

Solitons in the Gauged Skyrme–Maxwell Model

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Abstract. We consider soliton solutions of the U(1) gauged Skyrme model with the pion mass term. The domain of existence of gauged Skyrmions is restricted from above by the value of the pion mass. Concentrating on the solutions of topological degree one, we find that coupling to the electromagnetic field breaks the symmetry of the configurations, the Skyrmions carrying both an electric charge and a magnetic flux, with an induced dipole magnetic moment. The Skyrmions also possess an angular momentum, which is quantized in the units of the electric charge. The mass of the gauged Skyrmions monotonically decreases with increase of the gauge coupling.

Key words: Skyrme–Maxwell model; gauged Skyrmions; topological solitons; Skyrmions

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The study of the classical soliton solutions in field theory can be traced back to the pioneering paper by Skyrme [74, 75] (for a review, see [19, 50, 51, 71]). The Skyrme model was introduced in 1961 as a simple version of the nonlinear sigma model in $3 + 1$ dimensions, which can be used as an effective theory of atomic nuclei. It has been shown by Witten [81, 82] that the Skyrme model can be derived from the $1/N_c$ expansion of the QCD low-energy effective Lagrangian. The Skyrmions are topological solitons, in this framework the topological charge corresponding to the physical baryon number.

The simplest (and original) version of the Skyrme model can be constructed for the SU(2) valued chiral field. Then the model contains only three free parameters which set the length and energy scales and the mass of the pion field, respectively. An appropriate fitting of these parameters together with the assumption that the slowly rotating Skyrmion can be considered as rigid body, allows to evaluate various quantities like, e.g., mean square radii, g -factors of nucleons, and their magnetic moments [7]. It turns out that the agreement with the corresponding experimental data is surprisingly better than what one would expect, being within reasonable accuracy for the usual choice of the values of the parameters (for a review see [19, 84]).

However, the standard version of the Skyrme model has limited success, as there are several problems with description of the nuclear masses. First, in order to describe the properties of the pion excitations, the model must be supplemented with a potential [6, 9, 46]. Various modifications of the model's potential were considered, which typically do not much affect the binding energy of the solitons, but may produce a dramatic change of the shape of Skyrmions [13, 14, 24, 33, 34, 35, 36, 49].

The soliton solutions of the original Skyrme model do not attain the topological bound, which yields a linear relation between the static energy of the Skyrmions and their topological charges,

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the interaction energy of the Skyrmions being about 10% of the mass of a single Skyrmion. On the contrary, the corresponding experimental data for nuclear binding energies are much lower, typically of the order of 1% of the mass of the nucleon. Since all binding energies vanish as the model approaches the topological bound, which is often referred to as the BPS bound, several modifications of the Skyrme model were proposed recently to improve the situation. The most radical version of the BPS Skyrme theory is constructed via truncation of the model, it contains only a sixth-order derivative term, which is the topological current density squared, and a potential [1, 2]. It has been also suggested to consider a modification of the Skyrme model which supports self-duality equations [27, 28, 29]. Another direction is related with extensions of the Skyrme model via inclusion of higher-order derivative terms [1, 2, 3, 30, 37, 52, 53], with various modifications of the potential and couplings.

On the other hand, addition of various terms simulating the contributions of scalar and vector mesons, also can lower the binding energies of Skyrmions [5, 31, 38, 42, 56, 57, 69, 83]. It was pointed out by Sutcliffe [77, 78, 79] that the BPS Skyrme model coupled to an infinite tower of vector mesons can be derived from the $4 + 1$ dimensional Yang–Mills theory via the Atiyah–Manton construction [8]. This approach, inspired by the holographic construction of the Skyrme model by Sakai and Sugimoto [62], provide a good approximation to the truncated BPS Skyrme theory.

Another extension of the Skyrme model can be obtained via $U(1)$ gauging of the Skyrme field [59, 60]. This modification was originally motivated by construction of a semiclassical model of Rubakov–Callan effect [20, 61], a process of monopole-catalyzed proton decay [21, 45]. It was pointed out that the coupling to the electromagnetic field gives the Skyrmions the electric charge, the electrostatic repulsion decreases the total energy of the configuration although the binding energy of gauged Skyrmions in a minimal version of the Skyrme–Maxwell model increases [59]. Notably, the Skyrme–Maxwell theory also can be derived in a holographic model via an expansion of a Yang–Mills field of calorons [23].

Let us remark that, however, the properties of the $U(1)$ gauged Skyrmions have been better understood in lower dimensions. As a $(2 + 1)$ -dimensional analog of Skyrmions, the so-called “*baby Skyrmions*” were proposed in a non-linear $O(3)$ sigma model with a fourth-order derivative term in $2 + 1$ dimensions [15, 16, 48]. This simplified theory emulates the conventional Skyrme model in many respects. In particular, it was used to study dynamics of the solitons [48, 58] and the isorotations of multisoliton configurations without any restrictions of symmetry [10, 39]. The planar Skyrme–Maxwell model was considered in [32, 63, 70]. An interesting observation is that in the strong coupling regime the magnetic flux coupled to the Skyrmion, is quantized, although there is no topological reason of that. Furthermore, electrically charged planar Skyrmions were studied in an extended model with a Chern–Simons term [54, 55, 64]. The gauged BPS baby-Skyrme model was considered recently in [4, 22]. Interesting examples of gauged topological solitons were also constructed in the $O(3)$ sigma model [68], and in the $U(1)$ gauged Faddeev–Skyrme–Maxwell theory [65, 73].

Returning to the (usual) Skyrme model in $3 + 1$ dimensions, it was pointed out that a small contribution of the energy of electromagnetic interaction in the Skyrme model may contribute to a tiny difference between the masses of the neutron and the proton ($m_n = 939.566$ MeV, $m_p = 938.272$ MeV, respectively) [25, 26]. Usually, the proton-neutron mass difference in the Skyrme model can be explained via some mechanism of the spin-isospin symmetry breaking associated with violation of the spherical symmetry of the soliton, with possible contribution of the electromagnetic interaction [44]. Another approach is related with isospin symmetry breaking induced via coupling of the Skyrme field to vector mesons [43, 66]. This effect can be modeled by adding to the Lagrangian an explicit symmetry breaking term [76] or via additional derivative terms [80]. We argue below that the coupling to the electrostatic field alone will break the symmetry of the Skyrme field, even in the sector of topological degree one.

In this paper we consider axially symmetric parametrization of the U(1) gauged Skyrminion of topological degree one. The existence of such solutions relies on the presence of the pion mass potential term, both the gauge coupling and electrostatic potential are restricted from above by the value of the pion mass.

Our results show that the coupling to the electromagnetic field may deform the configurations. We shall construct explicit examples of electrically charged axially symmetric Skyrminions coupled to a circular magnetic flux and investigate their properties.

This paper is organised as follows. In Section 1, we introduce the model, define both the topological and electric charges and consider linearized perturbations of the fields. In Section 2, we present the axially-symmetric ansatz for the Skyrme field and for the electromagnetic field. In Section 3, we define the boundary conditions to be used in finding the numerical solutions. In Section 4, we discuss the numerical results. Finally, in Section 5, we present our conclusions and further remarks.

