Symmetry, Integrability and Geometry: Methods and Applications SIGMA **19** (2023), 025, 18 pages

The B_2 Harmonic Oscillator with Reflections and Superintegrability

Charles F. DUNKL

Department of Mathematics, University of Virginia,
PO Box 400137, Charlottesville VA 22904-4137, USA
E-mail: *cfd5z@virginia.edu*
URL: https://uva.theopenscholar.com/charles-dunkl

Received October 27, 2022, in final form April 17, 2023; Published online April 25, 2023
https://doi.org/10.3842/SIGMA.2023.025

Abstract. The two-dimensional quantum harmonic oscillator is modified with reflection terms associated with the action of the Coxeter group B_2, which is the symmetry group of the square. The angular momentum operator is also modified with reflections. The wavefunctions are known to be built up from Jacobi and Laguerre polynomials. This paper introduces a fourth-order differential-difference operator commuting with the Hamiltonian but not with the angular momentum operator; a specific instance of superintegrability. The action of the operator on the usual orthogonal basis of wavefunctions is explicitly described. The wavefunctions are classified according to the representations of the group: four of degree one and one of degree two. The identity representation encompasses the wavefunctions invariant under the group. The paper begins with a short discussion of the modified Hamiltonians associated to finite reflection groups, and related raising and lowering operators. In particular, the Hamiltonian for the symmetric groups describes the Calogero–Sutherland model of identical particles on the line with harmonic confinement.

Key words: Dunkl harmonic oscillator; dihedral symmetry; superintegrability; Laguerre polynomials; Jacobi polynomials

2020 Mathematics Subject Classification: 81R12; 37J35; 33C45; 81Q05

1 Introduction

The two-dimensional quantum harmonic oscillator is modified with reflection terms associated with the action of the Coxeter group B_2, the symmetry group of the square. The wavefunctions are known to be built up from Jacobi and Laguerre polynomials. This paper introduces a fourth-order differential-difference operator commuting with the Hamiltonian but not with the angular momentum operator; a specific instance of superintegrability. The action of the operator on an orthogonal basis of wavefunctions is explicitly described. The wavefunctions are not in general invariant under the group, rather are classified by the representations of the group: four of degree one and one of degree two. The group-invariant wavefunctions of the B_2 oscillator and its superintegrability have been studied by Tremblay et al. [10, 11], Quesne [9].

First the general background on finite reflection groups and root systems, Dunkl operators, and the associated Hamiltonian is described. In particular, the Calogero–Sutherland model of N identical particles on a line with r^{-2} interaction and harmonic confinement comes from the symmetric group (Lassalle [8], Baker and Forrester [1]). In the general situation, there are raising and lowering operators which can be used to construct operators commuting with the Hamiltonian and the group action. After this the development turns to dihedral groups (type $I_2(k)$) and the use of a complex coordinate system, which simplifies the description of rotations. Some general formulas are specialized to this setting.

The description of the wavefunctions of B_2 and of the action of specific operators is in Sections 4 and 6. The important operators are the Hamiltonian $\mathcal{H}$, the angular momentum $\mathcal{J}$ and a new operator $\mathcal{K}$ which commutes with $\mathcal{H}$ and the group action but not with $\mathcal{J}^2$. This property constitutes superintegrability. There are a number of different classes of wavefunctions, requiring frequent case-by-case analysis. The explicit action of $\mathcal{K}$ on the orthogonal basis of wavefunctions is found in Section 7.

In the appendix, there are details on some proofs, and a sketch of a symbolic computation method of proving relations involving polynomials and Dunkl operators.

2 Reflection groups and a harmonic oscillator

In $\mathbb{R}^N$ the inner product is $\langle x,y\rangle := \sum_{i=1}^N x_iy_i$ and $\|x\|^2 = \langle x,x\rangle$. If $v \neq 0$, then the reflection σ_v along v is defined by

$$x\sigma_v := x - 2\frac{\langle x,v\rangle}{\|v\|^2}v.$$

This is an isometry $\|x\sigma_v\|^2 = \|x\|^2$ and an involution $\sigma_v^2 = I$. The set of fixed points ($x\sigma_v = x$) is the hyperplane $\{x\colon \langle x,v\rangle = 0\}$. A finite root system is a subset R of nonzero elements of $\mathbb{R}^N$ satisfying $u,v \in R$ implies $u\sigma_v \in R$. We restrict consideration to reduced root systems, that is if $u, cu \in R$, then $c = \pm 1$. Define $W(R)$ to be the group generated by $\{\sigma_v\colon v \in R\}$; this is a finite subgroup of the orthogonal group $O_N(\mathbb{R})$. There is a decomposition of R into R_+ (the positive roots) and R_-; this relies on choice of a vector u such that $\langle u,v\rangle \neq 0$ for all $v \in R$ then set $R_+ = \{v \in R\colon \langle u,v\rangle > 0\}$. Since $\sigma_v = \sigma_{-v}$, the set R_+ can be used to index the reflections in $W(R)$. The set of reflections σ_v decomposes into conjugacy classes (W orbits) $\sigma_u \sim \sigma_v$ if $u = vw$ for some $w \in W(R)$. A *multiplicity function* κ_v is a function on R which is constant on each conjugacy class, usually here $\kappa_v \geq 1$. Set $\gamma_\kappa := \sum_{v\in R_+}\kappa_v$. Define the Dunkl operator ($1 \leq i \leq N$)

$$\mathcal{D}_if(x) := \frac{\partial}{\partial x_i}f(x) + \sum_{v\in R_+}\kappa_v\frac{f(x) - f(x\sigma_v)}{\langle x,v\rangle}v_i.$$

Then $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$ for all i, j (Dunkl [2], also see Dunkl and Xu [4, Theorem 6.4.8]). Let

$$\nabla = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_N}\right), \qquad \Delta = \sum_{i=1}^N\left(\frac{\partial}{\partial x_i}\right)^2 \qquad \text{and} \qquad \nabla_\kappa = (\mathcal{D}_1,\dots,\mathcal{D}_N).$$

The Dunkl Laplacian is $\Delta_\kappa := \sum_{i=1}^N \mathcal{D}_i^2$ and

$$\Delta_\kappa f(x) = \Delta f(x) + \sum_{v\in R_+}\kappa_v\left(2\frac{\langle\nabla f(x),v\rangle}{\langle x,v\rangle} - \|v\|^2\frac{f(x)-f(x\sigma_v)}{\langle x,v\rangle^2}\right).$$

This leads to the modified Schrödinger equation (with parameter $\omega > 0$)

$$\mathcal{H}\psi := \big(\omega^2\|x\|^2 - \Delta_\kappa\big)\psi = E\psi.$$

The exponential ground state is $g(x) := \exp\big(-\frac{\omega}{2}\|x\|^2\big)$, as can be seen from the transformation

$$g^{-1}\big(\omega^2\|x\|^2 - \Delta_\kappa\big)(fg) = -\Delta_\kappa f + \omega(N + 2\gamma_\kappa + 2\langle x,\nabla\rangle)f,$$

which implies $\big(\omega^2\|x\|^2 - \Delta_\kappa\big)g = \omega(N+2\gamma_\kappa)g$. An equivalent expression is

$$g^{-1}\mathcal{H}g = -\Delta_\kappa + \omega\sum_{i=1}^N(x_i\mathcal{D}_i + \mathcal{D}_ix_i). \tag{2.1}$$

Denote the set of polynomials on $\mathbb{R}^N$ by $\mathcal{P}$ and the set of polynomials homogeneous of degree n by $\mathcal{P}_n$ (that is, $p(cx) = c^n p(x)$ for $c \in \mathbb{R}$). Let $\mathcal{H}_{\kappa,n} = \{p \in \mathcal{P}_n \colon \Delta_\kappa p = 0\}$ (*κ-harmonic* polynomials). We find eigenfunctions of $g^{-1}\mathcal{H}g$ of the form $p(x)q\big(\omega\|x\|^2\big)$ with $p \in \mathcal{H}_{\kappa,n}$ (thus $\Delta_\kappa p = 0$ and $\langle x, \nabla\rangle p = np$). This gives the differential equation (where $t = \omega\|x\|^2$)

$$t\frac{\mathrm{d}^2}{\mathrm{d}t^2}q + \left(n + \frac{N}{2} + \gamma_\kappa - t\right)\frac{\mathrm{d}}{\mathrm{d}t}q - \frac{1}{4}\left(2n + N + 2\gamma_\kappa - \frac{E}{\omega}\right)q = 0$$

and the solution is the Laguerre polynomial $q(t) = L_m^{(\alpha)}(t)$, $m = 0, 1, 2, \dots$, $\alpha = \gamma_\kappa + n + \frac{N}{2} - 1$, $E = \omega(N + 2\gamma_\kappa + 2n + 4m)$. Note E depends on $\deg(pq) = n + 2m$. The Laguerre polynomial of degree n and index $\alpha > -1$ satisfies

$$L_n^{(\alpha)}(t) := \frac{(\alpha+1)_n}{n!}\sum_{j=0}^{n}\frac{(-n)_j}{(\alpha+1)_j}\frac{t^j}{j!},$$
$$\int_0^\infty L_m^{(\alpha)}(t)L_k^{(\alpha)}(t)t^\alpha \mathrm{e}^{-t}\,\mathrm{d}t = \delta_{mk}\frac{\Gamma(\alpha+1+m)}{m!} = \delta_{mk}\Gamma(\alpha+1)\frac{(\alpha+1)_m}{m!}.$$

The Pochhammer symbol is $(a)_n = \prod_{i=1}^{n}(a + i - 1)$ (or $(a)_0 = 1$ and $(a)_{n+1} = (a+n)(a)_n$).

There is an orthogonality structure which uses the $W(R)$-invariant weight function

$$h_\kappa(x) := \prod_{v\in R_+}|\langle x, v\rangle|^{\kappa_v}$$

positively homogeneous of degree γ_κ. The orthogonality $\mathcal{H}_{\kappa,n} \perp \mathcal{H}_{\kappa,m}$ for $n \neq m$ holds with respect to the measure $h_\kappa(x)^2\mathrm{d}\mu(x)$ on the sphere $S^{N-1} := \{x \colon \|x\| = 1\}$, where μ is the rotation-invariant surface measure. There is a key result on adjoints: suppose p, q are sufficiently smooth and have exponential decay then (with $1 \le i \le N$)

$$\int_{\mathbb{R}^N}(\mathcal{D}_i p)q h_\kappa^2\,\mathrm{d}m = -\int_{\mathbb{R}^N} p(\mathcal{D}_i q)h_\kappa^2\,\mathrm{d}m, \tag{2.2}$$

where $\mathrm{d}m$ is Lebesgue measure on $\mathbb{R}^N$ (see [4, Theorem 7.7.10]). Thus the adjoint of $\mathcal{D}_i$ is defined on a dense subspace of $L^2\big(\mathbb{R}^N, h_\kappa^2\mathrm{d}m\big)$ and $\mathcal{D}_i^* = -\mathcal{D}_i$. This meaning of adjoint will be used throughout. Furthermore, the conjugate of $\mathcal{H}$ is

$$h_\kappa\big(-\Delta_\kappa + \omega^2\|x\|^2\big)h_\kappa^{-1} = -\Delta + \omega^2\|x\|^2 + \sum_{v\in R_+}\frac{\kappa_v(\kappa_v - \sigma_v)\|v\|^2}{\langle x, v\rangle^2},$$

(details of the derivation are in Appendix A) a Schrödinger equation with the potential

$$V(x) = \omega^2\|x\|^2 + \sum_{v\in R_+}\frac{\kappa_v(\kappa_v - \sigma_v)\|v\|^2}{\langle x, v\rangle^2},$$

which includes reflections. The ground state is $h_\kappa g$. For the special case where R is the root system of type A_{N-1} and $W(R) = \mathcal{S}_N$ (the symmetric group), this potential occurs in the Calogero–Sutherland model of N identical particles on a line with r^{-2} interaction potential and harmonic confinement. There is a closely related model of N identical particles on a circle with r^{-2} interaction, called the trigonometric model. The wavefunctions are Jack polynomials in the variables $x_j = \mathrm{e}^{\mathrm{i}\theta_j}$, $1 \le j \le N$. Lapointe and Vinet [7] defined raising and lowering operators and found Rodrigues formulas for the Jack polynomials arising in this model. The Jack polynomials can be used as bases for generalized Hermite (Lassalle [8]) and Laguerre polynomials, which occur as wavefunctions in types A and B models on the line (Baker and Forrester [1], also see [4, Section 11.6.3]).

