

Planar Orthogonal Polynomials as Type I Multiple Orthogonal Polynomials

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Abstract. A recent result of S.-Y. Lee and M. Yang states that the planar orthogonal polynomials orthogonal with respect to a modified Gaussian measure are multiple orthogonal polynomials of type II on a contour in the complex plane. We show that the same polynomials are also type I orthogonal polynomials on a contour, provided the exponents in the weight are integer. From this orthogonality, we derive several equivalent Riemann–Hilbert problems. The proof is based on the fundamental identity of Lee and Yang, which we establish using a new technique.

Key words: planar orthogonal polynomials; multiple orthogonal polynomials; Riemann–Hilbert problems; Hermite–Padé approximation; normal matrix model

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*Dedicated to Alexander Its
on the occasion of his 70th birthday*

1 Introduction

This work is inspired by Lee and Yang’s paper [17], which showed that planar orthogonal polynomials can be viewed as multiple orthogonal polynomials of type II on a contour in the complex plane. Their work extends an earlier result of Balogh, Bertola, Lee, and McLaughlin [3].

We show that the same polynomials are also multiple orthogonal polynomials of type I if the exponents in the weight are positive integers, unlike in the situation studied in [17], where these exponents are arbitrary positive real numbers. We also present a novel, more transparent, technique to transform planar orthogonality into orthogonality on a contour. Before we begin, note that the title of our paper differs from the title of [17] only in one letter, yet this makes a considerable difference in the arguments used.

The polynomials in question are orthogonal with respect to a modified Gaussian measure,

$$\mu_W(dz) = \frac{1}{\pi} |W(z)|^2 e^{-|z|^2} \text{Leb}(dz), \quad (1.1)$$

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where Leb denotes the Lebesgue measure on \mathbb{C} (identified with \mathbb{R}^2) and the weight W reads

$$W(z) = \prod_{j=1}^p (z - a_j)^{c_j}, \quad z \in \mathbb{C}, \quad (1.2)$$

where $p \in \mathbb{N}$, the c_j are positive real numbers, and the a_j are distinct complex numbers.

If the c_j are not necessarily integer, one needs to specify the branch cuts and fix the branches in order to render (1.2) unambiguous. This complicates the analysis, and we return to such a general scenario only episodically. In contrast, if all c_j are positive integers, W becomes a polynomial of degree $c = \sum_{j=1}^p c_j$ and (1.2) extends to the whole complex plane \mathbb{C} . This is the situation of our primary concern.

Denote the scalar product corresponding to (1.1) by

$$\langle f, g \rangle_W = \int_{\mathbb{C}} f(z) \overline{g(z)} \mu_W(dz). \quad (1.3)$$

Then, the n -th degree monic orthogonal polynomial P_n with respect to μ_W can be uniquely recovered by solving a linear system of equations for its coefficients,

$$\int_{\mathbb{C}} P_n(z) \overline{z^k} \mu_W(dz) = 0, \quad k = 0, \dots, n-1. \quad (1.4)$$

The motivation for studying planar orthogonal polynomials comes from the theory of non-Hermitian random matrices, in particular from those related to the normal matrix model. In this model, the eigenvalues of an $n \times n$ normal matrix have the joint density

$$\frac{1}{Z_n} \prod_{j < k} |z_k - z_j|^2 \prod_{j=1}^n e^{-V(z_j)}, \quad (1.5)$$

where V is the potential of the model and Z_n is a normalization constant. The eigenvalues form a determinantal point process with the correlation kernel constructed in terms of the planar orthogonal polynomials orthogonal with respect to the one-particle weight $e^{-V(z)}$. The case (1.1)–(1.2) corresponds to

$$V(z) = |z|^2 - 2 \sum_{j=1}^p c_j \log |z - a_j|. \quad (1.6)$$

In particular, for integer c_j 's the probability law corresponding to (1.5)–(1.6) can be interpreted as that of a Ginibre ensemble of size $n + c$ conditioned on having an eigenvalue of multiplicity c_j at a_j for each $j = 1, \dots, p$.

The determinantal structure in (1.5) allows for a complete description of the eigenvalue correlation functions at the finite size n in terms of the correlation kernel, which in turn can be used to study the large n behavior of both the polynomials and the eigenvalues. In such studies, one typically replaces V by nV in (1.5) to obtain a balance between the “repulsion” and “confinement” present in the determinantal model (1.5). We refer to the surveys [7, Section 5], [13, Chapter 6] and references therein for more information on the normal matrix model.

In the analogous situation of Hermitian random matrices, the eigenvalue correlations are described by orthogonal polynomials on the real line. The theory of such polynomials is well-developed, and as a result the corresponding ensembles are understood much better than their non-Hermitian counterparts. One basic result is that the eigenvalues of Hermitian matrices and the zeros of the corresponding orthogonal polynomials (both real) have the same limiting behavior as $n \rightarrow \infty$ (e.g., see [9]). More subtle results on the universality of local eigenvalue

statistics were established using the characterization of orthogonal polynomials on the real line via a 2×2 matrix-valued Riemann–Hilbert problem (e.g., see [12]), followed by the Deift–Zhou steepest descent analysis (e.g., see [10, 11]).

The planar case, on the other hand, is more intricate. One has to distinguish between the asymptotic behavior of the random eigenvalues governed by (1.5) (with V replaced by nV) and the limiting behavior of the zeros of the corresponding orthogonal polynomials. While it is known that the eigenvalues fill out a two-dimensional domain called the droplet, the understanding of the asymptotic behavior of the zeros of the planar orthogonal polynomials is rather limited. Results exist when the classical Hermite, Laguerre and Gegenbauer polynomials appear as the planar orthogonal polynomials (e.g., see [1, 15, 21]), as well as for some special cases where the planar orthogonality can be reformulated as (multiple) orthogonality on a contour and the Riemann–Hilbert techniques can be used (e.g., see [3, 4, 5, 6, 8, 16]). For example, in [3] the situation (1.1)–(1.2) with $p = 1$ is considered. In this case, the Hermitian planar orthogonality can be transformed to non-Hermitian orthogonality on a contour due to a special identity [3, Lemma 3.1], and rigorous analysis is possible. The same identity was used in [22] in a study on moments of complex Ginibre matrices.

Multiple orthogonality plays a role in [6, 16, 18], where it can be treated by using large size Riemann–Hilbert problems (e.g., see [20]).

The common feature of the examples above is that the zeros accumulate along a one-dimensional curve (or a system of such curves) in the complex plane, known as the motherbody. We conjecture that this is a general phenomenon for all real analytic potentials, including those in (1.6). The results of [17, 18] support this conjecture. Indeed, the planar orthogonality corresponding to (1.5) with V given by

$$V(z) = n|z|^2 - 2 \sum_{j=1}^p c_j \log |z - a_j|$$

is studied in [17, 18] for the case of fixed c_j 's independent of n . The droplet turns out to be the unit disk, and the motherbody is supported on a multiple Szegő curve that depends on the a_j .