1 The model

The basic and original version of the Skyrme model in $(3 + 1)$ -Minkowski spacetime is defined by the Lagrangian [74, 75]

$$\mathcal{L}_{\text{Sk}} = \frac{f_\pi^2}{16} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32a_0^2} \text{Tr} ([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2), \quad (1.1)$$

where the Skyrme field $U(x)$ takes values in S^3 , the group manifold of SU(2). In the context of application of the Skyrme model as a candidate model of nuclear physics, the parameter f_π may be interpreted as the pion decay constant and a_0 is a dimensionless constant which can be defined from experimental data. The simple model (1.1) enjoys the chiral $\text{SU}(2)_L \times \text{SU}(2)_R$ global internal symmetry which acts of the Skyrme field through the action $U \rightarrow g_L U g_R^{-1}$, $\forall g_L, g_R \in \text{SU}(2)$. In addition, the flat metric corresponds to $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, with the Cartesian coordinates (x, y, z) , while (r, θ, φ) are the spherical ones.

The Lagrangian of the Skyrme model, in its minimal form (1.1), has just two terms, a usual sigma model term quadratic in derivatives and a Skyrme term quartic in derivatives. A potential term can be included in the Skyrme model, the most common choice being the pion mass term [6]

$$V = \frac{m_\pi^2 f_\pi^2}{8} \text{Tr}(U - \mathbb{1}).$$

Here the parameter m_π defines the asymptotic decay of the Skyrme field, its value being calibrated to fit the physical pion mass. The standard values of the parameters of model are fixed to make contact with the experimental data for protons and pions [6]: $f_\pi = 108$ MeV, $a_0 = 4.84$ and $m_\pi = 138$ MeV. The inclusion of the potential term stabilizes the model with respect to isorotations, beyond the rigid-body approximation [11, 12].

The requirement of finiteness of energy leads to restriction that the matrix-valued field U approaches the vacuum at all points at spatial infinity, $U \xrightarrow{\vec{r} \rightarrow \infty} \mathbb{1}$. This boundary condition compactifies the domain space $\mathbb{R}^3 \rightarrow S^3$ and breaks the full chiral $\text{SU}(2)_L \times \text{SU}(2)_R$ symmetry to the diagonal subgroup. The Skyrme field becomes a map $U: S^3 \mapsto S^3$ and the corresponding topological degree B can be written as

$$B = -\frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \text{Tr}(R_i R_j R_k). \quad (1.2)$$

where $R_i = (\partial_i U) U^\dagger$ is the $\mathfrak{su}(2)$ -valued current. The charge (1.2) is interpreted as the baryon number.

One can construct a gauged version of the Skyrme model by gauging of the U(1) subgroup of the SU(2) global symmetry, associated with the generator of its Cartan subalgebra, see [21, 59, 60]. The covariant derivative of the Skyrme field is defined as

$$D_\mu U = \partial_\mu U - ieA_\mu[Q, U],$$

where the charge matrix is $Q \equiv \frac{1}{2}(\frac{1}{3}\mathbb{1} + \tau_3) = \text{diag}(\frac{2}{3}, -\frac{1}{3})$. Then the full Lagrangian of the U(1) gauged Skyrme model can be written as

$$\begin{aligned} \mathcal{L} = & \frac{f_\pi^2}{16} \text{Tr} (D_\mu U D^\mu U^\dagger) + \frac{1}{32a_0^2} \text{Tr} ([D_\mu U U^\dagger, D_\nu U U^\dagger]^2) \\ & + \frac{m_\pi^2 f_\pi^2}{8} \text{Tr}(U - \mathbb{1}) - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \end{aligned} \quad (1.3)$$

while the electric charge is $Q_e = \int d^3x \partial_i \mathcal{F}^{i0}$. It is convenient to scale away the parameters f_π, a_0 by introducing the energy and length scales $f_\pi/(4a_0)$ and $2/(a_0 f_\pi)$, respectively. The rescaled pion mass parameter is $m = 2m_\pi/(a_0 f_\pi)$ and we absorb the parameter a_0 into the redefined gauge potential, $A_\mu \equiv a_0 \mathcal{A}_\mu$, $F_{\mu\nu} \equiv a_0 \mathcal{F}_{\mu\nu}$ with the gauge coupling $g \equiv e/a_0$.

In terms of these units the Skyrme–Maxwell Lagrangian (1.3) becomes

$$\mathcal{L} = \frac{1}{2} \text{Tr} (D_\mu U D^\mu U^\dagger) + \frac{1}{16} \text{Tr} ([D_\mu U U^\dagger, D_\nu U U^\dagger]^2) + m^2 \text{Tr}(U - \mathbb{1}) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu}. \quad (1.4)$$

In the static gauge the electric charge is now reduced to

$$Q_e = -\frac{1}{a_0} \int d^3x \partial_i^2 A_0 = -\frac{1}{a_0} \oint d\vec{S} \cdot \vec{\nabla} A_0. \quad (1.5)$$

1.1 Linearized perturbations

The Skyrme field can be decomposed into the scalar meson field ϕ_0 and the pion isotriplet ϕ_k via

$$U = \phi_0 \mathbb{1} + i \sum_{k=1}^3 \phi_k \tau_k, \quad (1.6)$$

where τ_k denotes the triplet of usual Pauli matrices, and the field $\phi^a = (\phi_0, \phi_k)$ is restricted to the unit sphere, $\phi^a \cdot \phi^a = 1$. In these component notations the Lagrangian for the U(1) gauged Skyrme model (1.4) can then be written as

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^a D^\mu \phi^a - \frac{1}{2} (D_\mu \phi^a D^\mu \phi^a)^2 + \frac{1}{2} (D_\mu \phi^a D_\nu \phi^a) (D^\mu \phi^b D^\nu \phi^b) \\ & - 2m^2 (1 - \phi_0), \end{aligned} \quad (1.7)$$

where

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - g A_\mu \varepsilon_{\alpha\beta} \phi^\beta, \quad D_\mu \phi^A = \partial_\mu \phi^A, \quad \alpha, \beta = 1, 2, \quad A = 0, 3.$$

In other words, the gauged Skyrme model (1.7) is invariant with respect to the local U(1) gauge transformations

$$U \rightarrow e^{ig\frac{\alpha}{2}\tau_3} U e^{-ig\frac{\alpha}{2}\tau_3}, \quad \text{or} \quad \phi_1 + i\phi_2 \rightarrow e^{-ig\alpha} (\phi_1 + i\phi_2), \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \quad (1.8)$$

where α is any real function of coordinates.

The vacuum of (1.7) corresponds to $U = \mathbb{1}$, $D_\mu \phi^a = 0$ and $F_{\mu\nu} = 0$. In the static gauge one can consider the vacuum boundary conditions

$$U(\infty) = \mathbb{1}, \quad A_0(\infty) = V, \quad A_i(\infty) = 0, \quad (1.9)$$

where V is a real constant. Note that the asymptotic value of the electric potential $A_0(\infty)$ can be adjusted via residual U(1) degrees of freedom, in particular, the transformations (1.8) with $\alpha = -Vt$ allows us to set $A_0(\infty) = 0$. In such a gauge the components of the charged pions field transform as $\phi_\alpha \rightarrow e^{i\omega t} \phi_\alpha$, with $\omega = gV$. In other words, in the Skyrme–Maxwell model (1.3) the isorotations of the Skyrmions are associated with the time-dependent gauge transformations [60]. Hereafter we will fix the boundary conditions (1.9) setting $\omega = 0$.