We need the basic commutation relations ($[A,B] := AB - BA$) for $a, b \in \mathbb{R}^N$:

$$[\langle a, \nabla_\kappa\rangle, \langle b, x\rangle] = \langle a, b\rangle + 2\sum_{v\in R_+} \kappa_v \frac{\langle a, v\rangle\langle b, v\rangle}{\|v\|^2}\sigma_v, \tag{2.3}$$

$$[\Delta_\kappa, \langle b, x\rangle] = 2\langle b, \nabla_\kappa\rangle, \qquad \left[\|x\|^2, \langle a, \nabla_\kappa\rangle\right] = -2\langle a, x\rangle. \tag{2.4}$$

Definition 2.1. For $a, b \in \mathbb{R}^N$, the angular momentum operator is $J_{a,b} := \langle a, x\rangle\langle b, \nabla_\kappa\rangle - \langle b, x\rangle\langle a, \nabla_\kappa\rangle$.

Proposition 2.2. *$J_{a,b} = \langle b, \nabla_\kappa\rangle\langle a, x\rangle - \langle a, \nabla_\kappa\rangle\langle b, x\rangle$; $J^*_{a,b} = -J_{a,b}$ and $[\mathcal{H}, J_{a,b}] = 0$.*

Proof. From (2.3), the commutator $[\langle a, x\rangle, \langle b, \nabla_\kappa\rangle] = -[\langle b, x\rangle, \langle a, \nabla_\kappa\rangle]$. This proves the first statement. By (2.2), $J^*_{a,b} = -\langle a, x\rangle\langle b, \nabla\kappa\rangle + \langle b, x\rangle\langle a, \nabla_\kappa\rangle = -J_{a,b}$. Next by (2.4),

$$\begin{aligned}
[\mathcal{H}, \langle b, \nabla_\kappa\rangle\langle a, x\rangle] &= \omega^2\left\{\|x\|^2\langle b, \nabla_\kappa\rangle\langle a, x\rangle - \langle b, \nabla_\kappa\rangle\|x\|^2\langle a, x\rangle\right\} \\
&\quad - \{\Delta_\kappa\langle b, \nabla_\kappa\rangle\langle a, x\rangle - \langle b, \nabla_\kappa\rangle\langle a, x\rangle\Delta_\kappa\} \\
&= \omega^2\left[\|x\|^2, \langle b, \nabla_\kappa\rangle\right]\langle a, x\rangle - \langle b, \nabla_\kappa\rangle[\Delta_\kappa, \langle a, x\rangle] \\
&= -2\omega^2\langle b, x\rangle\langle a, x\rangle - 2\langle b, \Delta_\kappa\rangle\langle a, \nabla_\kappa\rangle,
\end{aligned}$$

and this expression is symmetric in a, b and thus $[\mathcal{H}, J_{a,b}] = 0$. ∎

Corollary 2.3. *$[\Delta_\kappa, J_{a,b}] = 0$ and $\left[\|x\|^2, J_{a,b}\right] = 0$.*

This family of angular momentum operators has been studied by Feigin and Hakobyan [6], especially in connection with the symmetric group and the Calogero–Moser model.

We introduce raising and lowering operators. These operators were used by Feigin [5] in his study of generalized Calogero–Moser models, which are constructed in terms of subdiagrams (certain subsets of roots) of the Coxeter diagram of $W(R)$. Note $\{A, B\} := AB + BA$.

Definition 2.4. For $a \in \mathbb{R}^N$, $a \neq 0$, let $A_a^\pm = \omega\langle a, x\rangle \pm \langle a, \nabla_\kappa\rangle$ and $H_a := \frac{1}{2}\{A_a^+, A_a^-\} = \omega^2\langle a, x\rangle^2 - \langle a, \nabla_\kappa\rangle^2$.

Proposition 2.5. *$(A_a^+)^* = A_a^-$; $g^{-1}A_a^+g = \langle a, \nabla_\kappa\rangle$ (lowering) and $g^{-1}A_a^-g = 2\omega\langle a, x\rangle - \langle a, \nabla_\kappa\rangle$ (raising); $H_a^* = H_a$ and $[\mathcal{H}, H_a] = 0$. Also $g^{-1}H_ag = \omega(\langle a, x\rangle\langle a, \nabla_\kappa\rangle + \langle a, \nabla_\kappa\rangle\langle a, x\rangle) - \langle a, \nabla_\kappa\rangle^2$.*

Proof. From (2.2), it follows that $(A_a^+)^* = A_a^-$ and $H_a^* = H_a$. The commutator

$$\left[\omega^2\langle a, x\rangle^2 - \langle a, \nabla_\kappa\rangle^2, \mathcal{H}\right] = -\left[\langle a, \nabla_\kappa\rangle^2, \omega^2\|x\|^2\right] - \left[\omega^2\langle a, x\rangle^2, \Delta_\kappa\right]$$

and expanding the right hand side with formulas (2.4) and $\left[A^2, B\right] = A[A, B] + [A, B]A$ shows $[H_a, \mathcal{H}] = 0$. ∎

Proposition 2.6. *Suppose $w \in W(R)$, then $w^{-1}H_aw = H_{aw}$; suppose $S \subset R_+$ and $S \cup (-S)$ is an $W(R)$-orbit (closed under $v \to vw$), then $\sum_{v\in S} H_v^k$ commutes with each $w \in W(R)$ for $k = 1, 2, 3, \ldots$.*

Proof. This follows from $\langle a, \nabla_\kappa\rangle w = w\langle aw, \nabla_\kappa\rangle$ (see [4, Proposition 6.4.3]) and

$$w(\langle aw, x\rangle p(x)) = \langle aw, xw\rangle p(xw) = \langle a, x\rangle wp(x)$$

(because $w \in O_N(\mathbb{R})$). ∎

This produces a collection of self-adjoint operators commuting with $W(R)$ and $\mathcal{H}$.

3 The dihedral groups

For $m = 3, 4, \ldots$, the dihedral group $I_2(m)$ is the symmetry group of the regular m-gon. We will use complex coordinates for $\mathbb{R}^2$:

$$z := x_1 + \mathrm{i}x_2, \qquad \overline{z} := x_1 - \mathrm{i}x_2.$$

Let $\zeta := \exp\left(\frac{2\pi\mathrm{i}}{m}\right)$, then the reflections in $I_2(m)$ are $\sigma_j\colon (z, \overline{z}) \to \left(\overline{z}\zeta^j, z\zeta^{-j}\right)$ $(0 \le j < m)$, and the rotations are $\rho_j\colon (z, \overline{z}) \to \left(z\zeta^j, \overline{z}\zeta^{-j}\right)$. Then $\sigma_k\sigma_j\sigma_k = \sigma_{2k-j}$ and $\rho_k^{-1}\sigma_j\rho_k = \sigma_{j+2k}$; when m is even, there are two conjugacy classes $\{\sigma_{2j}\}$ and $\{\sigma_{2j+1}\}$ with $0 \le j \le \frac{m}{2} - 1$. The real root vector for σ_j is $v_j := \left(\sin\left(\frac{\pi j}{m}\right), -\cos\left(\frac{\pi j}{m}\right)\right)$ and $\langle x, v_j\rangle = \frac{\mathrm{i}}{2}\exp\left(-\frac{j\pi\mathrm{i}}{m}\right)\left(z - \zeta^j\overline{z}\right)$. The Dunkl operators are $\left(\partial_z := \frac{\partial}{\partial z}, \partial_{\overline{z}} := \frac{\partial}{\partial \overline{z}}\right)$

$$\begin{aligned} Tf(z) &= \partial_z f(z) + \sum_{j=0}^{m-1} \kappa_j \frac{f(z) - f\left(\overline{z}\zeta^j\right)}{z - \overline{z}\zeta^j} = \frac{1}{2}(\mathcal{D}_1 - \mathrm{i}\mathcal{D}_2)f, \\ \overline{T}f(z) &= \partial_{\overline{z}} f(z) - \sum_{j=0}^{m-1} \kappa_j \frac{f(z) - f\left(\overline{z}\zeta^j\right)}{z - \overline{z}\zeta^j}\zeta^j = \frac{1}{2}(\mathcal{D}_1 + \mathrm{i}\mathcal{D}_2)f. \end{aligned}$$

These imply $\Delta_\kappa = 4T\overline{T}$. If m is odd, then $\kappa_j = \kappa$; if m is even then $\kappa_{2j} = \kappa_0$ and $\kappa_{2j+1} = \kappa_1$ for all j. Denote H_{v_j} by H_j and let $\widehat{H}_j = g^{-1}H_j g$. In the complex coordinates,

$$\langle v_j, x\rangle = \frac{\mathrm{i}}{2}\exp\left(-\frac{j\pi\mathrm{i}}{m}\right)\left(z - \zeta^j\overline{z}\right), \qquad \langle v_j, \nabla_\kappa\rangle = -\mathrm{i}\left(\exp\frac{j\pi\mathrm{i}}{m}\right)\left(T - \zeta^{-j}\overline{T}\right),$$

$\left(\text{note } \left(\exp\frac{j\pi\mathrm{i}}{m}\right)^2 = \zeta^j\right)$, thus

$$\langle v_j, x\rangle\langle v_j, \nabla_\kappa\rangle = \frac{1}{2}\left(z - \zeta^j\overline{z}\right)\left(T - \zeta^{-j}\overline{T}\right)$$

and

$$\langle v_j, \nabla_\kappa\rangle^2 = -\zeta^j\left(T - \zeta^{-j}\overline{T}\right)^2 = -\zeta^j T^2 + 2T\overline{T} - \zeta^{-j}\overline{T}^2.$$

Then using (2.1),

$$\begin{aligned} \widehat{H}_j &= \zeta^j T^2 - 2T\overline{T} + \zeta^{-j}\overline{T}^2 + \frac{\omega}{2}\left\{\left(z - \zeta^j\overline{z}\right)\left(T - \zeta^{-j}\overline{T}\right) + \left(T - \zeta^{-j}\overline{T}\right)\left(z - \zeta^j\overline{z}\right)\right\}, \\ g^{-1}\mathcal{H}g &= -4T\overline{T} + \omega\left\{zT + \overline{z}\overline{T} + Tz + \overline{T}\overline{z}\right\}. \end{aligned} \tag{3.1}$$

In $\mathbb{R}^2$ there is only one angular momentum operator (up to scalar multiplication), namely $x_1\mathcal{D}_2 - x_2\mathcal{D}_1 = \mathrm{i}\left(zT - \overline{z}\overline{T}\right)$. Set $\mathcal{J} := zT - \overline{z}\overline{T}$.