In the scenario when the c_j grow linearly with n and $p \geq 2$, the Riemann–Hilbert problem in [18] has not been analyzed successfully yet. The multiple orthogonality of type I that we discovered, as we will show, leads to several different Riemann–Hilbert problems. Our hope is that one of them will help to carry through with the steepest descent analysis.

If all c_j are integer valued, the planar orthogonal polynomials can also be expressed as ratios of determinants as shown in [2]. The determinants are growing in size as the c_j increase, and therefore this determinantal formula may not be particularly useful for asymptotic analysis.

We finally remark that the important work of Hedenmalm and Wennman [14] provides the asymptotic behavior of the planar orthogonal polynomials in the exterior of the droplet and on its boundary (even slightly inside, under certain assumptions including those of the real analyticity of the boundary). This, however, does not give any information about the motherbody since it is inside the droplet (see [14, Remark 1.6(c)]).

2 Statement of result

Our main result is Theorem 2.1 below. It gives a number of properties that are equivalent to the planar orthogonality corresponding to (1.1)–(1.4) in the case the c_j are positive integers. Below, we use the conjugate W^* of W defined by

$$W^*(z) = \overline{W(\bar{z})} = \prod_{j=1}^p (z - \bar{a}_j)^{c_j}, \quad z \in \mathbb{C}. \quad (2.1)$$

We also use \mathcal{D}_z to denote the derivative operator with respect to z , and then $W^*(\mathcal{D}_z)$ is the differential operator

$$W^*(\mathcal{D}_z) = \prod_{j=1}^p (\mathcal{D}_z - \bar{a}_j)^{c_j}. \quad (2.2)$$

Theorem 2.1. *Let W be given by (1.2) where all c_j are positive integers (so that W is a polynomial of degree $c = \sum_{j=1}^p c_j$), and let the a_j be distinct complex numbers. Then, the following properties are equivalent for a monic polynomial P_n of degree n ,*

- (a) P_n is the planar orthogonal polynomial on \mathbb{C} with weight (1.1).
- (b) P_n satisfies

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \phi_k(z) dz = 0, \quad k = 0, 1, \dots, n-1, \quad (2.3)$$

where γ is a closed contour around the origin and

$$\phi_k(z) = \int_0^{\bar{z} \times \infty} W^*(u) u^k e^{-uz} du, \quad (2.4)$$

where the path of integration in (2.4) goes from 0 to ∞ along the ray $\arg u = \arg \bar{z}$.

- (c) One has

$$W^*(\mathcal{D}_z) [P_n(z) W(z)] = O(z^n) \quad (2.5)$$

as $z \rightarrow 0$.

- (d) There exist polynomials Q_j of $\deg Q_j \leq c_j - 1$, for $j = 1, \dots, p$, such that

$$P_n(z) W(z) + \sum_{j=1}^p Q_j(z) e^{\bar{a}_j z} = O(z^{n+c}) \quad (2.6)$$

as $z \rightarrow 0$.

Remark 2.2. The property (2.3) of the planar orthogonal polynomials has already been obtained by Lee and Yang [17]. They assume that the points a_j are distinct, non-zero, and with different arguments modulo 2π , and show that (2.3) leads to multiple orthogonality of type II (see also Section 6 below) for real positive c_j 's. In this general situation, because of the branch cuts, the contour γ in (2.3) has to pass through all a_j and can no longer be an arbitrary contour around the origin.

The proof of Theorem 2.1 relies on the *fundamental identity* of Lee and Yang [17, Proposition 1]. We state and prove it in the next section. Our proof is based on a new technique and is of independent interest. The proof in [17] (see also the proof of [3, Lemma 3.1]) goes as follows. One first restricts the integral in (1.4) to a large disk D_R and then applies Stokes' theorem to rewrite the new integral over D_R as an integral over the boundary ∂D_R . Then, (2.3) follows by a contour deformation argument and by passing to the limit $R \rightarrow \infty$.

In our proof, we only rely on most basic and elementary facts of complex analysis and avoid the use of Stokes' theorem. We write the integral over \mathbb{C} in polar coordinates and, by analyticity, deform the angular integral to an integral over γ . The final step is to switch the angular and the radial integrals by Fubini's theorem.

Theorem 2.1 is proved in Section 4. Part (c) of Theorem 2.1 is a very concise representation of the planar orthogonality in part (a), and thus is of interest in its own right. The equivalence with part (d) leads directly to the multiple orthogonality of type I as we explain in Section 5. We show that this orthogonality leads to three different, though closely related, Riemann–Hilbert problems. The latter uniquely characterize the planar orthogonal polynomial P_n and the auxiliary polynomials Q_j . We point out in passing that none of these Riemann–Hilbert problems turns out to be related to the Riemann–Hilbert problem in [17] (corresponding to type II orthogonality) in a canonical way (see also Section 7). In Section 6, we focus on the type II multiple orthogonality of Lee and Yang [17]. We give the corresponding proof in the case of polynomial W , which is essentially the same as the proof in [17] but more transparent since no branch cuts for W are necessary.

3 Fundamental identity

Akin to (2.1), we define the conjugate of a function Q by

$$Q^*(z) = \overline{Q(\bar{z})}, \quad z \in \mathbb{C}. \quad (3.1)$$

If Q is analytic in a certain domain Ω , then Q^* is also analytic however in the conjugate domain $\Omega^* = \{z \in \mathbb{C} \mid \bar{z} \in \Omega\}$. If Q is a polynomial, then Q^* is the polynomial whose coefficients are the complex conjugates of those of the polynomial Q .

The following result is due to Lee and Yang [17, Proposition 1]. As already stated above, we give a different proof. For the sake of clarity, we first deal with the case of polynomial W .

Proposition 3.1. *Let P and Q be polynomials and suppose that the c_j in (1.2) are positive integers. Then, the fundamental identity holds,*

$$\begin{aligned} \langle P, Q \rangle_W &= \frac{1}{\pi} \int_{\mathbb{C}} P(z) \overline{Q(\bar{z})} |W(z)|^2 e^{-|z|^2} \text{Leb}(dz) \\ &= \frac{1}{2\pi i} \oint_{\gamma} P(z) W(z) \int_0^{\bar{z} \times \infty} W^*(u) Q^*(u) e^{-uz} du dz, \end{aligned} \quad (3.2)$$

where γ is a simple closed contour that goes around the origin once in the counterclockwise direction and the path for the u integral goes from 0 to ∞ along the ray $\arg u = \arg \bar{z}$.