The asymptotic expansion of the fields around the vacuum (1.9) yields $U_{\text{ex}} \sim (1 - v_0)\mathbb{1} + iv_k \tau_k$ and $A_\mu = a_\mu + V\delta_{0\mu}$, where v_k and a_μ are perturbative excitations of the triplet of pions fields and the electromagnetic potential, respectively, while v_0 is the field excitation of the scalar component ϕ_0 . Since about the vacuum $\phi_0 \sim (1 - v_0)$, $\phi_k \sim v_k$, the constraint on the components of the scalar field yields $v_k^2 = 1 - (1 - v_0)^2$. Thus, $v_k^2 \approx 2v_0 + O(v_0^2)$ and $\partial_i v_0 \approx v_k \partial_i v_k + O(v_0^2, v_0 \partial_i v_0)$ and the fluctuations of the v_0 -field correspond to the second order of expansion of the fields around the vacuum.

Hence, the linearized perturbations of the fields around the vacuum are

$$U_{\text{ex}} = \mathbb{1} + iv_k \tau_k + O(v_k^2), \quad A_\mu = a_\mu + V\delta_{0\mu}.$$

Note that the perturbations of the pion mass term correspond to the second order of expansion of the fields v_k , since $2(1 - \phi_0) \sim 2v_0 \sim v_k^2 + O(v_k^3)$, and the linearized perturbation of the term $D_\mu \phi^a$ yields a leading contribution to the dynamics of the scalar excitations. Then the asymptotic expansion of the Lagrangian (1.7) in the static gauge gives

$$\mathcal{L}_{\text{pert}} = -[f^2 + (\partial_i v_k)^2 + m^2 v_k^2 - g^2 V^2 (v_1^2 + v_2^2)],$$

where $f^2 = (\partial_i a_0)^2 - \partial_i a_j (\partial_i a_j - \partial_j a_i)$ is the contribution of the Maxwell term. The corresponding linearized equation for the fluctuations of the scalar fields v_a is

$$\partial_i^2 v_a - [m^2 v_a - g^2 V^2 (v_1 \delta_{a1} + v_2 \delta_{a2})] = 0.$$

Therefore, the effective mass of charged pions, associated with the excitations $v_\pm = \frac{1}{\sqrt{2}}(v_1 \pm v_2)$, is $m_{\text{eff}}^\pm = \sqrt{m^2 - g^2 V^2}$, while the mass of the uncharged component, associated with excitation of the ϕ_3 field, is slightly higher,¹ $m_{\text{eff}}^{(v_3)} = m$.

Localized massive scalar modes with exponentially decaying tail may exist if $m > 0$ and

$$|gV| \leq m.$$

In the critical case $|gV| = m$ the asymptotic of charged modes possess a dipole as a leading contribution, similar to the neutral mode v_3 in the massless limit. Thus, the pattern of interaction between gauged Skyrmions becomes rather involved, it may include both long-range and dipole forces, as well as short-range Yukawa interactions.

¹This is not in especially good agreement with the experimental values of physical masses of pions, $m_{\pi^\pm} = 139.570$ MeV, $m_{\pi_0} = 134.977$ MeV. However, we can expect that the quantum corrections to the masses of excitations, which will also take into account a quartic pion interaction and contributions from vector mesons, may improve the situation.

1.2 Topological density and the stress-energy tensor

The U(1) gauge covariant generalization of the usual topological charge (1.2) associated to the transformation (1.8) can be constructed by replacing $\partial_i \rightarrow D_i$ in the expression (1.2), i.e.,

$$B_g = -\frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \text{Tr} (D_i U U^{-1} D_j U U^{-1} D_k U U^{-1}).$$

and then subtracting the functional $B_{\text{mag}} = \int d^3x \frac{ig}{32\pi^2} (\varepsilon_{ijk} F_{jk}) \text{Tr} (\tau_3, \partial_i U U^{-1})$. Due the fact that B_g and B_{mag} are invariant by the transformations (1.8), the same follows for the topological charge of the Skyrme–Maxwell model, which is so defined by

$$Q = B_g - B_{\text{mag}} = B + \int d^3x \partial_i \Lambda_i,$$

where

$$\Lambda_i = -\frac{ig}{16\pi^2} \varepsilon_{ijk} A_j \text{Tr} (\{\tau_3, \partial_k U\} U^{-1}).$$

Thus, the topological charge of the Skyrme–Maxwell model is an usual winding number (1.2), plus a surface term which only depends on the boundary conditions of the fields [23, 59]. In the abelian Skyrme–Maxwell model (1.3) with the boundary conditions (1.9) the flux of Λ_i is vanishing² [59].

Therefore, the topological charge of the U(1) gauged Skyrmons is still defined by the Skyrme map (1.2), i.e.,

$$Q = B.$$

In terms of the decomposition (1.6), the topological charge is expressed as

$$\begin{aligned} B &= \int d^3x q(\vec{r}) = -\frac{1}{12\pi^2} \int d^3x \varepsilon_{abcd} \varepsilon_{ijk} \phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d, \\ \Lambda_i &= -\frac{g}{4\pi^2} \varepsilon_{ijk} A_j \varepsilon_{AB} \phi_A \partial_k \phi_B, \end{aligned} \quad (1.10)$$

where $q(\vec{r})$ is the topological charge density and the indices $A, B = 0, 3$.

The stress-energy tensor of the model (1.3) can be obtained by variation of the action with respect to Minkowski metric $\eta^{\mu\nu}$, it gives

$$T^{\mu\nu} = T_{(M)}^{\mu\nu} + T_{(S)}^{\mu\nu},$$

where the electromagnetic contribution of the Maxwell term is

$$T_{(M)}^{\mu\nu} = -2F^{\mu\sigma} F^\nu{}_\sigma + \frac{\eta^{\mu\nu}}{2} F_{\alpha\beta} F^{\alpha\beta},$$

and the stress-energy tensor of the U(1)-gauged Skyrmons is

$$\begin{aligned} T_{(S)}^{\mu\nu} &= 2[D^\mu \phi_a D^\nu \phi^a - (D^{[\mu} \phi^a D^{\alpha]} \phi^b)(D^{[\nu} \phi_a D_{\alpha]} \phi_b)] \\ &\quad - \eta^{\mu\nu} \left((D_\alpha \phi_a)^2 - \frac{1}{2} (D_{[\alpha} \phi_a D_{\beta]} \phi_b)^2 - 2m^2(1 - \phi_0) \right). \end{aligned} \quad (1.11)$$

²However, in the SU(2) gauged Skyrme model, it can unwind the Skyrmon [17, 18]. This yields a simple classical model of monopole catalysis of nucleon decay (the Rubakov–Callan effect).

