^*$ (they correspond to the two intersection points of the horocircles $\widehat{u}, \widehat{u}_i^*$), giving altogether four tangent direction \dot{u} at u .

5.2.1 2nd order ODEs

Let the three-dimensional $\mathcal{I} \subset \mathcal{U} \times \mathcal{Z}$ be the solution set of equations (5.1). The natural projections $\mathcal{I} \rightarrow \mathcal{U}$, $\mathcal{I} \rightarrow \mathcal{Z}$ define a double fibration (2.1) so that the fibers of one fibration project to paths in the base of the other fibration, defining a pair of path geometries. This is a curved version of the more familiar classical point-line projective duality. A major difference between this curved case and the flat case is that the symmetry group drops from $\mathrm{PSL}(3, \mathbb{R})$ to $\mathrm{PSL}(2, \mathbb{R})$. A path geometry in the xy plane, such as the one given by equations (5.1) and (5.2) (or equivalently, equations (5.3)), is determined uniquely by a 2nd order ODE

$$y'' = F(x, y, y').$$

To find it, one takes the defining equation, say (5.3), and its two derivatives wrt x , assuming $y = y(x)$. Eliminating a, b and solving for y'' , we obtain

$$y'' = (xy' - y)^3. \quad (5.9)$$

The dual path geometry, in the ab plane, is obtained by a similar process, for a function $b = b(a)$. Eliminating x, y and solving for b'' , we obtain

$$b'' = \epsilon \frac{(\sqrt{1+b^2} - \epsilon)(1+b^2)}{b}, \quad \epsilon = \pm 1$$

(so there are in fact two equations). These are two examples of (dual) 2D path geometries with a three-dimensional (local) group of symmetries, classified in 1896 by Tresse [18, p. 76].

The equation (5.9) is cubic in y' , as so it defines a projective structure. This projective structure is metrisable (see [5]) as the integral curves of (5.9) are unparametrised geodesics of a (pseudo) Riemannian metric. There is in fact a three parameter family projectively equivalent metrics with these paths as unparametrized geodesics. We can arrange for one of these metrics to be invariant under rotations about the origin in the xy plane. In polar coordinates it is given by

$$ds^2 = \frac{dr^2}{(1+r^4)^2} + \frac{r^2 d\theta^2}{1+r^4}, \quad \text{where } x + iy = re^{i\theta}.$$

¹The resultant of two polynomials of one variable is a polynomial in their coefficients, that vanishes iff they have a common root. The resultant of $a_0 + a_1P + a_2P^2$, $b_0 + b_1P + b_2P^2$ is $(a_0b_2 - a_2b_0)^2 - (a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1)$.

6 Dancing conics

Let $N \subset \mathbb{P}^2 \times \mathbb{P}^5$ be a set of non-incident pairs (a, A) , where $a \in \mathbb{P}^2$ and A is an irreducible conic. We say that two pairs (a, A) and (b, B) are *dancing* if there exists a conic C which passes through six points $(a, b, A \cap B)$ in \mathbb{P}^2 , where $A \cap B$ are the four intersections of the conics (A, B) . Note that the dancing condition on N is “similar” to that given in Definition 2.1 for path geometries, in that both conditions define co-dimension one cones in TN and TM respectively: generically there is no line through a three given points, and there is no conic through a six given points.

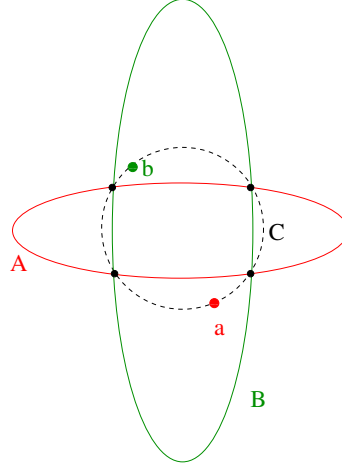


Figure 4. Dancing conics.