Proof. Write the left-hand side of (3.2) in polar coordinates,

$$\langle P, Q \rangle_W = \frac{1}{\pi i} \int_0^{\infty} \oint_{C_r} P(z) Q^*(\bar{z}) |W(z)|^2 \frac{dz}{z} r e^{-r^2} dr, \quad (3.3)$$

where the z -integral is taken along the circle C_r of radius r around the origin. Observe that $\bar{z} = r^2/z$ for $z \in C_r$. In view of (3.1), we can write the following chain of identities,

$$|W(z)|^2 = W(z) W^*(\bar{z}) = W(z) W^*(r^2/z), \quad z \in C_r.$$

Since also $Q^*(\bar{z}) = Q^*(r^2/z)$ for $z \in C_r$, the formula (3.3) becomes

$$\langle P, Q \rangle_W = \frac{1}{\pi i} \int_0^{\infty} \oint_{C_r} P(z) Q^*(r^2/z) W(z) W^*(r^2/z) \frac{dz}{z} r e^{-r^2} dr. \quad (3.4)$$

Recall that all c_j are positive integers, thus W and W^* are polynomials and the integrand in (3.4) is meromorphic in z with a sole pole at $z = 0$. Then, by Cauchy's theorem, the contour

can be deformed from C_r to a contour γ that goes around the origin in the counterclockwise direction once and is independent of r . We use Fubini's theorem to get

$$\langle P, Q \rangle_W = \frac{1}{\pi i} \oint_{\gamma} P(z)W(z) \int_0^{\infty} Q^*(r^2/z)W^*(r^2/z) \frac{re^{-r^2}}{z} dr dz. \quad (3.5)$$

Changing variables in the inner integral, $u = r^2/z$, we arrive at (3.2). \blacksquare

Our method of proof easily extends to a more general setting. Assume that the c_j are positive real numbers, not necessarily integer as before. As in [17], we restrict ourselves to the case that all a_j are non-zero, distinct, and have different arguments modulo 2π . For convenience, order the a_j so that $0 \leq \arg a_1 < \arg a_2 < \dots < \arg a_p < 2\pi$. We can still define W by the same formula (1.2) as earlier, however it is imperative one restrict the domain by making cuts. Following [17], we choose to cut along the rays

$$B = \bigcup_{j=1}^p \{z \in \mathbb{C} \mid z = a_j t, t \geq 1\}. \quad (3.6)$$

The domain $\mathbb{C} \setminus B$ is simply connected. Fixing a branch for the power functions in (1.2) renders W analytic in this domain. Note that such a choice of the branches does not affect $|W(z)|^2$, which is assumed to be extended by continuity to the whole complex plane \mathbb{C} .

To formulate the analogue of Proposition 3.1, we will set

$$\Omega = \mathbb{C} \setminus \bigcup_{j=1}^p \{z \in \mathbb{C} \mid z = a_j t, t \geq 0\}, \quad (3.7)$$

which is a union of sectors separated by the rays $\arg z = \arg a_j$ for $j = 1, \dots, p$.

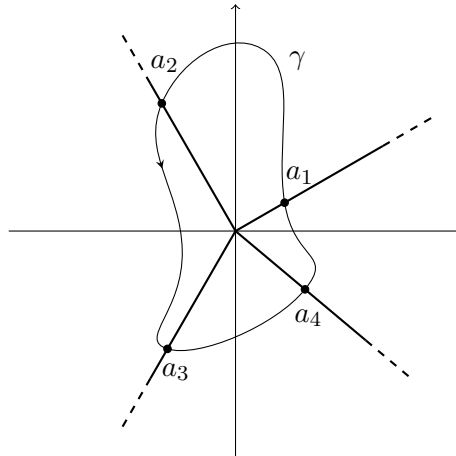


Figure 1. The domain Ω in (3.7) and an example contour γ .

The following is [17, Proposition 1].

Proposition 3.2. *Suppose that the c_j are positive real numbers, but not necessarily integers, and the a_j are non-zero, distinct complex numbers that have different arguments modulo 2π . Define Ω as in (3.7). Then (3.2) still holds, provided that γ is a counterclockwise-oriented contour in $\Omega \cup \{a_1, \dots, a_p\}$, going around the origin once (e.g., see Figure 1).*

Remark 3.3. Note that there is no need to make a cut for a certain a_j if the corresponding $c_j \in \mathbb{N}$. Nevertheless, we do so for the sake of notational convenience.

Remark 3.4. If $a_1 = 0$, the proposition still holds with the same Ω as in (3.7); however, one needs to choose γ in $\Omega \cup \{a_2, \dots, a_p\}$ instead. Indeed, first note that Remark 3.3 is still applicable. Hence, without loss of generality $c_1 \notin \mathbb{N}$. One needs to modify B in (3.6) by including an additional cut from 0 to ∞ , transversal to all the other cuts. Then, in a similar way as before, one can fix a branch of $W(z)$ in the new simply-connected domain $\mathbb{C} \setminus B$. Note that the definition of Ω in (3.7) does not change, however one of the rays collapses to the point $z = 0$. Finally, observe that z^{c_1} effectively cancels from (3.8), and the proof goes through in the same way except the fact that $Q^*(r^2/z)$ still may have a pole at zero and thus γ must not pass through it. The latter explains why $a_1 = 0$ was excluded from $\Omega \cup \{a_1, \dots, a_p\}$.

Proof of Proposition 3.2. If not all of the c_j are integers, then we need branch cuts to define W and W^* . For every $r > 0$, we have

$$W(z)W^*(r^2/z) = \prod_{j=1}^p (z - a_j)^{c_j} \left(\frac{\frac{r^2}{|a_j|^2} a_j - z}{z/\bar{a}_j} \right)^{c_j}, \quad (3.8)$$

which is initially only analytic in Ω . If $r \neq |a_j|$ for every $j = 1, \dots, p$, then there is an analytic continuation across the open straight line segment from a_j to $\frac{r^2}{|a_j|^2} a_j$ along the ray $\arg z = \arg a_j$ for every $j = 1, \dots, p$. That is, $W(z)W^*(r^2/z)$ has an analytic continuation to

$$\Omega_r := \Omega \cup \left\{ (1-t)a_j + t \frac{r^2}{|a_j|^2} a_j \mid 0 < t < 1 \right\}.$$

Observe that $\frac{r^2}{|a_j|^2} a_j$ is the image of a_j under the reflection about the circle C_r . Thus, C_r is contained in Ω_r .

With these preparations in mind, we follow the proof of Proposition 3.1. The identity (3.4) still holds. Then, by the above and Cauchy's theorem, for each $r > 0$ we are allowed to deform C_r to a contour γ as in the statement. Since γ is independent of r , we can apply Fubini's theorem to (3.4). This yields (3.5), and we can proceed as in the proof of Proposition 3.1. ■

Note that it is not difficult to extend the proofs of Propositions 3.1 and 3.2 to other types of weights, mutatis-mutandis. For instance, if

$$\mu(dz) = \frac{1}{\pi} |W(z)|^2 \frac{\text{Leb}(dz)}{(1 + |z|^2)^\alpha}, \quad \alpha > 1 + c,$$

(which generalizes spherical ensembles, e.g., see [7, Section 2.5]), one gets

$$\langle P, Q \rangle_W = \frac{1}{2\pi i} \oint_{\gamma} P(z)W(z) \int_0^{z \times \infty} \frac{Q^*(u)W^*(u)}{(1 + uz)^\alpha} du dz,$$

which holds as long as $\deg P + \deg Q < 2\alpha - 2c - 2$ so that the integral over \mathbb{C} converges.