In the static gauge the Hamiltonian of the Skyrme–Maxwell model can be written as

$$\mathcal{H}_{\text{static}} = \mathcal{H}_1 + \mathcal{H}_2, \quad (1.12)$$

where \mathcal{H}_1 and \mathcal{H}_2 are non-negative terms given by

$$\begin{aligned} \mathcal{H}_1 &= \frac{1}{2}|F_{ij}|^2 + |D_i\phi_a|^2 + \frac{1}{2}|D_{[i}\phi_a D_{j]}\phi_b|^2 + 2m^2(1 - \phi_0), \\ \mathcal{H}_2 &= |\partial_i A_0|^2 + g^2 A_0^2 M_\phi^2, \quad \text{with} \quad M_\phi^2 \equiv \left[(1 + |\partial_i \phi_A|^2)|\phi_\alpha|^2 + \frac{1}{4}|\partial_i(|\phi_A|^2)|^2 \right]. \end{aligned}$$

where $|\partial_i A_0|^2 = \partial_i A_0 \partial_i A_0$, and so on. Clearly, the last term in \mathcal{H}_2 yields the Gauss law. The function $g^2 M_\phi^2$ behaves like a spatially dependent square mass for the electric potential, which appears in the interior of the Skyrmion and vanishes asymptotically. The remaining gauge degrees of freedom can be fixed imposing the Coulomb gauge, $\partial_i A_i = 0$. Then the static Maxwell equations for the electric and magnetic potentials can be written as

$$\partial_j^2 A_0 = g^2 M_\phi^2 A_0, \quad \partial_j^2 A_i = -g\varepsilon_{\alpha\beta}\phi_\beta \left[(1 + D_j\phi_a D_j\phi_a) D_i\phi_\alpha - (D_j\phi_a D_j\phi_\alpha) D_i\phi_a \right].$$

2 Axially symmetric ansatz and effective Lagrangian

An obvious correspondence between the gauge transformations (1.8) and isorotations of the components of the Skyrme field suggests that, in general, gauged Skyrmion of topological degree one is not spherically symmetric [59]. Hence we consider a general axially symmetric ansatz (see [12, 41, 47, 60]):

$$\phi_1 + i\phi_2 = \psi_1(r, \theta)e^{in\varphi}, \quad \phi_3 = \psi_2(r, \theta), \quad \phi_0 = \psi_3(r, \theta), \quad (2.1)$$

where n an integer (so-called ‘‘vorticity’’) and the sigma-model constraint $\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$ is imposed. The gauge field is parameterized by the two potentials (magnetic and electric, respectively)

$$A \equiv A_\mu dx^\mu = A_\varphi(r, \theta)d\varphi + A_0(r, \theta)dt. \quad (2.2)$$

All five functions which parameterize the ansatz (2.1), (2.2) depend on the radial variable r and the polar angle θ .

Substituting the axial ansatz (2.1), (2.2) into the Hamiltonian of the Skyrme–Maxwell model (1.12) and into the Lagrangian (1.7), we obtain

$$L = -2\pi \int d\theta dr r^2 \sin\theta [\mathcal{F}^- + \mathcal{L}_2^- + \mathcal{L}_4^- + 2m^2(1 - \psi_3)], \quad (2.3)$$

$$H = 2\pi \int d\theta dr r^2 \sin\theta [\mathcal{F}^+ + \mathcal{L}_2^+ + \mathcal{L}_4^+ + 2m^2(1 - \psi_3)], \quad (2.4)$$

where

$$\begin{aligned} \mathcal{F}^\mp &\equiv \frac{1}{r^2 \sin^2\theta} \left(A_{\varphi,r}^2 + \frac{A_{\varphi,\theta}^2}{r^2} \right) \mp \left(A_{0,r}^2 + \frac{A_{0,\theta}^2}{r^2} \right), \\ \mathcal{L}_2^\mp &\equiv \psi_{a,r}^2 + \frac{\psi_{a,\theta}^2}{r^2} + \psi_1^2 \left(\frac{(n + gA_\varphi)^2}{r^2 \sin^2\theta} \mp g^2 A_0^2 \right), \\ \mathcal{L}_4^\mp &\equiv \frac{1}{r^2} [(\psi_{3,\theta}\psi_{2,r} - \psi_{2,\theta}\psi_{3,r})^2 + (\psi_{2,\theta}\psi_{1,r} - \psi_{1,\theta}\psi_{2,r})^2 + (\psi_{3,\theta}\psi_{1,r} - \psi_{1,\theta}\psi_{3,r})^2] \\ &\quad + \psi_1^2 \left(\frac{(n + gA_\varphi)^2}{r^2 \sin^2\theta} \mp g^2 A_0^2 \right) \left(\psi_{a,r}^2 + \frac{\psi_{a,\theta}^2}{r^2} \right). \end{aligned}$$

Here a comma denotes partial differentiation, i.e., $A_{\varphi,r} \equiv \frac{\partial A_\varphi}{\partial r}$, etc. The corresponding static field equations can be obtained from the variation of the effective Lagrangian (2.3) with respect to the functions parametrizing the ansatz (2.1), (2.2). In particular, one finds the reduced Maxwell equations for the magnetic and electric potentials:

$$\partial_r^2 A_\varphi + \frac{\partial_\theta^2 A_\varphi - \partial_\theta A_\varphi \cot \theta}{r^2} = g(n + gA_\varphi)K, \quad (2.5)$$

$$\partial_i^2 A_0 = g^2 A_0 K, \quad \text{with} \quad K \equiv \psi_1^2 \left[1 + \psi_{a,r}^2 + \frac{\psi_{a,\theta}^2}{r^2} \right]. \quad (2.6)$$

Further, the linearized equations for the pion fields for large r become

$$\partial_r^2 \psi_1 + \frac{2\partial_r \psi_1}{r} - (m^2 - g^2 V^2) \psi_1 = 0, \quad \partial_r^2 \psi_2 + \frac{2\partial_r \psi_2}{r} - m^2 \psi_2 = 0,$$

where $V = A_0(r \rightarrow \infty)$. Thus, the fields asymptotically decay as $\psi_1 \sim e^{-\sqrt{m^2 - g^2 V^2} r}$ and $\psi_2 \sim e^{-mr}$ and localized solutions exist if $|m| \geq |gV|$. Finally, substituting the ansatz (2.1) into the Skyrme topological charge (1.2) gives $B = n$ [72]. Hereafter we restrict our consideration to the gauged Skyrmions of degree one setting $n = 1$.

The total mass-energy of the configuration is defined as the volume integral over all space of the T_{00} component of the energy-momentum tensor, $E = \int d^3x T_0^0$, which also includes a purely electromagnetic contribution. The total angular momentum of the gauged Skyrminion is defined as $J = \int d^3x T_0^\varphi$. Using the field equations and the definition of the topological charge (1.2), one can show that the angular momentum of the gauged Skyrminion is classically quantized in units set by the electric charge [60]. Indeed, using the axially symmetric ansatz (2.1) and (2.2) the component $T_{0\varphi}$ of the stress-energy tensor (1.11) can be written as

$$T_{0\varphi} = 2[\partial_i A_0 \partial_i A_\varphi + gA_0(n + gA_\varphi)K],$$

where K is defined by (2.6). However, using the Euler-Lagrange equation for the electric potential (2.6) we can write $T_{0\varphi}$ as a total derivative, i.e.,

$$T_{0\varphi} = 2 \left[\partial_i A_0 \partial_i A_\varphi + \left(\frac{n}{g} + A_\varphi \right) \partial_i^2 A_0 \right] = 2\partial_i \left[\left(\frac{n}{g} + A_\varphi \right) \partial_i A_0 \right].$$

Therefore, making use of the equation (1.5), the expression for the angular momentum of the gauged Skyrminion can be written as

$$J = \int d^3x T_0^\varphi = -2 \oint_\infty d\vec{S} \cdot \vec{\nabla} A_0 \left(\frac{n}{g} + A_\varphi \right) = -\frac{2n}{g} \oint_\infty d\vec{S} \cdot \vec{\nabla} A_0 = 2na_0 \frac{Q_e}{g},$$

where we take into account that $\frac{n}{g} + A_\varphi \rightarrow \frac{n}{g}$, as $r \rightarrow \infty$. Since the dimensions of the energy and the length are given by $f_\pi/(4a_0)$ and $2/(f_\pi a_0)$, respectively, the dimension of angular momentum is $1/(2a_0^2)$. Using the definition $g = e/a_0$, we can write the angular momentum in the original units of the U(1) gauged Skyrme model (1.3) as $J = n \frac{Q_e}{e}$.