The κ-harmonic polynomials can be found in [4, Section 7.6]; they are expressed in terms of Gegenbauer, respectively Jacobi, polynomials, in case of odd m, respectively even m.

4 Orthogonal basis of wavefunctions for B_2

Henceforth, we specialize to the group $B_2 = I_2(4)$. The formulas in the previous section apply with $\zeta = \mathrm{i}$. Let $\gamma_\kappa = 2\kappa_0 + 2\kappa_1$. The weight function $h_\kappa = \left|z^2 - \overline{z}^2\right|^{\kappa_0}\left|z^2 + \overline{z}^2\right|^{\kappa_1}$. The group has five irreducible representations: four of degree one and one of degree two. The four multiplicative characters satisfy $\chi_0(\sigma_k) = 1$, $\chi_1(\sigma_k) = (-1)^k$, $\chi_2(\sigma_k) = (-1)^{k+1}$, $\chi_3(\sigma_k) = -1$, $0 \le k \le 3$. The basis of wavefunctions (solutions of $\mathcal{H}\psi = 2\omega(n + 1 + \gamma_\kappa)\psi$) are denoted $\psi_{n-j,j}$ where the subscript refers to a dominant monomial in the polynomial part (ignoring g) and monomials $z^a\overline{z}^b$ $(a+b=n)$ are ordered by $|a-b|$. The factor g in the wavefunctions will be omitted and we use operators in the form $g^{-1}Ag$ acting on polynomials.

The basis functions are all expressed in the following:

$$\begin{aligned} R_n^{(\alpha,\beta)}(z) &:= (z\overline{z})^{2n} P_n^{(\alpha,\beta)}\left(\frac{z^4+\overline{z}^4}{2z^2\overline{z}^2}\right) \\ &= \frac{(-1)^n}{2^{2n}n!}\sum_{j=0}^{n}\binom{n}{j}(-n-\alpha)_{n-j}(-n-\beta)_j\left(z^2-\overline{z}^2\right)^{2j}\left(z^2+\overline{z}^2\right)^{2n-2j}. \end{aligned}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ of degree n and indices α, β can be defined as (see [4, Proposition 4.14])

$$P_n^{(\alpha,\beta)}(t) := \frac{(\alpha+1)_n}{n!}\left(\frac{1+t}{2}\right)^n \sum_{j=0}^{n}\frac{(-n)_j(-n-\beta)_j}{(\alpha+1)_j\ j!}\left(\frac{t-1}{t+1}\right)^j;$$

this formula leads to the expression stated above. Then define

$$\begin{aligned} p_{4n,00}(z) &:= R_n^{(\kappa_0-1/2,\kappa_1-1/2)}(z), \\ p_{4n,11}(z) &:= \left(z^4-\overline{z}^4\right)R_{n-1}^{(\kappa_0+1/2,\kappa_1+1/2)}(z), \\ p_{4n+2,10}(z) &:= \left(z^2+\overline{z}^2\right)R_n^{(\kappa_0-1/2,\kappa_1+1/2)}(z), \\ p_{4n+2,01}(z) &:= \left(z^2-\overline{z}^2\right)R_n^{(\kappa_0+1/2,\kappa_1-1/2)}(z). \end{aligned}$$

These are of isotype χ_0, χ_3, χ_1, χ_2, respectively. The L^2-norms are necessary for normalization, and are derived from

$$\begin{aligned} &\int_0^{\pi/2}\sin^{2\alpha}\theta\cos^{2\beta}\theta P_n^{(\alpha-\frac12,\beta-\frac12)}(\cos 2\theta)P_k^{(\alpha-\frac12,\beta-\frac12)}(\cos 2\theta)\,\mathrm{d}\theta \\ &\qquad = \frac12\delta_{nk}B\left(\alpha+\frac12,\beta+\frac12\right)\frac{\left(\alpha+\frac12\right)_n\left(\beta+\frac12\right)_n(\alpha+\beta+n)}{n!(\alpha+\beta+1)_n(\alpha+\beta+2n)}. \end{aligned}$$

The beta function B is defined by a definite integral and satisfies $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$. Denote for polynomials $p(z,\overline{z})$

$$\begin{aligned} \|p\|_{\mathbb{T}}^2 &:= \int_{-\pi}^{\pi}\left|p\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 h_\kappa\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2\mathrm{d}\theta, \\ \|p\|^2 &:= \int_0^{\infty}\exp\left(-\omega r^2\right)r^{2\gamma_\kappa+1}\mathrm{d}r\int_{-\pi}^{\pi}\left|p\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 h_\kappa\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2\mathrm{d}\theta, \end{aligned}$$

then

$$\|1\|_{\mathbb{T}}^2 = 2^{\gamma_\kappa+1}B\left(\kappa_0+\frac12,\kappa_1+\frac12\right).$$

The squared norms are

$$\begin{aligned} \|p_{4n,00}\|_{\mathbb{T}}^2 &= \frac{\left(\kappa_0+\frac12\right)_n\left(\kappa_1+\frac12\right)_n(\kappa_0+\kappa_1+n)}{n!(\kappa_0+\kappa_1+1)_n(\kappa_0+\kappa_1+2n)}\|1\|_{\mathbb{T}}^2, \\ \|p_{4n,11}\|_{\mathbb{T}}^2 &= 16\frac{\left(\kappa_0+\frac12\right)_n\left(\kappa_1+\frac12\right)_n(\kappa_0+\kappa_1+n+1)}{(n-1)!(\kappa_0+\kappa_1+1)_n(\kappa_0+\kappa_1+2n)}\|1\|_{\mathbb{T}}^2, \\ \|p_{4n+2,10}\|_{\mathbb{T}}^2 &= \frac{4\left(\kappa_0+\frac12\right)_n\left(\kappa_1+\frac32\right)_n}{n!(\kappa_0+\kappa_1+1)_n(\kappa_0+\kappa_1+2n+1)}\|1\|_{\mathbb{T}}^2, \\ \|p_{4n+2,01}\|_{\mathbb{T}}^2 &= \frac{4\left(\kappa_0+\frac32\right)_n\left(\kappa_1+\frac12\right)_n}{n!(\kappa_0+\kappa_1+1)_n(\kappa_0+\kappa_1+2n+1)}\|1\|_{\mathbb{T}}^2. \end{aligned}$$

For the odd degrees,

$$p_{4n+1}(z) := \left\{ p_{4n,00}(z) + \frac{1}{4} p_{4n,11}(z) \right\}, \tag{4.1}$$

$$p_{4n+3}(z) := z\left\{ \left(n + \kappa_0 + \frac{1}{2}\right) p_{4n+2,10}(z) + \left(n + \kappa_1 + \frac{1}{2}\right) p_{4n+2,01}(z) \right\}. \tag{4.2}$$

From the orthogonality relations $p_{4n,00} \perp p_{4n,11}$ and $p_{4n+2,10} \perp p_{4n+2,01}$ (different isotypes),

$$\|p_{4n+1}\|_{\mathbb{T}}^2 = \|p_{4n,00}\|_{\mathbb{T}}^2 + \frac{1}{16}\|p_{4n,11}\|_{\mathbb{T}}^2 = \frac{\left(\kappa_0 + \frac{1}{2}\right)_n \left(\kappa_1 + \frac{1}{2}\right)_n}{n!(\kappa_0 + \kappa_1 + 1)_n} \|1\|_{\mathbb{T}}^2,$$

$$\begin{aligned} \|p_{4n+3}\|_{\mathbb{T}}^2 &= \left(n + \kappa_0 + \frac{1}{2}\right)^2 \|p_{4n+2,10}\|_{\mathbb{T}}^2 + \left(n + \kappa_1 + \frac{1}{2}\right)^2 \|p_{4n+2,01}\|_{\mathbb{T}}^2 \\ &= 4\frac{\left(\kappa_0 + \frac{1}{2}\right)_{n+1} \left(\kappa_1 + \frac{1}{2}\right)_{n+1}}{n!(\kappa_0 + \kappa_1 + 1)_n} \|1\|_{\mathbb{T}}^2. \end{aligned}$$

Next we list the orthogonal basis, which involves the Laguerre polynomials. The subscript notation may appear strange, but it makes it easy to identify the isotype and every possibility of $(n - j, j)$ can be found by suitably replacing n (the trailing factor g is understood),

$$\begin{aligned} \psi_{4n+j,j}(z) &= p_{4n,00}(z) L_j^{(\gamma_\kappa+4n)}(\omega z\overline{z}), \\ \psi_{4n+2+j,j}(z) &= p_{4n+2,10}(z) L_j^{(\gamma_\kappa+4n+2)}(\omega z\overline{z}), \\ \psi_{j,4n+j}(z) &= p_{4n,11}(z) L_j^{(\gamma_\kappa+4n)}(\omega z\overline{z}), \\ \psi_{j,4n+2+j}(z) &= p_{4n+2,01}(z) L_j^{(\gamma_\kappa+4n+2)}(\omega z\overline{z}). \end{aligned}$$

In this list, $\sigma_0\psi_{2n-j,j} = \psi_{2n-j,j}$ and $\sigma_0\psi_{j,2n-j} = -\psi_{j,2n-j}$, for $0 \le j \le n$ ($j < n$ for the second case). For odd degrees,

$$\begin{aligned} \psi_{4n+1+j,j}(z) &= p_{4n+1}(z) L_j^{(\gamma_\kappa+4n+1)}(\omega z\overline{z}), \\ \psi_{4n+3+j,j}(z) &= p_{4n+3}(z) L_j^{(\gamma_\kappa+4n+3)}(\omega z\overline{z}), \\ \psi_{j,4n+1+j}(z) &= \sigma_0\psi_{4n+1+j,j}(z), \\ \psi_{j,4n+3+j}(z) &= \sigma_0\psi_{4n+3+j,j}(z). \end{aligned}$$

By construction, $\mathcal{H}\psi_{n-j,j} = E_n\psi_{n-j,j}$, where the energy eigenvalue is $E_n := 2\omega(n+2\kappa_0+2\kappa_1+1)$. The squared norms of the ψ follow from the formula

$$\|\psi\|^2 = \frac{1}{2}\omega^{-(n+\gamma_\kappa+1)} \frac{\Gamma(\gamma_\kappa + n + \ell + 1)}{\ell!} \|p\|_{\mathbb{T}}^2,$$

where $p(z)$ is homogeneous of degree n and $\psi(z) = p(z)L_\ell^{(\gamma_\kappa+n)}(\omega z\overline{z})$. When $\omega = 1$, the wavefunctions $\psi_{n-j,j}$ are eigenfunctions of the Dunkl transform with eigenvalue $(-\mathrm{i})^n$ (see [4, Theorem 7.7.5]).