4 Proof of Theorem 2.1

Proof. (a) \Leftrightarrow (b): Taking $Q(z) = z^k$ in the identity (3.2) and recalling the definition (2.4) of ϕ_k , we obtain

$$\langle P, z^k \rangle_W = \frac{1}{2\pi i} \oint_{\gamma} P(z)W(z)\phi_k(z) dz. \quad (4.1)$$

This identity (4.1) shows that (a) and (b) in Theorem 2.1 are equivalent.

(b) \Leftrightarrow (c): From basic properties of the Laplace transform, we get

$$\int_0^{\bar{z} \times \infty} W^*(u) u^k e^{-uz} du = W^*(-\mathcal{D}_z)(-\mathcal{D}_z)^k \left[\frac{1}{z} \right] = W^*(-\mathcal{D}_z) \left[\frac{k!}{z^{k+1}} \right],$$

Thus, recalling (2.4), we can write

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \phi_k(z) dz = \frac{k!}{2\pi i} \oint_{\gamma} P_n(z) W(z) W^*(-\mathcal{D}_z) \left[\frac{1}{z^{k+1}} \right] dz$$

for any polynomial P_n . Integrating by parts, we find

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \phi_k(z) dz = \frac{k!}{2\pi i} \oint_{\gamma} \frac{W^*(\mathcal{D}_z)[P_n(z)W(z)]}{z^{k+1}} dz,$$

as there are no boundary terms on the closed contour γ and $W^*(\mathcal{D}_z)$ is the adjoint of $W^*(-\mathcal{D}_z)$. Thus, P_n satisfies (2.3) if and only if

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{W^*(\mathcal{D}_z)[P_n(z)W(z)]}{z^{k+1}} dz = 0, \quad k = 0, \dots, n-1. \quad (4.2)$$

The identities (4.2) mean that the coefficient before z^k of the polynomial $W^*(\mathcal{D}_z)[P_n(z)W(z)]$ vanishes for $k = 0, \dots, n-1$, and we conclude that items (b) and (c) are equivalent.

In the proof that (c) and (d) are equivalent, we are going to use three basic properties of the operator $W^*(\mathcal{D}_z)$ from (2.2). These properties are stated separately for the ease of reference.

Property W_1 . $W^*(\mathcal{D}_z)$ is a linear differential operator with kernel

$$\ker W^*(\mathcal{D}_z) = \left\{ \sum_{j=1}^p Q_j(z) e^{\bar{a}_j z} \mid \deg Q_j \leq c_j - 1 \text{ for } j = 1, \dots, p \right\}. \quad (4.3)$$

The kernel is a vector space over \mathbb{C} and its dimension is $c = \sum_{j=1}^p c_j$.

Property W_2 . If P is a polynomial then $W^*(\mathcal{D}_z)[P(z)]$ is a polynomial and

$$\deg W^*(\mathcal{D}_z)[P(z)] = \deg P(z), \quad (4.4)$$

provided all $a_j \neq 0$. If $a_1 = 0$ and $\deg P(z) \geq c_1$, then

$$\deg W^*(\mathcal{D}_z)[P(z)] = \deg P(z) - c_1. \quad (4.5)$$

Property W_3 . $W^*(\mathcal{D}_z)$ reduces the order of vanishing at $z = 0$ by $c = \deg W$, provided the order of vanishing is greater than or equal to c . Namely, for every non-negative integer n , and analytic function F with $F(z) = O(z^{n+c})$ as $z \rightarrow 0$, we have

$$W^*(\mathcal{D}_z)[F(z)] = O(z^n) \quad (4.6)$$

as $z \rightarrow 0$.

(d) \Rightarrow (c): Suppose P_n satisfies (2.6). Then, by applying $W^*(\mathcal{D}_z)$ and using Properties W_1 and W_3 , we obtain

$$W^*(\mathcal{D}_z)[P_n(z)W(z)] = W^*(\mathcal{D}_z)[O(z^{n+c})] = O(z^n), \quad (4.7)$$

which is (2.5). Thus, (d) implies (c).

(c) \Rightarrow (d): In the proof, we assume without loss of generality that $a_1 = 0$ and $c_1 \geq 0$. This includes the case that W has no zero at the origin since one can always set $c_1 = 0$.

Consider the linear mapping $\pi: \ker(W^*(\mathcal{D}_z)) \rightarrow \mathbb{C}^c$ given by

$$\ker(W^*(\mathcal{D}_z)) \ni Q(z) = \sum_{j=0}^{\infty} q_j z^j \xrightarrow{\pi} (q_0, \dots, q_{c_1-1}, q_{n+c_1}, \dots, q_{n+c-1}) \in \mathbb{C}^c. \quad (4.8)$$

We claim that π is injective, which will imply that π is surjective since it is a linear mapping between vector spaces of the same dimension c , as follows from (4.8) and Property W_1 . Suppose $Q \in \ker(W^*(\mathcal{D}_z))$ with $\pi(Q) = (0, \dots, 0) \in \mathbb{C}^c$. Then,

$$Q(z) = \sum_{j=c_1}^{n+c_1-1} q_j z^j + O(z^{n+c}) \quad (4.9)$$

as $z \rightarrow 0$. From (4.6) and $W^*(\mathcal{D}_z)[Q(z)] = 0$, it then follows that

$$W^*(\mathcal{D}_z) \left[\sum_{j=c_1}^{n+c_1-1} q_j z^j \right] = O(z^n) \quad (4.10)$$

as $z \rightarrow 0$. Because of (4.4)–(4.5), we have that the left-hand side of (4.10) is a polynomial of degree $\leq n-1$, and due to (4.10) it has a zero at $z = 0$ of order at least n . Hence, (4.10) vanishes identically, and thus $\sum_{j=c_1}^{n+c_1-1} q_j z^j$ belongs to the kernel of $W^*(\mathcal{D}_z)$. By Property W_1 , the kernel contains polynomials up to degree $c_1 - 1$ but no polynomials of higher degrees. Hence, $q_j = 0$ for $j = c_1, \dots, n + c_1 - 1$. Consequently, by (4.9) we have that $Q(z) = O(z^{n+c})$ as $z \rightarrow 0$. In particular, $Q(0) = Q'(0) = \dots = Q^{(c-1)}(0) = 0$. Thus $Q \in \ker(W^*(\mathcal{D}_z))$ is a solution of a homogeneous constant coefficient linear ODE of order c with c vanishing initial conditions. The uniqueness theorem for such ODEs then yields $Q \equiv 0$, which justifies the claim.