In addition to M , J and Q_e , another quantity of interest is the magnetic dipole moment μ_m , which is computed from the far field expression of the magnetic potential, $A_\varphi \rightarrow \frac{\mu_m}{r} \sin^2 \theta + O(r^{-2})$.

Also, as a measure of the deformation degree of the configurations, we consider the ratio

$$\epsilon = \frac{\sqrt{\langle x^2 \rangle}}{\sqrt{\langle z^2 \rangle}}, \quad \text{with} \quad \langle x^2 \rangle = \frac{\int d^3x x^2 q(\vec{r})}{\int d^3x q(\vec{r})}, \quad \langle z^2 \rangle = \frac{\int d^3x z^2 q(\vec{r})}{\int d^3x q(\vec{r})}, \quad (2.7)$$

where $\langle |\rho| \rangle$ and $\langle |z| \rangle$ are the mean dimensions of the axially symmetric gauged Skyrminion, defined as averages over the topological charge density $q(\vec{r})$ (1.10).

3 Numerical scheme and the boundary conditions

To find numerical solutions of field equations we used the software package CADSOOL based on the Newton–Raphson algorithm [67]. The calculations are performed on an equidistant grid in spherical coordinates r and θ . Typical grids we used have sizes 70×60 . In our numerical scheme we map the infinite interval of the variable r onto the compact radial coordinate $x = \frac{r}{1+r} \in [0 : 1]$. Estimated numerical errors are of order of 10^{-5} .

The system of the field equations of the Skyrme–Maxwell model represents a set of five coupled elliptic partial differential equations with mixed derivatives, to be solved numerically subject to the appropriate boundary conditions. As usual, they follow from the condition of regularity of the fields on the symmetry axis and symmetry requirements, as well as the condition of finiteness of the energy of the system. Explicitly, we impose

$$\begin{aligned} \psi_1|_{r=0} = 0, & \quad \psi_2|_{r=0} = 0, & \quad \psi_3|_{r=0} = -1, & \quad A_\varphi|_{r=0} = 0, & \quad \partial_r A_0|_{r=0} = 0, \\ \psi_1|_{r=\infty} = 0, & \quad \psi_2|_{r=\infty} = 0, & \quad \psi_3|_{r=\infty} = 1, & \quad A_\varphi|_{r=\infty} = 0, & \quad A_0|_{r=\infty} = V, \end{aligned}$$

and

$$\psi_1|_{\theta=0,\pi} = 0, \quad \partial_\theta \psi_2|_{\theta=0,\pi} = 0, \quad \partial_\theta \psi_3|_{\theta=0,\pi} = 0, \quad A_\varphi|_{\theta=0,\pi} = 0, \quad \partial_\theta A_0|_{\theta=0,\pi} = 0.$$

The solutions also possess a reflection symmetry with respect to the equatorial plane, i.e.,

$$\partial_\theta \psi_1|_{\theta=\pi/2} = 0, \quad \psi_2|_{\theta=\pi/2} = 0, \quad \partial_\theta \psi_3|_{\theta=\pi/2} = 0, \quad \partial_\theta A_\varphi|_{\theta=\pi/2} = 0, \quad \partial_\theta A_0|_{\theta=\pi/2} = 0.$$

4 Numerical results

For a fixed value of the pion mass $m = 1$ the solutions of the gauged Skyrme model depend on two continuous parameters, the values the gauge coupling g and the electric potential at infinity $A_0(\infty) = V$. Localized solutions of the model (1.7) exist as $|gV| \leq m = 1$.

In our approach, we have first considered a computation of the ungauged $B = 1$ spherically symmetric Skyrmion (with $g = V = 0$), which is used as an input for following numerical calculations. Then both g and V are increased in small steps. Setting $V = 0$ and increasing the gauge coupling constant g yields a branch of gauged static Skyrmions coupled to a toroidal magnetic flux and $A_0 \equiv 0$. Notably, there is no purely electrically charged solutions, all configurations being equipped with a toroidal magnetic flux which induces a dipole magnetic moment of the gauged Skyrmion.

First, we confirm an observation [59] that for any non-zero value of the gauge coupling g the spherical symmetry of the $B = 1$ ungauged Skyrmion is broken, the U(1) gauged Skyrmions are axially symmetric. In Figure 1, we exhibited some typical examples of the isosurfaces of the energy density and the angular momentum density at $V = 0.1$ and $m = 1$ and some set of values of the gauge coupling. The deformation of the Skyrmion is maximal as V becomes very small while the coupling constant g increases within allowed range, see Figure 1. In such a limit, the electrostatic energy is at least two orders of magnitude smaller than the magnetic energy, as shown in Figure 4 (right upper plot). The total energy of the gauged Skyrmion decreases as g increases, since the toroidal magnetic flux squeezes the configuration towards the symmetry axis.

For relatively small values of the parameter V , the energy of electrostatic repulsion remains much smaller than the magnetic energy of the gauged Skyrmion. The Figure 2 exhibits the profiles in the (xy) -plane of the ϕ_0 -component of the Skyrme field, the electrostatic potential A_0 and of the magnetic potential A_φ , for $V = 0.1$ and several values of the gauge coupling constant g .

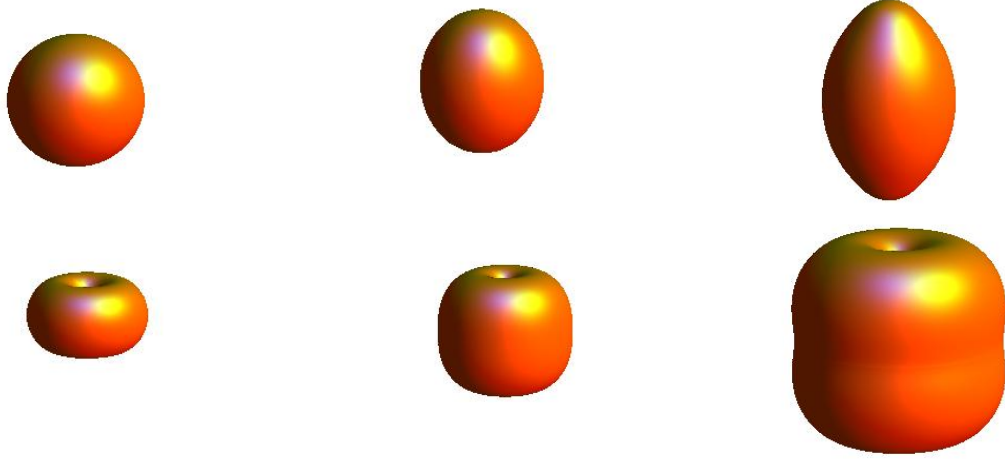


Figure 1. Isosurfaces of the total energy density distributions (top row) and the angular momentum density (bottom row) of the gauged $B = 1$ Skyrmion for $g = 0.1$ (left figures), $g = 2$ (middle figures) and $g = 10$ (right figures) at $V = 0.1$ and $m = 1$.