5 Some self-adjoint operators

5.1 General properties

In this section, we are concerned with finding the action of operators which commute with $\mathcal{H}$ on the basis functions described above. Suppose A is such an operator and A is self-adjoint (in $L^2(\mathbb{R}^2, h_\kappa^2 \mathrm{d}m_2)$, then for any $(n-j,j)$ the polynomial $A\psi_{n-j,j}$ is an eigenfunction of $\mathcal{H}$ with the same eigenvalue $2\omega(n+1+\gamma_\kappa)$ and has an expansion $\sum_{i=0}^n c_i\psi_{n-i,i}$ (note that $\mathrm{d}m_2$ denotes the $\mathbb{R}^2$ Lebesgue measure and equals $r\mathrm{d}r\ \mathrm{d}\theta$ for $z = r\mathrm{e}^{\mathrm{i}\theta}$). Suppose it is known that the top-degree (nonzero) monomials $z^{n-i}\overline{z}^i$ in $A\psi_{n-j,j}$ satisfy $m \le i \le N-m$, then $\psi_{n-k,k}$ for $k < m$ or $n-m < k \le n$ cannot appear in the expansion of $A\psi_{n-j,j}$. This is an implicit inductive argument: if z^n and $\overline{z}^n$ do not appear, then neither $\psi_{n,0}$ nor $\psi_{0,n}$ can appear in the expansion, now consider $z^{n-1}\overline{z}$ and $z\overline{z}^{n-1}$, $\psi_{n-1,1}$ and $\psi_{1,n-1}$ and so on. It also follows that it suffices to consider the top-degree terms to find the coefficients of the expansion. The top degree terms of $\psi_{n-j,j}$ are scalar multiples of $z^{n-j}\overline{z}^j \pm z^j\overline{z}^{n-j}$ if n is even, and of $z^{n-j}\overline{z}^j$ (or $z^j\overline{z}^{n-j}$) if n is odd and $n-j > j$ (or $n-j < j$).

Definition 5.1. For a polynomial $p(z,\overline{z})$, let $\mathcal{C}(p, z^m\overline{z}^n)$ denote the coefficient of $z^m\overline{z}^n$ in the expansion of p. If p can be expanded in a series of wavefunctions, then $\mathcal{C}(p,\psi_{n-j,j})$ denotes the coefficient of $\psi_{n-j,j}$.

Suppose, as above, that $[A,\mathcal{H}] = 0$ and A is self-adjoint, then

$$\mathcal{C}(A\psi_{n-j,j},\psi_{n-k,k})\|\psi_{n-k,k}\|^2 = \overline{\mathcal{C}}(A\psi_{n-k,k},\psi_{n-j,j})\|\psi_{n-j,j}\|^2; \tag{5.1}$$

generally the coefficients we use are real and the complex conjugate on $\mathcal{C}$ can be omitted. In particular, $\mathcal{C}(A\psi_{n-j,j},\psi_{n-k,k}) = 0$ implies $\mathcal{C}(A\psi_{n-k,k},\psi_{n-j,j}) = 0$.

5.2 Angular momentum

Here $\mathcal{J} = zT - \overline{z}\overline{T}$, from Proposition 2.2 we have $\mathcal{J}^* = -\mathcal{J}$ and $[\mathcal{J},\mathcal{H}] = 0$. Also $g^{-1}\mathcal{J}g = \mathcal{J}$ (because $\mathcal{J}(z\overline{z})^k = 0$). We determine the effect of $\mathcal{J}$ on $\psi_{n-j,j}$ by considering the dominant top-degree monomials. This suffices because $[\mathcal{J},\Delta_\kappa] = 0$ and the image of a κ-harmonic polynomial under $\mathcal{J}$ is κ-harmonic, and there are only two (independent) κ-harmonic polynomials of each degree (≥ 1)

$$\begin{aligned}
\mathcal{J}z^n &= nz^n + \sum_{j=0}^{3} \kappa_{j \bmod 2} \frac{z^n - (\mathrm{i}^j\overline{z})^n}{z - \mathrm{i}^j\overline{z}}(z + \mathrm{i}^j\overline{z}) \\
&= nz^n + 2(\kappa_0+\kappa_1)z^n + \big\{\kappa_0(1+(-1)^n) + \kappa_1\big(\mathrm{i}^n + (-\mathrm{i})^n\big)\big\}\overline{z}^n + \cdots \\
&= (n+2\kappa_0+2\kappa_1)z^n + \big(1+(-1)^n\big)\big(\kappa_0 + \mathrm{i}^n\kappa_1\big)\overline{z}^n + \cdots,
\end{aligned}$$

omitting terms like $z^{n-j}\overline{z}^j$ with $1 \le j < n$. Also

$$\mathcal{J}\overline{z}^n = -(n+2\kappa_0+2\kappa_1)\overline{z}^n - \big(1+(-1)^n\big)\big(\kappa_0+\mathrm{i}^n\kappa_1\big)z^n + \cdots$$

because $\sigma_0\mathcal{J}\sigma_0 = -\mathcal{J}$. If n is odd, then

$$\mathcal{J}z^n = (n+2\kappa_0+2\kappa_1)z^n + \cdots \qquad \text{and} \qquad \mathcal{J}^2z^n = (n+2\kappa_0+2\kappa_1)^2z^n + \cdots.$$

Thus

$$\begin{aligned}
\mathcal{J}\psi_{n+j,j} &= \mathcal{J}\big(p_n(z)L_j^{(\gamma_\kappa+n)}(\omega z\overline{z})\big) = (\mathcal{J}p_n(z))L_j^{(\gamma_\kappa+n)}(\omega z\overline{z}) = (n+2\kappa_0+2\kappa_1)\psi_{n+j,j}, \\
\mathcal{J}\psi_{j,n+j} &= -(n+2\kappa_0+2\kappa_1)\psi_{j,n+j}.
\end{aligned}$$

If $n = 0 \bmod 4$, then

$$\begin{aligned}
\mathcal{J}z^n &= (n+2\kappa_0+2\kappa_1)z^n + 2(\kappa_0+\kappa_1)\overline{z}^n + \cdots, \\
\mathcal{J}z^n &= -(n+2\kappa_0+2\kappa_1)\overline{z}^n - 2(\kappa_0+\kappa_1)\overline{z}^n + \cdots
\end{aligned}$$

and thus

$$\mathcal{J}^2 z^n = (n+2\kappa_0+2\kappa_1)\mathcal{J} z^n + 2(\kappa_0+\kappa_1)\mathcal{J}\overline{z}^n = n(n+4\kappa_0+4\kappa_1)z^n + \cdots,$$
$$\mathcal{J}^2\overline{z}^n = n(n+4\kappa_0+4\kappa_1)\overline{z}^n + \cdots.$$

If $n = 2 \bmod 4$, then

$$\mathcal{J} z^n = (n+2\kappa_0+2\kappa_1)z^n + 2(\kappa_0-\kappa_1)\overline{z}^n + \cdots,$$
$$\mathcal{J} z^n = -(n+2\kappa_0+2\kappa_1)\overline{z}^n - 2(\kappa_0-\kappa_1)\overline{z}^n + \cdots,$$
$$\mathcal{J}^2 z^n = (n+4\kappa_0)(n+4\kappa_1)z^n + \cdots,$$
$$\mathcal{J}^2 \overline{z}^n = (n+4\kappa_0)(n+4\kappa_1)\overline{z}^n + \cdots.$$

Thus

$$\mathcal{J}^2\psi_{4n+j,j} = 16n(n+\kappa_0+\kappa_1)\psi_{4n+j,j}, \tag{5.2}$$
$$\mathcal{J}^2\psi_{j,4n+j} = 16n(n+\kappa_0+\kappa_1)\psi_{j,4n+j}, \tag{5.3}$$
$$\mathcal{J}^2\psi_{4n+2+j,j} = 4(2n+2\kappa_0+1)(2n+2\kappa_1+1)\psi_{4n+2+j,j}, \tag{5.4}$$
$$\mathcal{J}^2\psi_{j,4n+2+j} = 4(2n+2\kappa_0+1)(2n+2\kappa_1+1)\psi_{j,4n+2+j}. \tag{5.5}$$

5.3 Raising and lowering operators

Formula (3.1) specializes to

$$\widehat{H}_j = \mathrm{i}^j T^2 - 2T\overline{T} + \mathrm{i}^{-j}\overline{T}^2 + \frac{\omega}{2}\big\{\big(z-\mathrm{i}^j\overline{z}\big)\big(T-\mathrm{i}^{-j}\overline{T}\big) + \big(T-\mathrm{i}^{-j}\overline{T}\big)\big(z-\mathrm{i}^j\overline{z}\big)\big\}.$$

Proposition 5.2. *$H_0 + H_2 = \mathcal{H} = H_1 + H_3$.*

Proof. We use the polynomial parts $\widehat{H}_j$. First

$$\widehat{H}_{2j} = -2T\overline{T} + \frac{\omega}{2}\big\{zT + Tz + \overline{z}\overline{T} + \overline{T}\overline{z}\big\} + (-1)^j\left(T^2 + \overline{T}^2 - \frac{\omega}{2}\big\{z\overline{T} + \overline{z}T + \overline{T}z + T\overline{z}\big\}\right),$$

thus $\widehat{H}_0 + \widehat{H}_2 = -4T\overline{T} + \omega\big\{zT + Tz + \overline{z}\overline{T} + \overline{T}\overline{z}\big\} = g^{-1}\mathcal{H}g$. Similarly, $\widehat{H}_1 + \widehat{H}_3 = g^{-1}\mathcal{H}g$ (using $\mathrm{i}^{2j+1} = (-1)^j\mathrm{i}$). ■

Corollary 5.3. *$[H_0, H_2] = 0$ and $[H_1, H_3] = 0$.*

Proof. $H_0H_2 = H_0(\mathcal{H} - H_0)$ and $[H_0, \mathcal{H}] = 0$ by Proposition 2.5. ■

The formula $w^{-1}H_a w = H_{aw}$ (see Proposition 2.6) implies $\rho_3\widehat{H}_0\rho_1 = \widehat{H}_2$. To find the effect of ρ_1 or ρ_3 consider the leading term in $\psi_{m+j,j} = p_m(z)L_j^{(\gamma_\kappa+m)}(\omega z\overline{z})$ (total degree $n = m+2j$), a scalar multiple of $z^{m+\ell}\overline{z}^\ell$; examination of each of the formulas for p_m shows that each monomial $z^a\overline{z}^b$ satisfies $a - b = m \bmod 4$, this also applies to $\psi_{j,m+j}$ except for the odd case where the leading term is $z^j\overline{z}^{j+m}$ and $a-b = -m \bmod 4$. Also $\rho_1(z^a\overline{z}^b) = (\mathrm{i}z)^a(-\mathrm{i}\overline{z})^b = \mathrm{i}^{a-b}z^a\overline{z}^b = \mathrm{i}^m z^a\overline{z}^b$. Thus by replacing m by $n-2j$, we obtain $\rho_1\psi_{n-j,j} = \mathrm{i}^{n-2j}\psi_{n-j,j}$; this applies to all n (if n is even then $\mathrm{i}^{n-2j} = \mathrm{i}^{2j-n}$). Replace i by $-\mathrm{i}$ to find $\rho_3\psi_{n-j,j}$.

Proposition 5.4. *Suppose $\widehat{H}_0\psi_{n-j,j} = \sum_{j=0}^n c_{j,i}\psi_{n-i,i}$, then (1) $c_{j,j} = \frac{1}{2}E_n$, (2) $i = j \bmod 2$ and $i \neq j$ implies $c_{i,j} = 0$, (3) $\widehat{H}_2\psi_{n-j,j} = \frac{1}{2}E_n\psi_{n-j,j} - \sum\{c_{j,i}\psi_{n-i,i} \colon j - i = 1 \bmod 2\}$.*

Proof. By hypothesis,

$$\begin{aligned} H_2\psi_{n-2j,j} &= \rho_3 H_0 \rho_1 \psi_{n-j,j} = \rho_3 \mathrm{i}^{n-2j}\sum_{i=0}^{n} c_{j,i}\psi_{n-i,i} = \mathrm{i}^{n-2j}\sum_{i=0}^{n} c_{j,i}\mathrm{i}^{-n+2i}\psi_{n-i,i} \\ &= \sum_{i=0}^{n}(-1)^{i-j}c_{j,i}\psi_{n-i,i}. \end{aligned}$$

From $H_0+H_2=\mathcal{H}$, it follows that $\sum_{i=0}^{n} c_{j,i}\big(1+(-1)^{j-i}\big)\psi_{n-i,i}=E_n\psi_{n-j,j}$. Thus $2c_{j,j}=E_n$ and $j-i=0 \bmod 2$ and $j\neq i$ implies $c_{j,i}=0$. ■

Since $\{\sigma_0,\sigma_2\}$ and $\{\sigma_1,\sigma_3\}$ are conjugacy classes, the operators $H_0^m+H_2^m$ and $H_1^m+H_3^m$ commute with $\mathcal{H}$ and with the action of the group for $m=1,2,3,\dots$ (Proposition 2.6). There is a fairly simple formula for $\sum_{i=0}^{3} H_i^2$. Note I denotes the identity in the group.