Now, assume that P_n satisfies (2.5), and write

$$P_n(z)W(z) = \sum_{j=0}^{n+c} p_j z^j.$$

Since π is surjective, there is $Q \in \ker(W^*(\mathcal{D}_z))$,

$$Q(z) = \sum_{j=0}^{\infty} q_j z^j,$$

such that $\pi(Q) = (-p_0, \dots, -p_{c_1-1}, -p_{n+c_1}, \dots, -p_{n+c-1})$. Then,

$$P_n(z)W(z) + Q(z) = \sum_{j=c_1}^{n+c_1-1} (p_j + q_j) z^j + O(z^{n+c}). \quad (4.11)$$

Applying $W^*(\mathcal{D}_z)$ to (4.11) and using (2.5), $W^*(\mathcal{D}_z)[Q(z)] = 0$, and (4.6), we obtain

$$W^*(\mathcal{D}_z) \left[\sum_{j=c_1}^{n+c_1-1} (p_j + q_j) z^j \right] = O(z^n) \quad (4.12)$$

as $z \rightarrow 0$. We find ourselves in the situation similar to that while proving the injectivity of π . It followed from (4.10) that $q_j = 0$ for $j = c_1, \dots, n + c_1 - 1$. In the exactly same way, now it follows from (4.12) that $p_j + q_j = 0$ for $j = c_1, \dots, n + c_1 - 1$.

Due to (4.11), we obtain

$$P_n(z)W(z) + Q(z) = O(z^{n+c}),$$

which is exactly (2.6) because $Q \in \ker(W^*(\mathcal{D}_z))$ must be of the form given in (4.3). This shows that (c) implies (d) and completes the proof of Theorem 2.1. ■

We present an alternative proof of (a) \Rightarrow (c) by making use of the Fourier transform of the measure μ_W in (1.1),

$$\widehat{\mu}_W(\zeta, \bar{\zeta}) = \int_{\mathbb{C}} e^{\bar{\zeta}z + \zeta\bar{z}} \mu_W(dz). \quad (4.13)$$

Alternative proof. (a) \Rightarrow (c): Apply the Wirtinger derivatives

$$\mathcal{D}_\zeta = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \mathcal{D}_{\bar{\zeta}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right),$$

where $\zeta = u + iv$, to (4.13), and observe that

$$\mathcal{D}_{\bar{\zeta}}^j \mathcal{D}_\zeta^k [\widehat{\mu}_W(\zeta, \bar{\zeta})] = \int_{\mathbb{C}} z^j \bar{z}^k e^{\bar{\zeta}z + \zeta\bar{z}} \mu_W(dz). \quad (4.14)$$

Due to linearity, (4.14) leads to

$$P_n(\mathcal{D}_{\bar{\zeta}}) \mathcal{D}_\zeta^k [\widehat{\mu}_W(\zeta, \bar{\zeta})] = \int_{\mathbb{C}} P_n(z) \bar{z}^k e^{\bar{\zeta}z + \zeta\bar{z}} \mu_W(dz).$$

Assuming that P_n is the planar orthogonal polynomial satisfying (1.4), we have

$$P_n(\mathcal{D}_{\bar{\zeta}}) \mathcal{D}_\zeta^k [\widehat{\mu}_W(\zeta, \bar{\zeta})] \Big|_{\zeta=0, \bar{\zeta}=0} = 0, \quad k = 0, \dots, n-1, \quad (4.15)$$

where we view ζ and $\bar{\zeta}$ as two independent variables. The latter is permissible as long as the Fourier transform $\widehat{\mu}_W(\cdot, \cdot)$, as a function of two complex variables, extends analytically to the neighborhood of $\{(\xi_1, \xi_2) \in \mathbb{C} \times \mathbb{C} \mid \xi_2 = \bar{\xi}_1\}$. This is the case we encounter below.

For μ_W from (1.1) with polynomial W , the Fourier transform can be computed as follows,

$$\begin{aligned} \widehat{\mu}_W(\zeta, \bar{\zeta}) &= \frac{1}{\pi} \int_{\mathbb{C}} e^{\bar{\zeta}z + \zeta\bar{z}} |W(z)|^2 e^{-|z|^2} \text{Leb}(dz) \\ &= \frac{1}{\pi} W(\mathcal{D}_{\bar{\zeta}}) W^*(\mathcal{D}_\zeta) \int_{\mathbb{R}^2} e^{2(ux+vy)} e^{-x^2-y^2} dx dy \\ &= W(\mathcal{D}_{\bar{\zeta}}) W^*(\mathcal{D}_\zeta) [e^{\zeta\bar{\zeta}}], \end{aligned} \quad (4.16)$$

where $\zeta = u + iv$, $z = x + iy$.

Inserting (4.16) into (4.15), and changing the order of the differential operators, we obtain

$$\mathcal{D}_\zeta^k W^*(\mathcal{D}_\zeta) (P_n W) (\mathcal{D}_{\bar{\zeta}}) [e^{\zeta\bar{\zeta}}] \Big|_{\zeta=0, \bar{\zeta}=0} = 0, \quad k = 0, \dots, n-1. \quad (4.17)$$

Since

$$(P_n W) (\mathcal{D}_{\bar{\zeta}}) [e^{\zeta\bar{\zeta}}] \Big|_{\bar{\zeta}=0} = P_n(\zeta) W(\zeta),$$

we see that (4.17) implies

$$\mathcal{D}_\zeta^k W^*(\mathcal{D}_\zeta) [P_n(\zeta) W(\zeta)] \Big|_{\zeta=0} = 0, \quad k = 0, \dots, n-1,$$

which is equivalent to (2.5). ■

This alternative proof will work for other types of measures μ as long as the Fourier transform $\widehat{\mu}$ has a simple enough representation. On the other hand, non-polynomial weights W will require the use of fractional derivatives, which are known to be non-local operators, causing substantial complications in this proof.

5 Type I multiple orthogonality

The property (2.6) in part (d) of Theorem 2.1 can be regarded as a Hermite–Padé approximation problem of type I at the origin. The general form of such approximation problems is the following. Given a collection of analytic functions (or, more generally, formal power series) f_0, \dots, f_p at $z = 0$ and a multi-index $\vec{n} = (n_0, \dots, n_p) \in \mathbb{N}^{p+1}$, find polynomials Q_0, Q_1, \dots, Q_p of degrees $\deg Q_j \leq n_j - 1$, $j = 0, 1, \dots, p$, such that

$$\sum_{j=0}^p Q_j(z) f_j(z) = O(z^{|\vec{n}|-1})$$

as $z \rightarrow 0$, where $|\vec{n}| = \sum_{j=0}^p n_j$. For more information on Hermite–Padé approximation and multiple orthogonal polynomials, we refer to [19] and references therein.

The problem (2.6) becomes a Hermite–Padé type I approximation problem for polynomials $Q_0 = P_n, Q_1, \dots, Q_p$, corresponding to the weights $f_0(s) = W(s)$, $f_1(s) = e^{\bar{a}_1 s}, \dots, f_p(s) = e^{\bar{a}_p s}$ and to the multi-index $\vec{n} = (n + 1, c_1, \dots, c_p)$. This leads directly to the type I multiple orthogonality. Indeed, the function on the left-hand side of (2.6) has vanishing Taylor coefficients up to and including order $n + c - 1$. By the Cauchy integral formula, this tells us that (2.6) is equivalent to

$$\frac{1}{2\pi i} \oint_{\gamma} \left(P_n(s) \frac{W(s)}{s^{n+c}} + \sum_{j=1}^p Q_j(s) \frac{e^{\bar{a}_j s}}{s^{n+c}} \right) s^k ds = 0, \quad k = 0, \dots, n + c - 1, \quad (5.1)$$

where γ is a simple closed contour around the origin. This is multiple orthogonality of type I on the contour γ with $p + 1$ weight functions $W(s)/s^{n+c}, e^{\bar{a}_1 s}/s^{n+c}, \dots, e^{\bar{a}_p s}/s^{n+c}$ which are meromorphic with a sole pole at $s = 0$.