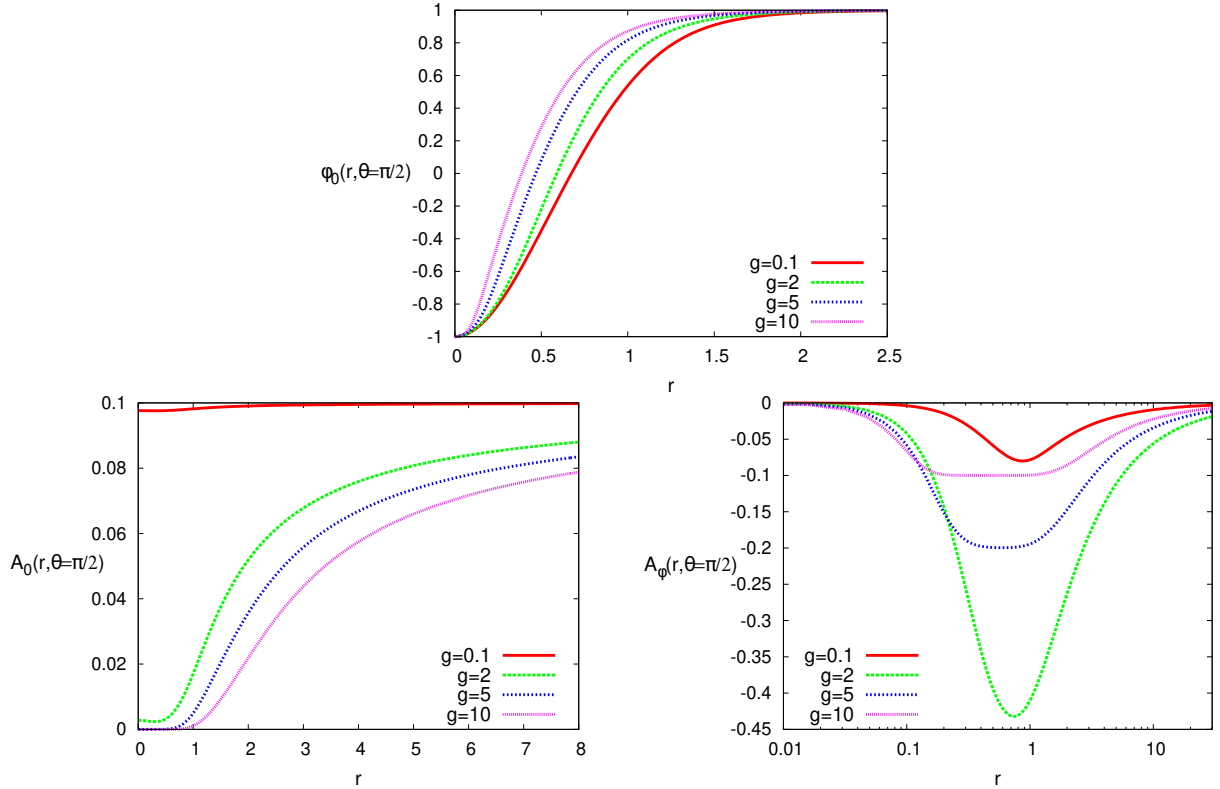


Figure 2. Gauged $B = 1$ Skyrmion: Profile functions of the component of the Skyrme field ϕ_0 , the electric potential A_0 and the magnetic potential A_φ of some illustrative solutions at the symmetry plane $\theta = \pi/2$ are plotted as functions of the radial coordinate for some set of values of g at $V = 0.1$.

Physically, the deformation of the Skyrmion is related with the occurrence of the circular magnetic flux in the equatorial plane. Its position is associated with a minimum of the magnetic potential A_φ . As the gauge coupling increases within its allowed range, the electrostatic potential A_0 in some central region decreases, approaching values close to zero for large enough g , see Figure 2, upper right plot.

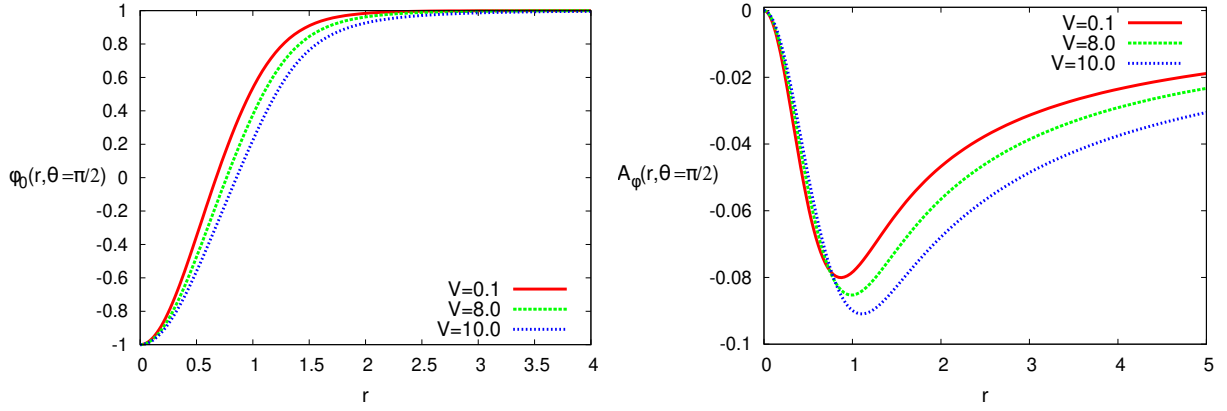


Figure 3. Gauged $B = 1$ Skyrmion: Profile functions of the component of the Skyrme field ϕ_0 and the magnetic potential A_φ of some illustrative solutions at the symmetry plane $\theta = \pi/2$ are plotted as functions of the radial coordinate for some set of values of V at $g = 0.1$.