Definition 5.5. Set $R:=(I+\kappa_0(\sigma_0+\sigma_2)+\kappa_1(\sigma_1+\sigma_3))^2-2\big(\kappa_0^2+\kappa_1^2\big)(1-\rho_2)$. This is an element of the center of the group algebra, that is, $[R,\sigma_j]=0$ for $0\le j\le 3$ and $[R,\rho_j]=0$ for $1\le j\le 3$.

Equivalently, $R=I+4\big(\kappa_0^2+\kappa_1^2\big)\rho_2+2\kappa_0(\sigma_0+\sigma_2)+2\kappa_1(\sigma_1+\sigma_3)+4\kappa_0\kappa_1(\rho_1+\rho_3)$.

Theorem 5.6. $H_0^2+H_2^2+H_1^2+H_3^2=\frac{3}{2}\mathcal{H}^2-2\omega^2\mathcal{J}^2-2\omega^2R$.

Proof. The details are presented in Appendix B. The idea is to use direct (computer-assisted) calculation. ■

The following defines the operator which is the main concern in the sequel; it will be shown to commute with $\mathcal{H}$ and the group action but not with angular momentum. The latter claim is proven by demonstrating that eigenfunctions of $\mathcal{J}^2$ are not preserved.

Definition 5.7. Set $\mathcal{K}:=H_0^2+H_2^2-H_1^2-H_3^2$, a fourth-order operator.

From Propositions 2.5 and 2.6, it follows that $[\mathcal{K},\mathcal{H}]=0$ and each of $H_0^2+H_2^2$, $H_1^2+H_3^2$ commutes with the group action.

Proposition 5.8. $\mathcal{K}=2(H_1H_3-H_0H_2)=-\frac{1}{2}\mathcal{H}^2+2\omega^2\mathcal{J}^2+2\omega^2R+4\big(H_0-\frac{1}{2}\mathcal{H}\big)^2$.

Proof. $\mathcal{K}=(H_0+H_2)^2-2H_0H_2-(H_1+H_3)^2+2H_1H_3$. Also $\mathcal{K}+\sum_{i=0}^{3}H_i^2=2\big(H_0^2+H_2^2\big)=2\mathcal{H}^2-4H_0H_2$. From Proposition 5.4, $H_2-\frac{1}{2}\mathcal{H}=-(H_0-\frac{1}{2}\mathcal{H})$ and $H_2H_0=\frac{1}{4}\mathcal{H}^2-\big(H_0-\frac{1}{2}\mathcal{H}\big)^2$. Thus

$$\begin{aligned} \mathcal{K} &= 2\mathcal{H}^2-4H_0H_2-\left(\frac{3}{2}\mathcal{H}^2-2\omega^2\mathcal{J}^2-2\omega^2R\right) \\ &= \frac{1}{2}\mathcal{H}^2+2\omega^2\mathcal{J}^2+2\omega^2R-4\left\{\frac{1}{4}\mathcal{H}^2-\left(H_0-\frac{1}{2}\mathcal{H}\right)^2\right\}; \end{aligned}$$

this completes the proof. ■

6 The expansion coefficients of H_0

6.1 General formulas

This section calculates the coefficients in $H_0\psi_{n-j,j}$. Start with

$$\begin{aligned} \widehat{H}_0 &= \big(T-\overline{T}\big)^2+\frac{\omega}{2}\big\{(z-\overline{z})\big(T-\overline{T}\big)+\big(T-\overline{T}\big)(z-\overline{z})\big\} \\ &= \big(T-\overline{T}\big)^2+\omega\big\{(z-\overline{z})\big(T-\overline{T}\big)+1+2\kappa_0\sigma_0+\kappa_1(\sigma_1+\sigma_3)\big\}. \end{aligned}$$

Let

$$A := (z-\overline{z})(\partial_z - \partial_{\overline{z}}) + 1 + 2\kappa_0,$$
$$B := \kappa_1\left\{(\sigma_1+\sigma_3) + (z-\overline{z})\left\{\frac{(1+\mathrm{i})}{z-\mathrm{i}\overline{z}}(1-\sigma_1) + \frac{(1-\mathrm{i})}{z+\mathrm{i}\overline{z}}(1-\sigma_3)\right\}\right\}$$

so that $\widehat{H}_0 = (T-\overline{T})^2 + \omega(A+B)$. The part of $(z-\overline{z})(T-\overline{T})$ corresponding to σ_0 is $2\kappa_0(z-\overline{z})\times\frac{1-\sigma_0}{(z-\overline{z})} = 2\kappa_0(1-\sigma_0)$. To determine the coefficients in $\widehat{H}_0\psi_{n-j,j} = \sum_{i=0}^{n} c_{j,i}\psi_{n-i,i}$, it suffices to consider the monomials of degree n in $\widehat{H}_0\psi_{n-j,j}$, that is, analyze $(A+B)\psi_{j,n-j}$. For a polynomial $p = \sum_{i=k}^{\ell} c_i z^{n-i}\overline{z}^i$ with $c_k \neq 0 \neq c_\ell$ let $D(p) = c_k z^{n-k}\overline{z}^k + c_\ell z^{n-\ell}\overline{z}^\ell$ (D for dominant terms). Then (for $2j \le n$)

$$D\big(Az^{n-j}\overline{z}^j\big) = -jz^{n-j+1}\overline{z}^{j-1} - (n-j)z^{n-j-1}\overline{z}^{j+1},$$
$$D\big(Az^j\overline{z}^{n-j}\big) = -jz^{j-1}\overline{z}^{n-j+1} - (n-j)z^{j+1}\overline{z}^{n-j-1}.$$

Let $2j \le n$ and $m := n-2j$, then

$$\begin{aligned} Bz^{n-j}\overline{z}^j &= \kappa_1(z\overline{z})^j\mathrm{i}^m(1+(-1)^m)\overline{z}^m \\ &\quad + \kappa_1(z\overline{z})^j(z-\overline{z})\sum_{k=0}^{m-1} z^{m-1-k}\overline{z}^k\big\{\mathrm{i}^k(1+\mathrm{i}) + (-\mathrm{i})^k(1-\mathrm{i})\big\} \\ &= \kappa_1(z\overline{z})^j\left\{2z^m + 2\varepsilon_{m-1}\overline{z}^m + 2\sum_{k=1}^{m-1} z^{m-k}\overline{z}^k(\varepsilon_k - \varepsilon_{k-1})\right\}, \end{aligned}$$

where $\varepsilon_k = (-1)^{\lfloor (k+1)/2\rfloor}$ and $\lfloor r\rfloor$ is the largest integer $\le r$. Thus

$$D\big(Bz^{n-j}\overline{z}^j\big) = 2\kappa_1\big(z^{n-j}\overline{z}^j + \varepsilon_{n-2j-1}z^j\overline{z}^{n-j}\big);$$

the same formula holds with $(z,\overline{z})$ replaced by $(\overline{z},z)$ (because $[B,\sigma_0]=0$). The special case $Bz^j\overline{z}^j = 2\kappa_1 z^j\overline{z}^j$.

Let $1 \le j \le n$, then by construction $D(\psi_{2n-j,j}) = \mathcal{C}\big(\psi_{2n-j,j}, z^{2n-j}\overline{z}^j\big)\big(z^{2n-j}\overline{z}^j + z^j\overline{z}^{2n-j}\big)$, similarly for $\psi_{j,2n-j}$; and from the above formulas, it follows that

$$D((A+B)\psi_{2n-j,j}) = -j\mathcal{C}(\psi_{2n-j,j}, z^{2n-j}\overline{z}^j)(z^{2n-j+1}\overline{z}^{j-1} + z^{j-1}\overline{z}^{2n-j+1}), \tag{6.1}$$
$$D((A+B)\psi_{j,2n-j}) = -j\mathcal{C}(\psi_{j,2n-j}, z^{2n-j}\overline{z}^j)(z^{2n-j+1}\overline{z}^{j-1} - z^{j-1}\overline{z}^{2n-j+1}), \tag{6.2}$$

the other terms are dominated by these. This implies that $\mathcal{C}(H_0\psi_{2n-j,j}, \psi_{2n-i,i}) = 0$ and $\mathcal{C}(H_0\psi_{j,2n-j}, \psi_{i,2n-i,}) = 0$ for $i < j-1$. Thus the nonzero coefficients occur only for $|j-i| \le 1$.

For the odd case, suppose $j \le n$, then

$$D(\psi_{2n+1-j,j}) = \mathcal{C}\big(\psi_{2n+1-j,j}, z^{2n+1-j}\overline{z}^j\big)z^{2n+1-j}\overline{z}^j + \mathcal{C}\big(\psi_{2n+1-j,j}, z^{j+1}\overline{z}^{2n-j}\big)z^{j+1}\overline{z}^{2n-j}.$$

This implies

$$\begin{aligned} D((A+B)\psi_{2n+1-j,j}) &= \big(-jz^{2n+2-j}\overline{z}^{j-1} + 2\varepsilon_{2n+2j}\kappa_1 z^j\overline{z}^{2n+1-j}\big)\mathcal{C}\big(\psi_{2n+1-j,j}, z^{2n+1-j}\overline{z}^j\big) \\ &\quad - (j+1)\mathcal{C}\big(\psi_{2n+1-j,j}, z^{j+1}\overline{z}^{2n-j}\big)z^j\overline{z}^{2n+1-j}. \end{aligned} \tag{6.3}$$

6.2 Even degree

To find $\mathcal{C}\big(\widehat{H}_0\psi_{2n-j,j},\psi_{2n-j+1,j-1}\big)$ and $\mathcal{C}\big(\widehat{H}_0\psi_{j,2n-j},\psi_{j-1,2n-j+1}\big)$, we used formulas (6.1) and (6.2). Then for $\mathcal{C}\big(\widehat{H}_0\psi_{2n-j,j},\psi_{2n-j-1,j+1}\big)$ and $\mathcal{C}\big(\widehat{H}_0\psi_{j,2n-j},\psi_{j+1,2n-j-1}\big)$, we used (5.1). The leading coefficients of $\psi_{n-j,j}$ are derived from

$$\mathcal{C}\big(R_n^{(\alpha,\beta)},z^{4n}\big)=\mathcal{C}\big(R_n^{(\alpha,\beta)},\overline{z}^{4n}\big)=\frac{1}{2^{2n}n!}(n+\alpha+\beta+1)_n,$$
$$\mathcal{C}\big(L_j^{(n+\gamma_\kappa)}(\omega z\overline{z}),z^j\overline{z}^j\big)=(-1)^j\frac{\omega^j}{j!}.$$