Proof of Theorem 2.5. The dancing condition is

$$(a^T A a)(b^T B b) - (a^T B a)(b^T A b) = 0, \quad (6.1)$$

where (by a slight abuse of notation) A above is a symmetric matrix representing a conic, and $a^T = [a^1, a^2, a^3]$ is a vector representing the homogeneous coordinates of $a \in \mathbb{P}^2$. To prove (6.1) note that the conic C must belong to the pencil $(A + tB)$, and contain the points (a, b) so that

$$a^T(A + tB)a = 0, \quad b^T(A + tB)b = 0.$$

Eliminating t between these two equations yields (6.1).

Let $\pi: N \rightarrow M$ be a projection given by $\pi(a, A) = (a, \alpha)$, where α is the polar line of A with respect to a . The fibres of π are three-dimensional. For example if a is a point in an affine space, and α is a line at infinity, then the fiber of (a, α) consists of all ellipses centred at a .

Consider the dancing condition (6.1) on N , and set $B = A + \epsilon \dot{A}$, and $b = a + \epsilon \dot{a}$, where ϵ is small. Substituting this in (6.1), and keeping the the second order in ϵ yields a quadratic condition

$$(a^T \cdot A \cdot a)(a^T \cdot \dot{A} \cdot \dot{a}) - (a^T \cdot \dot{A} \cdot a)(a^T \cdot A \cdot \dot{a}) \quad (6.2)$$

(where \cdot denotes the matrix multiplication). The corresponding (degenerate) conformal structure on M is

$$G = (a^T \cdot A \cdot a)(a^T \cdot dA \odot da) - (a^T \cdot dA \cdot a) \odot (a^T \cdot A \cdot da).$$

We can use inhomogeneous coordinates on N with the pair (a, A) represented by $a = (x, y, 1)$ and the symmetric matrix A with $A_{33} = 1$. The $SL(3, \mathbb{R})$ invariance allows us without loss of

generality to take the pair (a, A) with $a_0 = (0, 0, 1)^T$, $A_0 = \text{diag}(-1, -1, 1)$. Then (6.2) reduces to the quadric

$$\dot{x}\dot{A}_{13} + \dot{y}\dot{A}_{23},$$

with signature $++--000$, as stated, and with kernel given by a tangent subspace $\dot{x} = \dot{A}_{13} = \dot{y} = \dot{A}_{23} = 0$. Now the tangent to the fiber of π at (a_0, A_0) is the kernel of the derivative of π at (a_0, A_0) . Since $\pi(a, A) = (a, a^T A)$, the derivative at (a_0, A_0) is

$$(\dot{a}, \dot{A}) \mapsto (\dot{a}^T A_0 + a_0^T \dot{A}, \dot{a}),$$

hence the kernel is given by $\dot{A}a_0 = \dot{a} = 0$, or $\dot{x} = \dot{A}_{13} = \dot{y} = \dot{A}_{23} = 0$ as stated. \blacksquare

Remark 6.1. One can check that the dancing condition on N is *not* the pull back of the dancing condition on M , although both are $\text{SL}(3, \mathbb{R})$ -invariant and horizontal w.r.t. $\pi: N \rightarrow M$. If it were, then if (a, A) and (b, B) are dancing in N , then $\pi(a, A)$, and $\pi(b, B)$ are dancing in M . We shall now argue that this is not the case. To see it, note that in homogeneous coordinates the projection π is given by $(a^i, A_{jk}) \rightarrow (a^i, \alpha_j = A_{jk}a^k)$. Let (a, A) and (b, B) belong to M . The intersection of polar lines $a^T A z = 0$, $b^T B z = 0$ is c , where $c^i = \epsilon^{ijk} A_{jp} B_{kq} a^p b^q$. This point is co-linear with (a, b) if $\epsilon_{ijk} c^i a^j b^k = 0$. Using the identity $\epsilon_{ijk} \epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$ yields

$$(a^T A a)(b^T B b) - (a^T B b)(b^T A a) = 0,$$

which is different than (6.1).

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