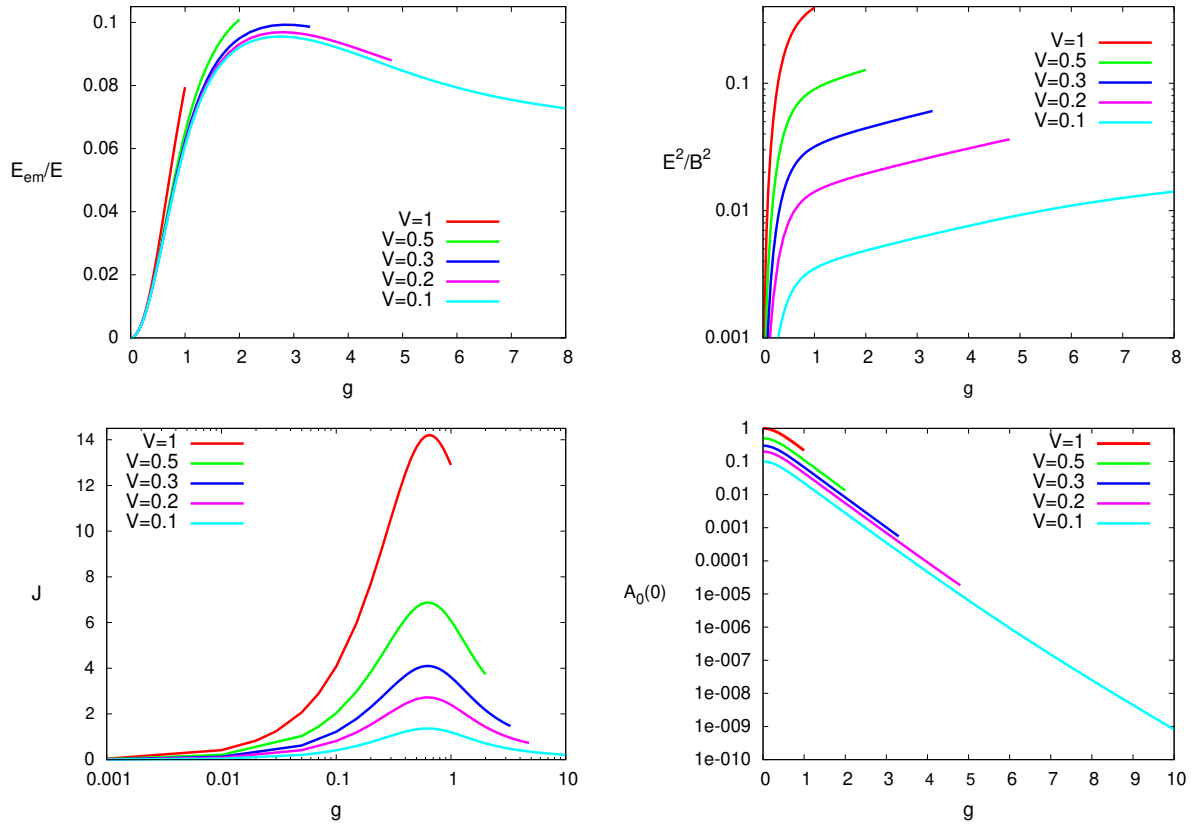


Figure 4. Gauged $B = 1$ Skyrmion: The ratio of the electromagnetic and total energies (upper left), the ratio of the electric and magnetic energies (upper right), the total angular momentum (lower left) and the value of the electric potential A_0 at the center of the Skyrmion (lower right) are plotted as functions of the gauge coupling constant g for some set of values of the parameter V at $m = 1$.

At the same time, in the strong coupling regime, the magnetic potential develops a sharp plateau in the equatorial plane, where $gA_\varphi + n \sim 0$ and $A_0 \sim 0$, see Figure 2 (with n the winding of the Skyrme field in the ansatz (2.1)). Clearly, within this domain, the electromagnetic fields described by the Maxwell equations (2.5), (2.6) are massless. Moreover, the electric field is pushed out of the core of the configurations, in this limit the electric charge density distribution

is almost spherically symmetric. The plateau further extends to some region as the gauge coupling grows up to the maximal possible value. This pattern resembles the Meissner effect: at some critical value of the gauge coupling $g_{\text{cr}} \approx 3.3$ the magnetic field is expelled from the core and the circular magnetic flux becomes quantized in units of the topological winding n .

It is interesting to remark that this situation is similar to that found for the U(1) gauged Hopfions in the Faddeev–Skyrme–Maxwell model [73] and for the gauged planar Skyrmons [32, 63, 70] and gauged O(3) lumps [68]. In the limit $V = 0$ the electric field is vanishing and the gauge coupling g may increase indefinitely. Effectively, this corresponds to the truncation of the system to the case of the abelian Higgs model. Then the linear string of magnetic flux generated by massless scalar excitations passes through the center of the soliton core, which is strongly deformed by the “supercurrent” located in the equatorial plane.

The electrostatic energy dominates when we set $g \ll 1$ and increase the parameter V up to its maximal possible value. The angular momentum of the configuration rapidly increases, as displayed in Figure 5, middle left plot. The electrostatic repulsion slightly increases the size of the core of the Skyrmion, see Figure 3. However, this effect is not so strong, as it is in the case of dominance of the magnetic energy.

In Figure 4, we show the ratio between the electromagnetic and total energy of gauged Skyrmion as defined by the functional (2.4), as a function of the gauge coupling g (upper left plot) for several values of V . For all considered range of parameters, the contribution of electromagnetic energy to E is less than 10 %, this ratio being maximal for $V \sim 0.5$ and $g \sim 2$.

Also, as the parameter V remains smaller than 2, the electrostatic energy of the configuration is smaller than the magnetic energy, see Figure 5, upper right plot. However, the energy stored in the electric field becomes more significant as V increases and the gauge coupling g remains relatively small. For any g , both the magnetic dipole moment and the deformation of the solutions increase with V , see Figure 5 (bottom panels). Moreover, note that for fixed V , the behaviour of μ_m as a function of g is non-monotonic.

Notably, both the energy and the angular momentum of gauged Skyrmion remain finite as $|gV| \rightarrow m$. This is similar to the situation which was observed for isospinning solitons in 3 + 1 dimensions [11, 12, 40]. On the contrary, in 2 + 1 dimensions both the energy and the angular momentum of isospinning Skyrmons diverge [10, 39].

Considering the dependency of the angular momentum and electric charge of the gauged Skyrmion, we observe that, for a fixed value of the electrostatic parameter V , both quantities become maximal at some value of the gauge coupling g . Further increase of the coupling leads to decrease of the angular momentum J , see Figure 4, lower left plot.

We can now attempt to apply our results to the physical properties of baryons. A standard approach is to make use of the Adkins–Nappi–Witten calibration setting the physical values of the Skyrme parameters as $f_\pi = 129$ MeV and $a_0 = 5.45$ [7]. The pion mass parameter is taken as $m = 0.526$, other calibrations are also suggested, see, e.g., [50].

Considering possible interpretation of gauged Skyrmons as baryons, we have to take into account that in natural units $\hbar = c = 1$ the physical charge of a proton is $Q_e^2 = 4\pi\alpha$ (see equation (1.5)), where $\alpha = 1/137$ is the fine structure constant. This condition yields a continuous set of values of parameters (g, V) , see inset plot in Figure 6.

We have shown above that the gauged Skyrmion can lower its classical mass by interacting with the electromagnetic field. Since, naively, a neutron could be identified with the ungauged $g = 0$ Skyrmion, one can attempt to evaluate the corresponding mass splitting and compare it with corresponding experimental data. However, real nuclei have non-zero magnetic dipole moments, therefore it will be consistent to consider a neutron as an electrically neutral gauged Skyrmion with $V = 0$ and a non-vanishing coupling g .