Then

$$\mathcal{C}\big(p_{4n,00},z^{4n}\big)=\mathcal{C}\big(p_{4n,00},\overline{z}^{4n}\big)=\frac{1}{2^{2n}n!}(n+\kappa_0+\kappa_1)_n,$$
$$\mathcal{C}\big(p_{4n,11},z^{4n}\big)=-\mathcal{C}\big(p_{4n,11},\overline{z}^{4n}\big)=\frac{1}{2^{2n-2}(n-1)!}(n+\kappa_0+\kappa_1+1)_{n-1}$$

and

$$\mathcal{C}\big(p_{4n,+2,10},z^{4n+2}\big)=\mathcal{C}\big(p_{4n+2,10},\overline{z}^{4n+2}\big)=\frac{1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_n,$$
$$\mathcal{C}\big(p_{4n+2,01},z^{4n+2}\big)=-\mathcal{C}\big(p_{4n+2,01},\overline{z}^{4n+2}\big)=\frac{1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_n.$$

First consider the even degree polynomials satisfying $\sigma_0 p=p$

$$\left(H_0-\frac{1}{2}E_{4n+2j}\right)\psi_{4n+j,j}=\omega^2\frac{n+\kappa_0+\kappa_1}{2n+\kappa_0+\kappa_1}\psi_{4n+j+1,j-1}$$
$$+\frac{(j+1)(2n+2\kappa_0-1)(4n+2\kappa_0+2\kappa_1+j)}{2(2n+\kappa_0+\kappa_1)}\psi_{4n+j-1,j+1},\tag{6.4}$$
$$\left(H_0-\frac{1}{2}E_{4n+2j+2}\right)\psi_{4n+2+j,j}=4\omega^2\frac{n+1}{2n+\kappa_0+\kappa_1+1}\psi_{4n+j+3,j-1}$$
$$+2\frac{(j+1)(2n+2\kappa_1+1)(4n+2\kappa_0+2\kappa_1+j+2)}{2n+\kappa_0+\kappa_1+1}\psi_{4n+1+j,j+1},\tag{6.5}$$

then the even degree polynomials satisfying $\sigma_0 p=-p$,

$$\left(H_0-\frac{1}{2}E_{4n+2j}\right)\psi_{j,4n+j}=4\omega^2\frac{n}{2n+\kappa_0+\kappa_1}\psi_{j-1,4n+j+1}$$
$$+\frac{(j+1)(2n+2\kappa_1-1)(4n+2\kappa_0+2\kappa_1+j)}{(2n+\kappa_0+\kappa_1)}\psi_{j+1,4n+j-1},\tag{6.6}$$
$$\left(H_0-\frac{1}{2}E_{4n+2j+2}\right)\psi_{j,4n+2+j}=\omega^2\frac{n+\kappa_0+\kappa_1+1}{2n+\kappa_0+\kappa_1+1}\psi_{j-1,4n+j+3}$$
$$+\frac{(j+1)(2n+2\kappa_0+1)(4n+2\kappa_0+2\kappa_1+j+2)}{2(2n+\kappa_0+\kappa_1+1)}\psi_{j+1,4n+1+j}.\tag{6.7}$$

Thus the matrix of H_0 in the bases $\{\psi_{2n-j,j}\colon 0\le j\le n\}$ and $\{\psi_{j,2n-j}\colon 0\le j<n\}$ is tridiagonal.

6.3 Odd degree

Formula (6.3) is used to find $\mathcal{C}\big(\widehat{H}_0\psi_{2n+1-j,j},\psi_{2n+2-j,j-1}\big)$ and $\mathcal{C}\big(\widehat{H}_0\psi_{2n+1-j,j},\psi_{j,2n+1-j}\big)$. We will show that the nonzero coefficients in $H_0\psi_{2n+1-j,j}=\sum_{i=0}^{2n+1}c_{j,i}\psi_{2n+1-i,i}$ occur at $i=j-1$,

$j, j+1, 2n+1-j$. Suppose m is odd, then $\psi_{m-j,j} = \sum_{i=j}^{m-j-1} a_i z^{m-i}\overline{z}^i$ and a_j, a_{m-j-1} are involved in finding $\mathcal{C}\big(H_0\psi_{m-j,j}, z^{m-j+1}\overline{z}^{j-1}\big)$ and $\mathcal{C}\big(H_0\psi_{m-j,j}, z^j\overline{z}^{m-j}\big)$. Then

$$\begin{aligned}\mathcal{C}(H_0\psi_{m+j,j},\psi_{m+1+j,j-1}) &= \frac{\mathcal{C}\big(H_0\psi_{m+j,j}, z^{m+1+j}\overline{z}^{j-1}\big)}{\mathcal{C}\big(\psi_{m+1+j,j-1}, z^{m+1+j}\overline{z}^{j-1}\big)}\\ &= -\frac{j\omega\mathcal{C}\big(\psi_{m+j,j}, z^{m+j}\overline{z}^{j}\big)}{\mathcal{C}\big(\psi_{m+1+j,j-1}, z^{m+1+j}\overline{z}^{j-1}\big)},\\ \mathcal{C}\big(H_0\psi_{m+j,j}, z^j\overline{z}^{m+j}\big) &= \mathcal{C}\big(H_0\psi_{m+j,j},\psi_{m+1+j,j-1}\big)\mathcal{C}\big(\psi_{m+1+j,j-1}, z^j\overline{z}^{m+j}\big)\\ &\quad+\mathcal{C}(H_0\psi_{m+j,j},\psi_{j,m+j})\mathcal{C}\big(\psi_{j,m+j}, z^j\overline{z}^{m+j}\big)\\ &= -\omega(j+1)\mathcal{C}\big(\psi_{m+j,j}, z^{j+1}\overline{z}^{m-1+j}\big)\\ &\quad+2(-1)^{(m+1)/2}\omega\kappa_1\mathcal{C}\big(\psi_{m+j,j}, z^{m+j}\overline{z}^{j}\big).\end{aligned}$$

The lower three lines are used to solve for $\mathcal{C}(H_0\psi_{m+j,j},\psi_{j,m+j})$. There are two cases: $m-2j = 4n+1, 4n+3$. First

$$\begin{aligned}\mathcal{C}\big(p_{4n+1}, z^{4n+1}\big) &= \mathcal{C}\big(p_{4n,00}, z^{4n}\big)+\frac{1}{4}\mathcal{C}\big(p_{4n,11}, z^{4n}\big) = \frac{1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_n,\\ \mathcal{C}\big(p_{4n+1}, z\overline{z}^{4n}\big) &= \mathcal{C}\big(p_{4n,00}, z^{4n}\big)-\frac{1}{4}\mathcal{C}\big(p_{4n,11}, z^{4n}\big) = \frac{\kappa_0+\kappa_1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_{n-1}\end{aligned}$$

and second

$$\begin{aligned}\mathcal{C}\big(p_{4n+3}, z^{4n+3}\big) &= \left(n+\kappa_0+\frac{1}{2}\right)\mathcal{C}\big(p_{4n+2,10}, z^{4n+2}\big)+\left(n+\kappa_1+\frac{1}{2}\right)\mathcal{C}\big(p_{4n+2,01}, z^{4n+2}\big)\\ &= \frac{1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_{n+1},\\ \mathcal{C}\big(p_{4n+3}, z\overline{z}^{4n+2}\big) &= \left(n+\kappa_0+\frac{1}{2}\right)\mathcal{C}\big(p_{4n+2,10}, z^{4n+2}\big)-\left(n+\kappa_1+\frac{1}{2}\right)\mathcal{C}\big(p_{4n+2,01}, z^{4n+2}\big)\\ &= \frac{\kappa_0-\kappa_1}{2^{2n}n!}(n+\kappa_0+\kappa_1+1)_n.\end{aligned}$$

Then

$$\begin{aligned}&\left(H_0-\frac{1}{2}E_{4n+1+2j}\right)\psi_{4n+1+j,j}\\ &\quad= \frac{\omega^2}{2n+\kappa_0+\kappa_1+1}\psi_{4n+2+j,j-1}+\frac{(j+1)(4n+2\kappa_0+2\kappa_1+j+1)}{2n+\kappa_0+\kappa_1}\psi_{4n+j,j+1}\\ &\quad\quad+\omega\left\{\frac{j(2n+2\kappa_0+1)}{2n+\kappa_0+\kappa_1+1}+\frac{2n(j+1)}{2n+\kappa_0+\kappa_1}-(2j+2\kappa_1+1)\right\}\psi_{j,4n+1+j},\end{aligned}\tag{6.8}$$

$$\begin{aligned}&\left(H_0-\frac{1}{2}E_{4n+3+2j}\right)\psi_{4n+3+j,j}\\ &\quad= \frac{4\omega^2(n+1)(n+\kappa_0+\kappa_1+1)}{2n+\kappa_0+\kappa_1+2}\psi_{4n+4+j,j-1}\\ &\quad\quad+\frac{(j+1)(4n+2\kappa_0+2\kappa_1+j+3)(2n+2\kappa_0+1)(2n+2\kappa_1+1)}{2n+\kappa_0+\kappa_1+1}\psi_{4n+2+j,j+1}\\ &\quad\quad+\omega\left\{\frac{(j+1)(2n+2\kappa_0+1)}{2n+\kappa_0+\kappa_1+1}-\frac{2j(n+1)}{2n+\kappa_0+\kappa_1+2}+(2j+2\kappa_1+1)\right\}\psi_{j,4n+3+j}.\end{aligned}\tag{6.9}$$

7 The expansion coefficients of $\mathcal{K}$

7.1 Even degree

The matrices of $H_0 - \frac{1}{2}\mathcal{H}$ with respect to the bases $\{\psi_{2n-j,j}\colon 0 \le j \le n\}$ ($\sigma_0 p = p$) and $\{\psi_{j,2n-j}\colon 0 \le j < n\}$ ($\sigma_0 p = -p$) are tridiagonal with zeroes on the main diagonal. If M is such a matrix, then the only nonzero elements in M^2 are $\left(M^2\right)_{i,i-2} = M_{i,i-1}M_{i-1,i-2}$, $\left(M^2\right)_{i,i+2} = M_{i,i+1}M_{i+1,i+2}$ and $\left(M^2\right)_{i,i} = M_{i,i-1}M_{i-1,i} + M_{i,i+1}M_{i+1,i}$. By use of the expansion coefficients of $H_0 - \frac{1}{2}\mathcal{H}$, we find the matrix for $\mathcal{K} = -\frac{1}{2}\mathcal{H}^2 + 2\omega^2\mathcal{J}^2 + 2\omega^2 R + 4\left(H_0 - \frac{1}{2}\mathcal{H}\right)^2$ (the first three terms act as scalars).

