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Compatible Poisson Brackets Associated with Elliptic Curves in $G(2,5)$

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Abstract. We prove that a pair of Feigin–Odesskii Poisson brackets on $\mathbb{P}^4$ associated with elliptic curves given as linear sections of the Grassmannian $G(2,5)$ are compatible if and only if this pair of elliptic curves is contained in a del Pezzo surface obtained as a linear section of $G(2,5)$.

Key words: Poisson bracket; bi-Hamiltonian structure; elliptic curve; triple Massey products

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1 Introduction

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

In this paper we continue to study compatible pairs among the Poisson brackets on projective spaces introduced by Feigin–Odesskii (see [1, 10]). Their construction associates with every stable vector bundle $\mathcal{V}$ of degree $n > 0$ and rank k on an elliptic curve E, a Poisson bracket on the projective space $\mathbb{P}H^0(E,\mathcal{V})^*$. We refer to such Poisson brackets as FO brackets of type $q_{n,k}$.

Two Poisson brackets are called *compatible* if the corresponding bivectors satisfy $[\Pi_1,\Pi_2] = 0$ (equivalently, any linear combination of these brackets is again Poisson). In [9], Odesskii and Wolf discovered 9-dimensional spaces of compatible FO brackets of type $q_{n,1}$ on $\mathbb{P}^{n-1}$ for each $n \geq 3$. Their construction was interpreted and extended in [3], where the authors showed that one gets compatible FO brackets if the elliptic curves are anticanonical divisors on a surface S and the stable bundles on them are restrictions of a single exceptional bundle on S that forms an exceptional pair with $\mathcal{O}_S$ (see [3, Theorem 4.4]). One can ask whether any two compatible FO brackets of type $q_{n,k}$ on $\mathbb{P}^{n-1}$ appear in this way. In [7] we have shown that this is the case for $k = 1$ (for some specific rational surfaces containing normal elliptic curves in projective spaces). In the present work, we consider the case of FO brackets of type $q_{5,2}$ on $\mathbb{P}^4$. Note that the question of finding bi-Hamiltonian structures with brackets of type $q_{5,2}$ was raised by Rubtsov in [11].

Let V be a 5-dimensional vector space. Consider the Plücker embedding

$$G(2,V) \to \mathbb{P}\left(\bigwedge^2 V\right).$$

It is well known that for a generic 5-dimensional subspace $W \subset \bigwedge^2 V$ the corresponding linear section

$$E_W := G(2,V) \cap \mathbb{P}W$$

is an elliptic curve. Furthermore, if $\mathcal{U} \subset V \otimes \mathcal{O}$ is the universal subbundle on $G(2,V)$, then one can check that the restriction $V_W := \mathcal{U}^\vee|_{E_W}$ is a stable bundle of rank 2 and degree 5 on E_W (see Lemma 2.2.1 below). Thus, we have the corresponding Feigin–Odesskii bracket of type $q_{5,2}$ on $\mathbb{P}H^0(E_W, V_W)^*$.

Furthermore, one can check that the restriction map

$$V^* = H^0(G(2,V), \mathcal{U}^\vee) \to H^0(E_W, V_W)$$

is an isomorphism (see Lemma 2.2.1). Thus, we get a Poisson bracket Π_W on $\mathbb{P}V$ (defined up to a rescaling).

On the other hand, we have a natural $\mathrm{GL}(V)$-invariant map

$$\pi_{5,2}\colon \bigwedge^5 \left(\bigwedge^2 V\right) \to H^0\left(\mathbb{P}V, \bigwedge^2 T\right) \otimes \det^2(V)$$

constructed as follows.

Note that we have a natural isomorphism $V \simeq H^0(\mathbb{P}V, T(-1))$, hence we get a natural map $V \otimes \mathcal{O}(1) \to T$, and hence, the composed map

$$\phi\colon\ W \otimes \mathcal{O}(2) \to \bigwedge^2 V \otimes \mathcal{O}(2) \to \bigwedge^2 T$$

on $\mathbb{P}V$. Taking the 5th exterior power of this map, we get a map

$$\bigwedge^5(\phi)\colon\ \det(W) \otimes \mathcal{O}(10) \to \bigwedge^5 \left(\bigwedge^2 T\right) \simeq \left(\bigwedge^2 T\right)^\vee \otimes \det^3(T),$$

where we used the identification $\det\left(\bigwedge^2 T\right) \simeq \det^3(T)$. Note that we have a nondegenerate pairing given by the exterior product,

$$\bigwedge^2 T \otimes \bigwedge^2 T \to \det(T),$$

hence, we have an isomorphism $\bigwedge^2 T \simeq \left(\bigwedge^2 T\right)^\vee \otimes \det(T)$, and we can rewrite the above map as

$$\det(W) \to \bigwedge^2 T \otimes \det^2(T)(-10) \simeq \bigwedge^2 T \otimes \det^2(V).$$