Our numerical evaluations of the ratio of energies of charged and uncharged Skyrmons, which can be identified as a proton and a neutron, respectively, show that, for a given value of the

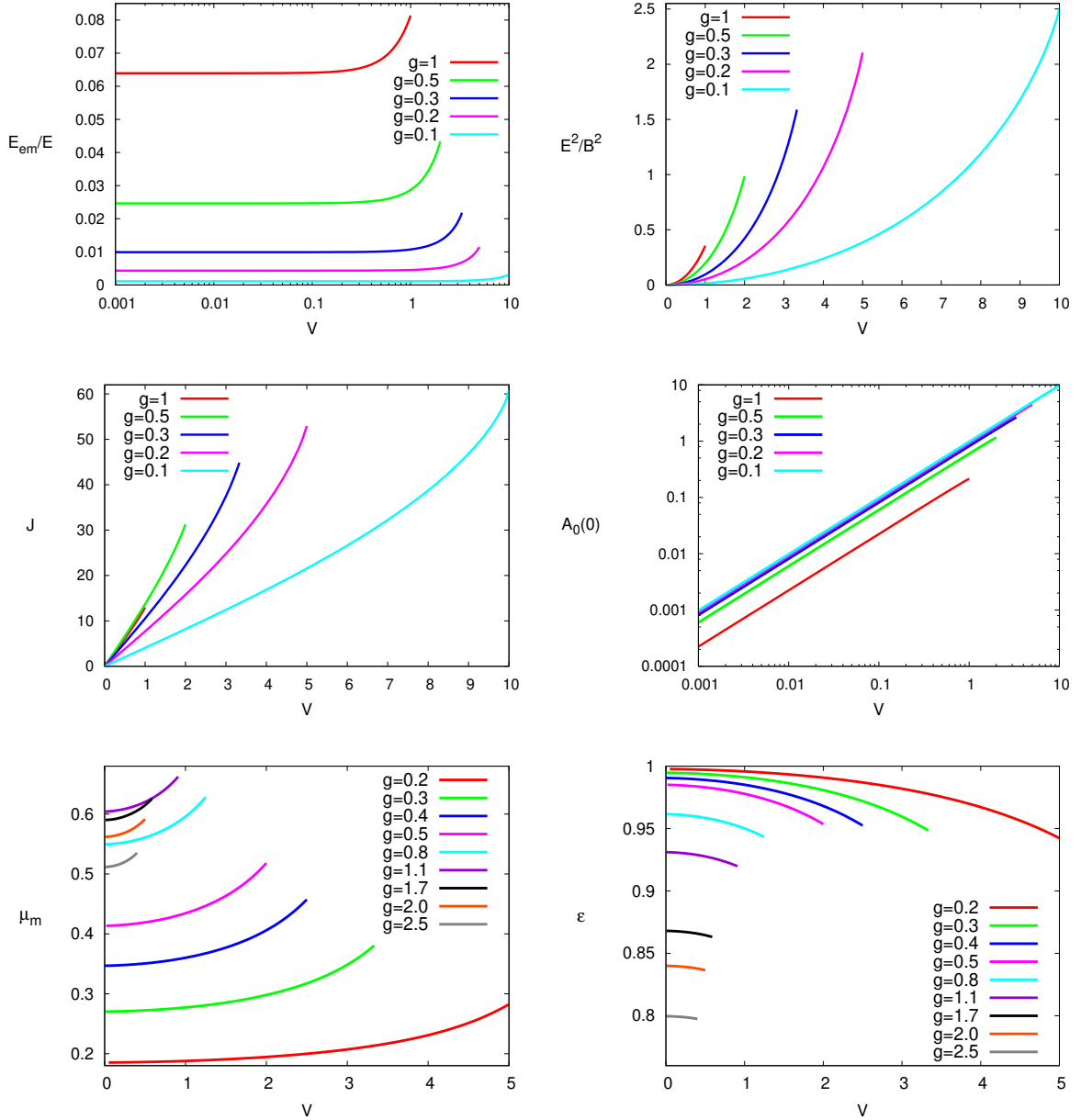


Figure 5. Gauged $B = 1$ Skyrmion: The ratio of the electromagnetic and total energies (upper left), the ratio of the electric and magnetic energies (upper right), the total angular momentum (middle left), the value of the electric potential A_0 at the center of the Skyrmion (middle right), the values of the dipole magnetic moment (bottom left) and the deformation parameter ϵ (2.7) (bottom right) are plotted as functions of the parameter V for some set of values of the gauge coupling constant g at $m = 1$.

gauge coupling g , the energy of a charged soliton is always higher, see Figure 6. Clearly, this is not in a good agreement with the experimental data, the proton is a bit lighter than a neutron. Subtraction of the contribution of electromagnetic energy does not change the situation, a proton and a neutron cannot be described in the gauged Skyrme model with the same set of parameters. Related evaluation of the ratio of magnetic moments of the $Q_e = 1$ (in units of $\sqrt{4\pi\alpha}$) and $Q_e = 0$ states with fixed value of g leads to an even worse disagreement, see Figure 6. We can conclude that more accurate description of nuclei in the Skyrme model can be possible only via appropriate adjustment of the parameters of the model, even for the states with baryon charge $B = 1$.

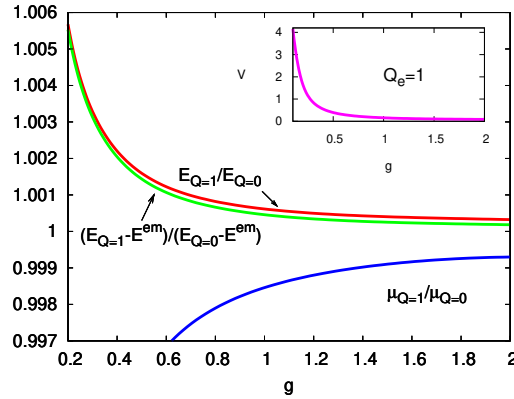


Figure 6. Gauged $B = 1$ Skyrmion: The ratio of energies of charged and uncharged Skyrmons and the ratio of the corresponding dipole magnetic moments are plotted as functions of the gauge coupling g at $m = 0.526$. The curve of dependency $g(V)$ for $Q_e = 1$ is displayed in the inset plot.

5 Conclusions

In this paper, we revisited the $U(1)$ gauged Skyrme–Maxwell model in $3 + 1$ dimensions and studied the solutions of topological degree one. We have shown that their domain of existence is restricted by the value of the pion mass parameter m , such that by analogy with the isospinning Skyrmons [12], the gauged Skyrmons do not exist in the model without a potential term. We confirm the observation of paper [59] that the coupling to the electromagnetic field violates the spherical symmetry of the configuration and induces a dipole magnetic moment of the Skyrmion, which carries both an electric charge and a (local) magnetic flux (but not a net one).

We find that gauged Skyrmons exist for all range of values of parameters of the model restricted by the condition $|gV| \leq m$. The upper critical value m yields two limiting cases, $|g| \gg |V|$ (magnetic limit) and $|V| \gg |g|$ (electrostatic limit), respectively. The gauged Skyrmion is strongly deformed in the magnetic limit, it becomes extremely elongated and stretched out by the circular magnetic flux. In the opposite limit the repulsive electrostatic interaction increases the size of the interior region of the soliton, however the deformation of the Skyrmion is not so strong as in the magnetic limit.

Our results show that, although the mass of a gauged Skyrmion can be decreased by interaction with the electromagnetic field, the proton-neutron mass difference cannot be described in the $U(1)$ gauged Skyrme model with the same fixed set of parameters.

Certainly, this is a first step toward complete investigation of the gauged Skyrmons, this study should be extended to the solutions of higher degrees and different geometry. One can expect the electrostatic repulsion and generation of magnetic fluxes may significantly affect the bounded configurations of gauged Skyrmons.

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