Both multiple orthogonality of type I and type II are characterized by Riemann–Hilbert problems that were identified by Van Assche, Geronimo, and Kuijlaars in [20]. For the particular case (5.1), the Riemann–Hilbert problem is of size $(p+2) \times (p+2)$ and its jump is on the contour γ . Below, we will use the following notation. For any oriented contour γ and a function Y defined in $\mathbb{C} \setminus \gamma$, we write Y_+ and Y_- to denote the limiting values of Y on γ from the left and right sides, respectively.

Riemann–Hilbert problem (type I multiple orthogonality for P_n, Q_1, \dots, Q_p). *Find solution $Y: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}^{(p+2) \times (p+2)}$ such that*

(1) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$;

$$(2) Y_+(z) = Y_-(z) J_Y(z) \text{ for } z \in \gamma, \text{ where } J_Y(z) = \begin{bmatrix} 1 & 0 & \cdots & 0 & W(z)/z^{n+c} \\ 0 & 1 & \cdots & 0 & e^{\bar{a}_1 z}/z^{n+c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & e^{\bar{a}_p z}/z^{n+c} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix};$$

$$(3) Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{bmatrix} z^n & 0 & \cdots & 0 & 0 \\ 0 & z^{c_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z^{c_p} & 0 \\ 0 & 0 & \cdots & 0 & z^{-(n+c)} \end{bmatrix} \text{ as } z \rightarrow \infty.$$

The unique solution of the Riemann–Hilbert problem has the polynomials P_n, Q_1, \dots, Q_p in its first row. Namely,

$$Y(z) = \begin{bmatrix} P_n(z) & Q_1(z) & \cdots & Q_p(z) & \frac{1}{2\pi i} \oint_{\gamma} \left(P_n(s) \frac{W(s)}{s^{n+c}} + \sum_{j=1}^p Q_j(s) \frac{e^{\bar{a}_j s}}{s^{n+c}} \right) \frac{ds}{s-z} \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \end{bmatrix}$$

for $z \in \mathbb{C} \setminus \gamma$. The other rows are filled with type I multiple orthogonal polynomials of slightly different degrees (for details, see [20]).

One can reduce the size of the Riemann–Hilbert problem as follows. Divide both sides of (2.6) by W and carry over P_n to the right-hand side. Then, we obtain

$$\sum_{j=1}^p Q_j(z) \frac{e^{\bar{a}_j z}}{W(z)} = -P_n(z) + O(z^{n+c}), \quad (5.2)$$

where we assume that $a_j \neq 0$ for all j . Then by Cauchy's formula we obtain

$$\frac{1}{2\pi i} \oint_{\gamma} \sum_{j=1}^p Q_j(s) \frac{e^{\bar{a}_j s}}{s^{n+c} W(s)} s^k ds = -\delta_{k,c-1}, \quad k = 0, \dots, c-1, \quad (5.3)$$

where γ is a counterclockwise-oriented closed contour going around the origin once, such that all the zeros of W lie in its exterior. This formulation of the type I multiple orthogonality leads to a Riemann–Hilbert problem of size $(p+1) \times (p+1)$.

Note that we use Y again to denote the solution of the Riemann–Hilbert problem, although this solution is different from the earlier one that we also called Y . We trust that this does not lead to any confusion.

Riemann–Hilbert problem (type I multiple orthogonality for Q_1, \dots, Q_p). *Find solution $Y: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}^{(p+1) \times (p+1)}$ such that*

(1) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$;

$$(2) Y_+(z) = Y_-(z) J_Y(z) \text{ for } z \in \gamma, \text{ where } J_Y(z) = \begin{bmatrix} 1 & \cdots & 0 & e^{\bar{a}_1 z} / (z^{n+c} W(z)) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & e^{\bar{a}_p z} / (z^{n+c} W(z)) \\ 0 & \cdots & 0 & 1 \end{bmatrix};$$

$$(3) Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{bmatrix} z^{c_1} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & z^{c_p} & 0 \\ 0 & \cdots & 0 & z^{-c} \end{bmatrix} \text{ as } z \rightarrow \infty.$$

The unique solution to this Riemann–Hilbert problem has the polynomials Q_1, \dots, Q_p in its last row,

$$Y(z) = \begin{bmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ Q_1(z) & \cdots & Q_p(z) & \frac{1}{2\pi i} \oint_{\gamma} \sum_{j=1}^p Q_j(s) \frac{e^{\bar{a}_j s}}{s^{n+c} W(s)} \frac{ds}{s-z} \end{bmatrix},$$

for $z \in \mathbb{C} \setminus \gamma$. Once we know Q_1, \dots, Q_p , then P_n can be recovered from (5.2). That is, $-P_n$ is the n -th partial sum of the Maclaurin series of $\sum_{j=1}^p Q_j(z) \frac{e^{\bar{a}_j z}}{W(z)}$.

As a remark we add that if $a_1 = 0$, then the O -term in (5.2) changes to $O(z^{n+c-c_1})$. The condition (5.3) and the Riemann–Hilbert problem similar to the above can still be written after appropriate modifications.

In a similar manner, we can single out one of the Q_j in (2.6). For the ease of notation, let us choose Q_p and rewrite (2.6) as

$$P_n(z)W(z)e^{-\bar{a}_p z} + \sum_{j=1}^{p-1} Q_j(z)e^{(\bar{a}_j - \bar{a}_p)z} = -Q_p(z) + O(z^{n+c}).$$

Then, by Cauchy's integral formula,

$$\frac{1}{2\pi i} \oint_{\gamma} \left(P_n(s) \frac{W(s)e^{-\bar{a}_p s}}{s^{n+c}} + \sum_{j=1}^{p-1} Q_j(s) \frac{e^{(\bar{a}_j - \bar{a}_p)s}}{s^{n+c}} \right) s^k ds = 0 \quad (5.4)$$

with $k = 0, \dots, n + c - c_p - 1$, where γ is an arbitrary closed contour around the origin.

The corresponding Riemann–Hilbert problem is as follows.