Theorem A. *For every* 5*-dimensional subspace* $W \subset \bigwedge^2 V$*, such that* $E_W := G(2,V) \cap \mathbb{P}W$ *is an elliptic curve, one has an equality*

$$\pi_{5,2}(\lambda_W) = \Pi_W \otimes \delta,$$

for some trivializations $\lambda_W \in \bigwedge^5 W$ *and* $\delta \in \det^2(V)$*.*

Theorem A is deduced from the existence of a formula for Π_W, depending linearly on the Plücker coordinates of W (which follows from the results of [3]), combined with a representation-theoretic argument employing the fact that the construction of Π_W is $\mathrm{GL}(V)$-equivariant.

Theorem B.

(i) *For* 5*-dimensional subspaces* $W, W' \subset \bigwedge^2 V$ *such that* E_W *and* $E_{W'}$ *are elliptic curves, the Poisson brackets* Π_W *and* $\Pi_{W'}$ *are compatible if and only if* $\dim W \cap W' \geq 4$*.*

(ii) *For any collection* (W_i) *of* 5*-dimensional subspaces in* $\bigwedge^2 V$*, the brackets* (Π_{W_i}) *are pairwise compatible if and only if either there exists a* 6*-dimensional subspace* $U \subset \bigwedge^2 V$ *such that each* W_i *is contained in* U*, or there exists a* 4*-dimensional subspace* $K \subset \bigwedge^2 V$ *such that each* W_i *contains* K*.*

The idea of proof is to analyze the vanishing $[\Pi_{W_1}, \Pi_{W_2}] = 0$ near a sufficiently generic point where Π_{W_1} vanishes. An important ingredient of the proof is a 2-dimensional distribution on $G(2,V)$ associated with $W \subset \bigwedge^2 V$: it corresponds to the rational map from $G(2,V)$ to $\mathbb{P}^4$ obtained as the composition of the Plucker embedding with the linear projection to $\mathbb{P}(\bigwedge^2 V/W)$ (see Section 3.3). The analysis of the vanishing of the Schouten bracket is used to prove that the elliptic curve E_{W_1} is everywhere tangent to the distribution associated with W_2, which implies the result.

Corollary C. *The maximal dimension of a linear subspace of Poisson brackets on $\mathbb{P}(V)$, where $\dim V = 5$, spanned by some FO brackets Π_W of type $q_{5,2}$, is 6.*

Theorems A and B suggest the following

Conjecture D. *Let $W \subset \bigwedge^2 V$ be a 5-dimensional subspace such that E_W is an elliptic curve. Consider the subspace*

$$T_W := \left(\bigwedge^4 W\right) \wedge \left(\bigwedge^2 V\right) \subset \bigwedge^5 \left(\bigwedge^2 V\right)$$

(the quotient of the latter subspace by $\bigwedge^5 W$ is exactly the image of the tangent space to the Grassmannian $G\big(5, \bigwedge^2 V\big)$ under Plücker embedding). Then the subspace of $\xi \in \bigwedge^5\big(\bigwedge^2 V\big)$ satisfying $[\pi_{5,2}(\xi), \Pi_W] = 0$ coincides with $T_W + \ker(\pi_{5,2})$.

Note that we know the inclusion one way: the subspace T_W is spanned by $\bigwedge^5(W')$ such that $\dim(W' \cap W) \geq 4$ and $E_{W'}$ is an elliptic curve, and by Theorems A and B, $\big[\pi_{5,2}\big(\bigwedge^5(W')\big), \Pi_W\big] = 0$.

2 Generalities

2.1 Feigin–Odesskii Poisson brackets of type $q_{n,k}$

Let E be an elliptic curve, with a fixed trivialization $\eta\colon \mathcal{O}_E \to \omega_E$, $\mathcal{V}$ a stable bundle on E of rank k and degree $n > 0$. We consider the corresponding Feigin–Odesskii Poisson bracket $\Pi = \Pi_{E,\mathcal{V}}$ of type $q_{n,k}$ on the projective space $\mathbb{P}H^1\big(E, \mathcal{V}^\vee\big)$ defined as in [10].

We will need the following definition of Π in terms of triple Massey products. For nonzero $\phi \in H^1\big(E, \mathcal{V}^\vee\big)$, we denote by $\langle\phi\rangle$ the corresponding line, and we use the identification of the cotangent space to $\langle\phi\rangle$ with $\langle\phi\rangle^\perp \subset H^0(E, \mathcal{V})$ $\big($where we use the Serre duality $H^0(E,\mathcal{V}) \simeq H^1\big(E, \mathcal{V}^\vee\big)^*\big)$.