To compute $\mathcal{K}\ \psi_{4n+j,j}$, we use the coefficients of the expansions of$H_0\psi_{4n+j+k,j-k}$ with $k = -1, 0, 1$ from (6.4), (6.5). Note $4n + j - 1 = (4(n-1)+2) + (j+1)$ and $4n + j + 1 = (4n+2) + (j-1)$; this indicates which types are involved. The scalars derive from E_{4n+2j}, $R\psi_{4n+j,j} = (1 + 2\kappa_0 + 2\kappa_1)^2\psi_{4n+j,j}$ and $\mathcal{J}^2\psi_{4n+j,j} = 16n(n + \kappa_0 + \kappa_1)\psi_{4n+j,j}$ (from (5.2))

$$
\begin{aligned}
&\mathcal{K}\psi_{4n+j,j} = A^0_{-1}(n)\psi_{4n+j+2,j-2} + A^0_0(n,j)\psi_{4n+j,j} + A^0_1(n,j)\psi_{4n+j-2,j+2},\\
&A^0_{-1}(n) = 16\omega^4\frac{(n+1)(n+\kappa_0+\kappa_1)}{(2n+\kappa_0+\kappa_1)(2n+\kappa_0+\kappa_1+1)},\\
&A^0_1(n,j) = 4\frac{(j+1)(j+2)(2n+2\kappa_0-1)(2n+2\kappa_1-1)}{(2n+\kappa_0+\kappa_1-1)(2n+\kappa_0+\kappa_1)}\\
&\qquad\times(4n+2\kappa_0+2\kappa_1+j-1)(4n+2\kappa_0+2\kappa_1+j),\\
&A^0_0(n,j) = -8\omega^2(\kappa_0-\kappa_1)\left\{2j + \frac{(n+1)j(j-1)}{2n+\kappa_0+\kappa_1+1} - \frac{n(j+1)(j+2)}{2n+\kappa_0+\kappa_1-1}\right\}.
\end{aligned}
$$

For $\mathcal{K}\psi_{4n+2+j,j}$, we use $4(n+1)+(j-1)$ and $4n+(j+1)$ for the adjacent labels, and $E_{4n+2+2j}$, $R\psi_{4n+2+j} = (1+2\kappa_0-2\kappa_1)^2\psi_{4n+2+j,j}$ and $\mathcal{J}^2\psi_{4n+2+j,j} = 4(2n+2\kappa_0+1)(2n+2\kappa_1+1)\psi_{4n+2+j,j}$ (from (5.4))

$$
\begin{aligned}
&\mathcal{K}\psi_{4n+2+j,j} = A^1_{-1}(n)\psi_{4n+4+j,j-2} + A^1_0(n,j)\psi_{4n+2+j,j} + A^1_1(n,j)\psi_{4n+j,j+2},\\
&A^1_{-1}(n) = 16\omega^4\frac{(n+1)(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1+2)(2n+\kappa_0+\kappa_1+1)},\\
&A^1_1(n,j) = 4\frac{(j+1)(j+2)(2n+2\kappa_0-1)(2n+2\kappa_1+1)}{(2n+\kappa_0+\kappa_1+1)(2n+\kappa_0+\kappa_1)},\\
&\qquad\times(4n+2\kappa_0+2\kappa_1+j+2)(4n+2\kappa_0+2\kappa_1+j+1)\\
&A^1_0(n,j) = 8\omega^2(\kappa_0+\kappa_1) - 8\omega^2(\kappa_0-\kappa_1-1)\left(2j+1+\frac{(n+1)j(j-1)}{2n+\kappa_0+\kappa_1+2} - \frac{n(j+1)(j+2)}{2n+\kappa_0+\kappa_1}\right).
\end{aligned}
$$

For $\mathcal{K}\psi_{j,4n+j}$, use the coefficients from (6.6), (6.7), the reversed labels from $\psi_{4n+j,j}$ and from (5.3)

$$
R\psi_{j,4n+j} = (1-2\kappa_0-2\kappa_1)^2\psi_{j,4n+j}, \mathcal{J}^2\psi_{j,4n+j} = 16n(n+\kappa_0+\kappa_1)\psi_{j,4n+j},
$$

then

$$
\begin{aligned}
&\mathcal{K}\psi_{j,4n+j} = B^0_{-1}(n)\psi_{j-2,4n+j+2} + B^0_0(n,j)\psi_{j,4n+j} + B^0_1(n,j)\psi_{j+2,4n+j-2},\\
&B^0_{-1}(n) = 16\omega^4\frac{n(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1)(2n+\kappa_0+\kappa_1+1)},\\
&B^0_1(n,j) = 4\frac{(j+1)(j+2)(2n+2\kappa_0-1)(2n+2\kappa_1-1)}{(2n+\kappa_0+\kappa_1-1)(2n+\kappa_0+\kappa_1)}\\
&\qquad\times(4n+2\kappa_0+2\kappa_1+j-1)(4n+2\kappa_0+2\kappa_1+j),
\end{aligned}
$$

$$B_0^0(n,j) = -8\omega^2(\kappa_0-\kappa_1)\left\{2j+2+\frac{nj(j-1)}{2n+\kappa_0+\kappa_1+1}-\frac{(n-1)(j+1)(j+2)}{2n+\kappa_0+\kappa_1-1}\right\}.$$

For $\mathcal{K}\psi_{j,4n+2+j}$, use the reversed labels from $\psi_{4n+2+j,j}$ and $R\psi_{j,4n++2j} = (1-2\kappa_0+2\kappa_1)^2\psi_{j,4n+j}$, $\mathcal{J}^2\psi_{j,4n+2+j} = 4(2n+2\kappa_0+1)(2n+2\kappa_1+1)\psi_{j,4n+2+j}$ (5.5)

$$\begin{aligned}
&\mathcal{K}\psi_{j,4n+2+j} = B_{-1}^1(n)\psi_{j-1,4n+3+j} + B_0^1(n,j)\psi_{j,4n+2+j} + B_1^1(n,j)\psi_{j+1,4n+j+1},\\
&B_{-1}^1(n) = 16\omega^4\frac{(n+1)(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1+2)(2n+\kappa_0+\kappa_1+1)},\\
&B_1^1(n,j) = 4\frac{(j+1)(j+2)(2n+2\kappa_0+1)(2n+2\kappa_1-1)}{(2n+\kappa_0+\kappa_1+1)(2n+\kappa_0+\kappa_1)}\\
&\qquad\qquad \times(4n+2\kappa_0+2\kappa_1+j+1)(4n+2\kappa_0+2\kappa_1+j+2),\\
&B_0^1(n,j) = -8\omega^2(\kappa_0+\kappa_1) - 8\omega^1(\kappa_0-\kappa_1+1)\\
&\qquad\qquad \times\left(2j+1+\frac{(n+1)j(j-1)}{2n+\kappa_0+\kappa_1+2}-\frac{n(j+1)(j+2)}{2n+\kappa_0+\kappa_1}\right).
\end{aligned}$$

That concludes the even degree case. One might notice that all the coefficients are bounded in $(n-j)$, except for the types $\psi_{2n-j,j}\to\psi_{2n-j-2,j+2}$, $j\le n-2$, and $\psi_{j,2n-j}\to\psi_{j+2,2n-j-2}$, $j\le n-3$, which are $O\big((n-j)^2\big)$.

7.2 Odd degree

The matrix M of $H_0-\frac{1}{2}\mathcal{H}$ with respect to the basis $\{\psi_{2n+1-j,j}\colon 0\le j\le 2n+1\}$ has nonzero entries on the subdiagonal $\{(i+1,i)\}$, the superdiagonal $\{(i,i+1)\}$ (with $0\le i\le 2n$) and the cross diagonal $\{(i,2n+1-i)\colon 0\le i\le 2n+1\}$. Because $\big[H_0-\frac{1}{2}\mathcal{H},\sigma_0\big]=0$, there is a symmetry property $M_{i,j}=M_{2n+1-i,2n+1-j}$. Then

$$\begin{aligned}
&\big(M^2\big)_{i,i+2} = M_{i,i+1}M_{i+1,i+2}, \qquad \big(M^2\big)_{i,i-2} = M_{i,i-1}M_{i-1,i-2},\\
&\big(M^2\big)_{i,i} = M_{i,i+1}M_{i+1,i} + M_{i,i-1}M_{i-1,i} + M_{i,2n+1-i}^2,\\
&\big(M^2\big)_{i,2n-i} = M_{i,i+1}(M_{i+1,2n-i}+M_{i,2n+1-i}),\\
&\big(M^2\big)_{i,2n+2-i} = M_{i,i-1}(M_{i-1,2n+2-i}+M_{i,2n+1-i})
\end{aligned}$$

(because $M_{2n+1-i,2n-i}=M_{i,i+1}$ and $M_{2n+1-i,2n+2-i}=M_{i,i-1}$). The two cases for $\psi_{2n+1-j,j}$ are $2n+1-2j=1,3\bmod 4$. The adjacent $(j\pm1)$ polynomials to $\psi_{4n+1+j,j}$ are $\psi_{4n+3+(j-1),j-1}$ and $\psi_{4(n-1)+3+(j+1),j+1}$. By use of (6.8), (6.9) and Section 5.2, $E_{4n+1+2j}, \mathcal{J}^2\psi_{4n+1+j,j} = (4n+1+2\kappa_0+2\kappa_1)^2$, $R\psi_{4n+1+j,j} = (1-4\kappa_0^2-4\kappa_1^2)\psi_{4n+1+j}$, we obtain

$$\begin{aligned}
&\mathcal{K}\psi_{4n+1+j,j} = A_{-2}(n)\psi_{4n+3+j,j-2} + A_0(n,j)\psi_{4n+1+j,j} + A_2(n,j)\psi_{4n-1+j,j+2}\\
&\qquad\qquad + A_1(n,j)\psi_{j+1,4n+j} + A_{-1}(n,j)\psi_{j-1,4n+2+j},\\
&A_{-2}(n) = 16\omega^4\frac{(n+1)(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1+2)(2n+\kappa_0+\kappa_1+1)},\\
&A_2(n,j) = 4\frac{(j+1)(j+2)(2n+2\kappa_0-1)(2n+2\kappa_1-1)}{(2n+\kappa_0+\kappa_1-1)(2n+\kappa_0+\kappa_1)}\\
&\qquad\qquad \times(4n+2\kappa_0+2\kappa_1+j+1)(4n+2\kappa_0+2\kappa_1+j),\\
&A_0(n,j) = -8\omega^2\frac{(\kappa_0+\kappa_1)(\kappa_0-\kappa_1)(2n+\kappa_0+\kappa_1+j+1)^2}{(2n+\kappa_0+\kappa_1)(2n+\kappa_0+\kappa_1+1)},\\
&A_1(n,j) = -8\omega(\kappa_0-\kappa_1)\frac{(j+1)(2n+\kappa_0+\kappa_1+j+1)(4n+2\kappa_0+2\kappa_1+j+1)}{(2n+\kappa_0+\kappa_1-1)_3},
\end{aligned}$$

$$A_{-1}(n,j) = -8\omega^3(\kappa_0+\kappa_1)\frac{(2n+\kappa_0+\kappa_1+j+1)}{(2n+\kappa_0+\kappa_1)_3}.$$

Omit A_{-2} if $j<2$, A_2 if $n=0$, A_1 if $n=0$, A_{-1} if $j<1$. In the special case $\psi_{1,0}=z$, only one term appears: $\mathcal{K}z = A_0(0,0)z$ and $A_0(0,0) = -8\omega^2(\kappa_0-\kappa_1)(\kappa_0+\kappa_1+1)$.