Riemann–Hilbert problem (type I multiple orthogonality for P_n, Q_1, \dots, Q_{p-1}). *Find solution $Y: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}^{(p+1) \times (p+1)}$ such that*

(1) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$;

$$(2) Y_+(z) = Y_-(z)J_Y(z), \quad z \in \gamma, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 & W(z)e^{-\bar{a}_p z}/z^{n+c} \\ 0 & 1 & \cdots & 0 & e^{(\bar{a}_1 - \bar{a}_p)z}/z^{n+c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & e^{(\bar{a}_{p-1} - \bar{a}_p)z}/z^{n+c} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix};$$

$$(3) Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{bmatrix} z^n & 0 & \cdots & 0 & 0 \\ 0 & z^{c_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z^{c_{p-1}} & 0 \\ 0 & 0 & \cdots & 0 & z^{-n-c+c_p} \end{bmatrix} \quad \text{as } z \rightarrow \infty.$$

The unique solution is

$$Y(z) = \begin{bmatrix} P_n(z) & Q_1(z) & \cdots & Q_{p-1}(z) & \frac{1}{2\pi i} \oint_{\gamma} \left(P_n(s) \frac{W(s)e^{-\bar{a}_p z}}{s^{n+c}} + \sum_{j=1}^{p-1} Q_j(s) \frac{e^{(\bar{a}_j - \bar{a}_p)s}}{s^{n+c}} \right) \frac{ds}{s-z} \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \end{bmatrix}$$

for $z \in \mathbb{C} \setminus \gamma$.

For $p = 1$, we recover the Riemann–Hilbert problem for the orthogonal polynomials from [3]. Indeed, if $p = 1$, $W(z) = (z - a)^c$, then (5.4) yields

$$\int_{\gamma} \frac{P_n(s)W(s)e^{-\bar{a}s}}{s^{n+c}} s^k ds = 0, \quad k = 0, \dots, n - 1,$$

which is a usual non-Hermitian orthogonality on a contour. The above Riemann–Hilbert problem reduces to the usual Riemann–Hilbert problem for orthogonal polynomials that is known from the work of Balogh, Bertola, Lee, and McLaughlin [3]. The first row of the solution is

$$Y(z) = \begin{bmatrix} P_n(z) & \frac{1}{2\pi i} \int_{\gamma} \frac{P_n(s)(s-a)^c e^{-\bar{a}s}}{s^{n+c}} \frac{ds}{s-z} \\ * & * \end{bmatrix}, \quad z \in \mathbb{C} \setminus \gamma.$$

6 Type II multiple orthogonality

The planar orthogonality corresponding to (1.1)–(1.2) is equivalent to the type II multiple orthogonality. This has been established by Lee and Yang in [17]. We give a proof in the case of polynomial W , which is essentially the same as the one in [17], except that the situation is more transparent due to the lack of branch cuts for W and ϕ_k .

Type II multiple orthogonality means that there exists functions w_1, \dots, w_p and non-negative integers n_1, \dots, n_p with $n = \sum_{k=1}^p n_k$ such that P_n is the unique polynomial of degree n satisfying

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) z^m w_k(z) dz = 0 \quad (6.1)$$

for $k = 1, \dots, p$ and $m = 0, \dots, n_k - 1$, where γ is a closed contour around the origin.

Theorem 6.1. *Suppose W is a polynomial. Given $n \in \mathbb{N}$, choose integers n_1, \dots, n_p such that*

$$\sum_{j=1}^p n_j = n, \quad \left\lfloor \frac{n}{p} \right\rfloor \leq n_j \leq \left\lceil \frac{n}{p} \right\rceil, \quad (6.2)$$

and define w_k , $k = 1, \dots, p$, by

$$w_k(z) = \int_0^{\bar{z} \times \infty} \prod_{j=1}^p (u - \bar{a}_j)^{c_j + n_j - \delta_{k,j}} e^{-uz} du. \quad (6.3)$$

Then, the planar orthogonal polynomial P_n is the unique monic polynomial of degree n that satisfies (6.1).

Note that (6.2) implies $|n_j - n_k| \leq 1$ for every $j, k = 1, \dots, p$. Euclid's division lemma yields

$$n = ap + b$$

for some $a, b \in \mathbb{N} \cup \{0\}$ such that $0 \leq b < p$. And it is clear that $a + 1$ and a will appear in (n_1, \dots, n_p) exactly b and $p - b$ times, respectively.

Proof of Theorem 6.1. Suppose that P_n is the degree n monic planar orthogonal polynomial. Take $k \in \{1, \dots, p\}$, and let m be an integer such that $0 \leq m \leq n_k - 1$. Carrying out m integration by parts in (6.3), we get

$$\begin{aligned} z^m w_k(z) &= (-1)^m \int_0^{\bar{z} \times \infty} \prod_{j=1}^p (u - \bar{a}_j)^{c_j + n_j - \delta_{k,j}} \left[\left(\frac{d}{du} \right)^m e^{-uz} \right] du \\ &= \int_0^{\bar{z} \times \infty} \left[\left(\frac{d}{du} \right)^m \prod_{j=1}^p (u - \bar{a}_j)^{c_j + n_j - \delta_{k,j}} \right] e^{-uz} du + \Pi_{k,m}(z), \end{aligned} \quad (6.4)$$

where $\Pi_{k,m}$ is a polynomial that comes from the boundary terms at $u = 0$.

Observe that

$$\left(\frac{d}{du}\right)^m \prod_{j=1}^p (u - \bar{a}_j)^{c_j + n_j - \delta_{k,j}} \quad (6.5)$$

is a polynomial in u of degree $\leq c + n - 1 - m$ with a zero of order $c_j + n_j - \delta_{k,j} - m$ at \bar{a}_j for every $j = 1, \dots, p$. From the definition (6.2) it follows that $|n_k - n_j| \leq 1$, which implies that $n_k \leq n_j + 1 - \delta_{k,j}$ for every $j = 1, \dots, p$. Using $m \leq n_k - 1$, we find that (6.5) has a zero at \bar{a}_j of order $\geq c_j$. Therefore, (6.5) is divisible by $W^*(u)$, and we can write

$$\left(\frac{d}{du}\right)^m \prod_{j=1}^p (u - \bar{a}_j)^{c_j + n_j - \delta_{k,j}} = W^*(u) Q_{k,m}(u), \quad (6.6)$$

where $Q_{k,m}$ is a polynomial of degree $\deg(Q_{k,m}) = n - 1 - m \leq n - 1$ and $Q_{k,m}$ has a zero at \bar{a}_j of order $n_j - \delta_{k,j} - m$ for every $j = 1, \dots, p$.

In view of (6.4) and (6.6), we then have

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) z^m w_k(z) dz &= \frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \left[\int_0^{\bar{z} \times \infty} W^*(u) Q_{k,m}(u) e^{-uz} du \right] dz \\ &\quad + \frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \Pi_{k,m}(z) dz. \end{aligned}$$

The second term vanishes because of Cauchy's theorem, and the remaining term vanishes because of part (b) in Theorem 2.1 and the fact that $\deg(Q_{k,m}) \leq n - 1$. Hence, (6.1) holds.