The adjacent $(j\pm1)$ polynomials to $\psi_{4n+3+j,j}$ are $\psi_{4(n+1)+1+(j-1),j-1}$ and $\psi_{4n+1+(j+1),j+1}$. By use of (6.8), (6.9) and $E_{4n+3+2j}$, $\mathcal{J}^2\psi_{4n+3+j,j} = (4n+3+2\kappa_0+2\kappa_1)^2$, $R\psi_{4n+3+j,j} = \big(1-4\kappa_0^2-4\kappa_1^2\big)\psi_{4n+3+j}$, we obtain

$$\begin{aligned}
\mathcal{K}\psi_{4n+3+j,j} &= B_{-2}(n)\psi_{4n+5+j,j-2} + B_0(n,j)\psi_{4n+3+j,j} + B_2(n,j)\psi_{4n+1+j,j+2}\\
&\quad + B_1(n,j)\psi_{j+1,4n+2+j} + B_{-1}(n,j)\psi_{j-1,4n+4+j},\\
B_{-2}(n) &= 16\omega^4\frac{(n+1)(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1+2)(2n+\kappa_0+\kappa_1+3)},\\
B_2(n,j) &= 4\frac{(j+1)(j+2)(2n+2\kappa_0+1)(2n+2\kappa_1+1)}{(2n+\kappa_0+\kappa_1+1)(2n+\kappa_0+\kappa_1)}\\
&\quad\times(4n+2\kappa_0+2\kappa_1+j+2)(4n+2\kappa_0+2\kappa_1+j+3),\\
B_0(n,j) &= -8\omega^2\frac{(\kappa_0+\kappa_1)(\kappa_0-\kappa_1)(2n+\kappa_0+\kappa_1+j+2)^2}{(2n+\kappa_0+\kappa_1+1)(2n+\kappa_0+\kappa_1+2)},\\
B_1(n,j) &= -8\omega(\kappa_0+\kappa_1)\frac{(2n+\kappa_0+\kappa_1+j+2)(4n+2\kappa_0+2\kappa_1+j+3)}{(2n+\kappa_0+\kappa_1)_3}\\
&\quad\times(j+1)(2n+2\kappa_0+1)(2n+2\kappa_1+1),\\
B_{-1}(n,j) &= -32\omega^3(\kappa_0-\kappa_1)\frac{(n+1)(2n+\kappa_0+\kappa_1+j+1)(n+\kappa_0+\kappa_1+1)}{(2n+\kappa_0+\kappa_1+1)_3}.
\end{aligned}$$

Omit B_{-2} if $j<2$, B_2 if $n=0, B_{-1}$ if $j=0$.

It is perhaps a surprise that the coefficients $A_0(n,j)$ and $B_0(n,j)$ are products of linear factors, in contrast to the even case where the neatest expressions for A_0^0, A_0^1, B_0^0, B_0^1 are partial fractions. In fact, all of the coefficients in this subsection are products of linear factors, which is not the case for some of the terms in $H_0\psi_{2n+1-j,j}$.

8 Conclusion

We described an orthogonal basis of wavefunctions in terms of Jacobi and Laguerre polynomials. Each of the basis elements is of a particular isotype, that is, involved in one of the five irreducible representations of the group B_2. We defined a fourth order differential-difference self-adjoint operator $\mathcal{K}$ which commutes with $\mathcal{H}$ but not with the angular momentum $\mathcal{J}^2$. This is an example of superintegrability. The action of $\mathcal{K}$ on the basis elements was found explicitly. It is known [9] that there are differential operators of degree $2k$ which demonstrate superintegrability for the two-parameter $I_2(2k)$ (even dihedral group) model. It does not appear straightforward to adapt the methods of this paper to the larger groups.

A Transformation of the Hamiltonian

This is a short proof of the formula

$$h_\kappa\big(-\Delta_\kappa+\omega^2\|x\|^2\big)h_\kappa^{-1} = -\Delta+\omega^2\|x\|^2+\sum_{v\in R_+}\frac{\kappa_v(\kappa_v-\sigma_v)\|v\|^2}{\langle x,v\rangle^2},$$

where $h_\kappa(x) := \prod_{v\in R_+} |\langle x,v\rangle|^{\kappa_v}$, the $W(R)$-invariant weight function used in $L^2\big(\mathbb{R}^N, h_\kappa^2\mathrm{d}m\big)$. For the Laplacian, we have

$$h_\kappa\Delta\big(fh_\kappa^{-1}\big)-\Delta f = fh_\kappa\Delta h_\kappa^{-1}+2h_\kappa\big\langle\nabla f,\nabla h_\kappa^{-1}\big\rangle$$
$$= f\sum_{v\in R_+}\kappa_v\frac{\|v\|^2}{\langle x,v\rangle^2}+f\sum_{i=1}^N\left(\sum_{v\in R_+}\frac{-\kappa_v v_i}{\langle x,v\rangle}\right)^2-2\sum_{v\in R_+}\kappa_v\frac{\langle\nabla f,v\rangle}{\langle x,v\rangle}$$

and

$$\sum_{i=1}^N\left(\sum_{v\in R_+}\frac{-\kappa_v v_i}{\langle x,v\rangle}\right)^2=\sum_{u,v\in R_+}\kappa_u\kappa_v\frac{\langle u,v\rangle}{\langle x,u\rangle\langle x,v\rangle}=\sum_{v\in R_+}\kappa_v^2\frac{\|v\|^2}{\langle x,v\rangle^2};$$

this follows from breaking up the double sum over rotations $w=\sigma_u\sigma_v$ and the identity σ_u^2 and applying a lemma [4, Lemma 6.4.6] the w terms vanish. Thus

$$h_\kappa\Delta\big(fh_\kappa^{-1}\big)-\Delta f = f\sum_{v\in R_+}\kappa_v(\kappa_v+1)\frac{\|v\|^2}{\langle x,v\rangle^2}-2\sum_{v\in R_+}\kappa_v\frac{\langle\nabla f,v\rangle}{\langle x,v\rangle}.$$

Also

$$\sum_{v\in R_+}2\kappa_v\frac{h_\kappa\big\langle\nabla\big(fh_\kappa^{-1}\big),v\big\rangle-\langle\nabla f,v\rangle}{\langle x,v\rangle}=-2f\sum_{u,v\in R_+}\kappa_u\kappa_v\frac{\langle u,v\rangle}{\langle x,u\rangle\langle x,v\rangle}=-2f\sum_{v\in R_+}\kappa_v^2\frac{\|v\|^2}{\langle x,v\rangle^2}.$$

The other part of $h_\kappa\Delta_\kappa\big(fh_\kappa^{-1}\big)$ contributes $-\sum_{v\in R_+}\kappa_v\frac{f-\sigma_v f}{\langle x,v\rangle^2}$ thus

$$h_\kappa\Delta_\kappa\big(fh_\kappa^{-1}\big)-\Delta f=\sum_{v\in R_+}\kappa_v\frac{\|v\|^2}{\langle x,v\rangle^2}\{(\kappa_v+1)f-2\kappa_v f-f+\sigma_v f\}$$
$$=-\sum_{v\in R_+}\kappa_v\frac{\|v\|^2}{\langle x,v\rangle^2}(\kappa_v f-\sigma_v f).$$

This proves the formula.

B Symbolic computation proofs

There is an analog $K(x,y)$ of the exponential function $\exp\langle x,y\rangle$ on $\mathbb{R}^N\times\mathbb{R}^N$ which satisfies $K(x,y)=K(y,x)$, $K(xw,yw)=K(x,y)$ for all $w\in W(R)$ and $\mathcal{D}_i^{(x)}K(x,y)=y_iK(x,y)$ (where $\mathcal{D}_i^{(x)}$ is the operator $\mathcal{D}_i$ acting on x, for $1\le i\le N$). The kernel exists for nonsingular parameters $\{\kappa_v\}$, which include the situation $\kappa_v\ge 0$. Suppose $p(x)$ is a polynomial then by the product rule

$$\mathcal{D}_i(p(x)K(x,y))=\left(y_ip(x)+\frac{\partial}{\partial x_i}p(x)\right)K(x,y)+\sum_{v\in R_+}\kappa_v\frac{p(x)-p(x\sigma_v)}{\langle x,v\rangle}K(x\sigma_v,y)v_i.$$

This formula together with $wK(x,y)=K(xw,y)=K\big(x,yw^{-1}\big)$ show how an element of the rational Cherednik algebra (an algebra of operators on polynomials generated by $\big\{\mathcal{D}_i^{(x)},x_i\colon 1\le i\le N\big\}\cup W(R)$) acts on a generic sum $\sum_{w\in W(R)}p_W(x,y)K(xw,y)$. It can be shown that if $\mathcal{T}$ is in the rational Cherednik algebra and $\mathcal{T}K(x,y)=0$, then $\mathcal{T}=0$ (see Dunkl [3]). For particular groups and operators, the calculation of $\mathcal{T}K(x,y)$ can be implemented in computer algebra.

The function K is an undefined function with argument $\langle x, y\rangle$ $\big($or $\langle x, yw^{-1}\rangle\big)$. To compute $\mathcal{D}_i^{(x)}K(xw,y) = \mathcal{D}_i^{(x)}K\big(x, yw^{-1}\big) = \big(yw^{-1}\big)_i K(xw,y)$ one applies $\frac{\partial}{\partial x_i}$ to $\langle x, yw^{-1}\rangle$, a straightforward calculation.

In the B_2 application with complex coordinates $z = x_1 + \mathrm{i}x_2$, $u = y_1 + \mathrm{i}y_2$, the inner product is $\langle x, y\rangle = \frac{1}{2}(z\overline{u} + \overline{z}u)$. As examples,

$$\begin{aligned}
&\overline{T}K\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right) = \frac{1}{2}uK\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right),\\
&T\left\{\big(z^2-\overline{z}^2\big)K\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right)\right\} = \left\{\frac{1}{2}\big(z^2-\overline{z}^2\big)\overline{u}+2z\right\}K\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right)\\
&\qquad\qquad\qquad + 2\kappa_0(z-\overline{z})K\left(-\frac{1}{2}(zu+\overline{zu})\right)\\
&\qquad\qquad\qquad + 2\kappa_0(z+\overline{z})K\left(\frac{1}{2}(zu+\overline{zu})\right),
\end{aligned}$$

and

$$K\left(-\frac{1}{2}(zu+\overline{zu})\right) = \sigma_2 K\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right), \qquad K\left(\frac{1}{2}(zu+\overline{zu})\right) = \sigma_0 K\left(\frac{1}{2}(z\overline{u}+\overline{z}u)\right).$$

This method is used to prove Theorem 5.6.

Acknowledgements

The author presented some of the material in a plenary lecture at the 34th International Colloquium on Group Theoretical Methods in Physics, Strasbourg, France, July 18–22, 2022. Also the author thanks the referees for their careful reading and constructive suggestions.

References

[1] Baker T.H., Forrester P.J., The Calogero–Sutherland model and generalized classical polynomials, *Comm. Math. Phys.* **188** (1997), 175–216, arXiv:solv-int/9608004.

[2] Dunkl C.F., Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[3] Dunkl C.F., Computing with differential-difference operators, *J. Symbolic Comput.* **28** (1999), 819–826.

[4] Dunkl C.F., Xu Y., Orthogonal polynomials of several variables, 2nd ed., *Encyclopedia Math. Appl.*, Vol. 155, Cambridge University Press, Cambridge, 2014.

[5] Feigin M., Generalized Calogero–Moser systems from rational Cherednik algebras, *Selecta Math. (N.S.)* **18** (2012), 253–281, arXiv:0809.3487.

[6] Feigin M., Hakobyan T., On Dunkl angular momenta algebra, *J. High Energy Phys.* **2015** (2015), no. 11, 107, 23 pages, arXiv:1409.2480.

[7] Lapointe L., Vinet L., Exact operator solution of the Calogero–Sutherland model, *Comm. Math. Phys.* **178** (1996), 425–452, arXiv:q-alg/9509003.

[8] Lassalle M., Polynômes de Hermite généralisés, *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), 579–582.

[9] Quesne C., Exchange operator formalism for an infinite family of solvable and integrable quantum systems on a plane, *Modern Phys. Lett. A* **25** (2010), 15–24, arXiv:0910.2151.

[10] Tremblay F., Turbiner A.V., Winternitz P., An infinite family of solvable and integrable quantum systems on a plane, *J. Phys. A* **42** (2009), 242001, 10 pages, arXiv:0904.0738.

[11] Tremblay F., Turbiner A.V., Winternitz P., Periodic orbits for an infinite family of classical superintegrable systems, *J. Phys. A* **43** (2010), 015202, 14 pages, arXiv:0910.0299.