Conversely, suppose that P_n satisfies (6.1) with w_k and n_k as in the statement of the theorem. Then, by (6.4) and (6.6), we get

$$\frac{1}{2\pi i} \oint_{\gamma} P_n(z) W(z) \int_0^{\bar{z} \times \infty} W^*(u) Q_{k,m}(u) e^{-uz} du dz = 0,$$

for $k = 1, \dots, p$ and $m = 0, \dots, n_k - 1$. Again, there is no contribution from the polynomial $\Pi_{k,m}$. This leads directly to (2.3), provided that the $Q_{k,m}$ are a basis of the vector space of polynomials of degree $\leq n - 1$. Then, Theorem 2.1 tells us that P_n is the degree n planar orthogonal polynomial.

The polynomials $Q_{k,m}$ for $k = 1, \dots, p$ and $m = 0, \dots, n_k - 1$ have degrees $\leq n - 1$, and because of (6.2) there are n of them. Thus, it suffices to prove that the $Q_{k,m}$ are linearly independent. Suppose that $\beta_{k,m}$ are complex numbers such that

$$\sum_{k=1}^p \sum_{m=0}^{n_k-1} \beta_{k,m} Q_{k,m} = 0. \quad (6.7)$$

We already noted in the first part of the proof that $Q_{k,m}$ has a zero at \bar{a}_j of exact order $n_j - \delta_{k,j} - m$. Hence, $Q_{k,m}(\bar{a}_j) \neq 0$ if and only if $n_j - \delta_{k,j} - m = 0$.

From (6.2), we have that $|n_j - n_k| \leq 1$, and since $m \leq n_k - 1$, it is then easy to see that $Q_{k,m}(\bar{a}_j) \neq 0$ if and only if $m = n_k - 1$, and either $k = j$, or $k \neq j$ and $n_k = n_j + 1$. Thus, by evaluating (6.7) at \bar{a}_j , we obtain

$$\beta_{j,n_j-1} Q_{j,n_j-1}(\bar{a}_j) + \sum_{\substack{k=1 \\ n_k=n_j+1}}^p \beta_{k,n_k-1} Q_{k,n_k-1}(\bar{a}_j) = 0. \quad (6.8)$$

Suppose $n_{j_0} = \lceil \frac{n}{p} \rceil$. Then, there are no indices k with $n_k = n_{j_0} + 1$, and (6.8) implies that $\beta_{j_0, n_{j_0}-1} = 0$ since $Q_{j_0, n_{j_0}-1}(\overline{a_{j_0}}) \neq 0$. Suppose $n_{j_0} = \lfloor \frac{n}{p} \rfloor$. Then, every k with $n_k = n_{j_0} + 1$ satisfies $n_k = \lceil \frac{n}{p} \rceil$, and we just proved that $\beta_{k, n_k-1} = 0$ for such k . Thus, by using (6.8) again we obtain that $\beta_{j_0, n_{j_0}-1} = 0$. Since j_0 can be arbitrary, we conclude that $\beta_{j, n_j-1} = 0$ for all $j = 1, \dots, p$.

The formula (6.8) reduces to

$$\sum_{k=1}^p \sum_{m=0}^{n_k-2} \beta_{k,m} Q_{k,m} = 0. \quad (6.9)$$

We continue by looking at the remaining polynomials $Q_{k,m}$ in (6.9) and observe that the only ones with $\frac{d}{dz} Q_{k,m} \big|_{z=\overline{a_j}} \neq 0$ are those with $m = n_k - 2$ and either $k = j$, or $k \neq j$ and $n_k = n_j + 1$. Arguing as before we find that $\beta_{j, n_j-2} = 0$ for every $j = 1, \dots, p$. Continuing in this way by considering the higher order derivatives, we ultimately find that $\beta_{k,m} = 0$ for all $k = 1, \dots, p$ and $m = 0, \dots, n_k - 1$, which shows that the polynomials $Q_{k,m}$ are indeed linearly independent. ■

The connection between Riemann–Hilbert problems and type II multiple orthogonality is well known, see Van Assche, Geronimo, and Kuijlaars [20]. Therefore, we arrive at the following Riemann–Hilbert problem of size $(p+1) \times (p+1)$, corresponding to Theorem 6.1 (first appeared in Lee and Yang [17]).

Riemann–Hilbert problem (type II multiple orthogonality). *Let w_1, \dots, w_p be given by formula (6.3). Find $Y: \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}^{(p+1) \times (p+1)}$ satisfying*

(1) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$;

(2) $Y_+(z) = Y_-(z)J_Y(z)$ for $z \in \gamma$, where $J_Y(z) = \begin{bmatrix} 1 & w_1(z) & \cdots & w_p(z) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix};$

(3) $Y(z) = (I + O(\frac{1}{z})) \begin{bmatrix} z^n & 0 & \cdots & 0 \\ 0 & z^{-n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{-n_p} \end{bmatrix}$ as $z \rightarrow \infty$.

The unique solution has P_n in its first row

$$Y(z) = \begin{bmatrix} P_n(z) & \frac{1}{2\pi i} \oint_{\gamma} P_n(s)w_1(s) \frac{ds}{s-z} & \cdots & \frac{1}{2\pi i} \oint_{\gamma} P_n(s)w_p(s) \frac{ds}{s-z} \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \quad (6.10)$$

for $z \in \mathbb{C} \setminus \gamma$.

7 Conclusion

It is known that the planar orthogonal polynomials orthogonal with respect to the measure (1.1) are multiple orthogonal polynomials of type II (see [17]). Assuming W is a polynomial weight,

we have shown that the same planar orthogonal polynomials are also multiple orthogonal polynomials of type I. It is remarkable that the planar orthogonality manifests in two different ways at once. We are not aware of any other examples of such a phenomenon except for the case $p = 1$, where both situations reduce to the usual orthogonality.

Generally speaking, multiple orthogonality of each type, I and II, is characterized by a Riemann–Hilbert problem. Moreover, there is a canonical correspondence between such Riemann–Hilbert problems. Indeed, if Y is given by (6.10) then not only does it solve a Riemann–Hilbert problem for multiple orthogonal polynomials of type II, but also its inverse transpose Y^{-t} contains multiple orthogonal polynomials of type I in each of its rows. In the course of the paper, we have established several different Riemann–Hilbert problems connected with orthogonality of type I. However, if $p \geq 2$, then the orthogonal polynomial P_n enters the inverse transpose of (6.10) only as part of a bigger algebraic expression and not by itself as it should in the case of the canonical correspondence; therefore, neither of our Riemann–Hilbert problems associated to multiple orthogonality of type I are related to the type II problem in a canonical way.

A major interest in stating the Riemann–Hilbert problems is to use them for the asymptotic analysis. We do not address this topic in the present paper, however we mention that Lee and Yang [18] used the Riemann–Hilbert problem corresponding to multiple orthogonality of type II for asymptotic analysis in the situation where the c_j are fixed and $n \rightarrow \infty$. It would be very interesting to deal with the case of varying weights, namely, when the c_j depend on n and tend to infinity at a rate proportional to n . For $p = 1$ this was accomplished in [3], and we hope that one of the Riemann–Hilbert problems in our paper can be useful for the case $p \geq 2$.

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