Lemma 2.1.1 ([3, Lemma 2.1])**.** *For $s_1, s_2 \in \langle\phi\rangle^\perp$ one has*

$$\Pi_\phi(s_1 \wedge s_2) = \langle \phi, MP(s_1, \phi, s_2)\rangle,$$

where MP denotes the triple Massey product for the arrows

$$\mathcal{O} \xrightarrow{s_2} \mathcal{V} \xrightarrow{\phi} \mathcal{O}[1] \xrightarrow{s_1} \mathcal{V}[1].$$

2.2 Formula for a family of complete intersections

Let X be a smooth projective variety of dimension n, $C \subset X$ a connected curve given as the zero locus of a regular section F of a vector bundle N of rank $n-1$, such that $\det(N)^{-1} \simeq \omega_X$. Then the normal bundle to C is isomorphic to $N|_C$, so by the adjunction formula, ω_C is trivial. Thus, if C is smooth, it is an elliptic curve. Assume that P is a vector bundle on X, such that the following cohomology vanishing holds:

$$H^i\left(X, \bigwedge^i N^\vee \otimes P\right) = H^{i-1}\left(X, \bigwedge^i N^\vee \otimes P\right) = 0 \qquad \text{for} \quad 1 \leq i \leq n-1. \tag{2.1}$$

We have the following Koszul resolution for $\mathcal{O}_C$:

$$0 \to \bigwedge^{n-1} N^\vee \to \cdots \to \bigwedge^2 N^\vee \xrightarrow{\delta_2(F)} N^\vee \xrightarrow{\delta_1(F)} \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

which induces a map $e_C \colon \mathcal{O}_C \to \bigwedge^{n-1} N^\vee[n-1]$ in the derived category of X. Here the differential $\delta_i(F)$ is given by the contraction with $F \in H^0(X, N)$, so it depends linearly on F.

Lemma 2.2.1.

(i) *The natural restriction map* $H^0(X, P) \to H^0(C, P|_C)$ *and the map*

$$\mathrm{Ext}^1(P, \mathcal{O}_C) \xrightarrow{e_C} \mathrm{Ext}^n\left(P, \bigwedge^{n-1} N^\vee\right) \simeq \mathrm{Ext}^n(P, \omega_X)$$

are isomorphisms. These maps are dual via the Serre duality isomorphisms

$$\mathrm{Ext}^1(P|_C, \mathcal{O}_C) \simeq H^0(C, P|_C)^*, \qquad \mathrm{Ext}^n(P, \omega_X) \simeq H^0(X, P)^*.$$

(ii) *Assume in addition that* $\mathrm{End}(P) = \mathbf{k}$ *and we have the following vanishing:*

$$\mathrm{Ext}^i\left(P, \bigwedge^i N^\vee \otimes P\right) = \mathrm{Ext}^{i-1}\left(P, \bigwedge^i N^\vee \otimes P\right) = 0 \qquad \text{for} \quad 1 \le i \le n-1. \tag{2.2}$$

Then the bundle $P|_C$ *is stable.*

Proof. (i) This is obtained from the Koszul resolution of $\mathcal{O}_C$. For example, the space $H^0(P \otimes \mathcal{O}_C)$ is computed by tensoring this resolution with P and using the spectral sequence

$$H^i\left(\bigwedge^j N^\vee\right) \Rightarrow H^{i-j}(P \otimes \mathcal{O}_C)$$

and the assumption (2.1).

(ii) Computing $\mathrm{Hom}(P|_C, P|_C) = \mathrm{Hom}(P, P|_C)$ using the Koszul resolution of $P|_C = P \otimes \mathcal{O}_C$, we get that it is 1-dimensional. Hence, $P|_C$ is stable. ∎

Now we can rewrite the formula of Lemma 2.1.1 for the FO-bracket $\Pi_{C, P|_C}$ on

$$\mathbb{P}H^1(C, P^\vee|_C) \simeq \mathbb{P}\,\mathrm{Ext}^n(P, \omega_X)$$

in terms of higher products on X (obtained by the homological perturbation from a dg-enhancement of $D^b(\mathrm{Coh}(X))$).

Proposition 2.2.2. *For nonzero* $\phi \in \mathrm{Ext}^n(P, \omega_X) \simeq \mathrm{Ext}^1_C(P|_C, \mathcal{O}_C)$*, and* $s_1, s_2 \in \langle \phi \rangle^\perp \subset H^0(X, P)$*, one has*

$$\Pi_{C, P|_C, \phi}(s_1 \wedge s_2) = \pm \left\langle \phi, \sum_{i=1}^{n} (-1)^i m_{n+2}(\delta_1(F), \dots, \delta_{i-1}(F), s_1, \delta_i(F), \dots, \delta_{n-1}(F), \phi, s_2) \right\rangle.$$

Proof. The computation is completely analogous to that of [8, Proposition 3.1], so we will only sketch it. First, one shows that our Massey product can be computed as the triple product m_3 for the arrows

$$\mathcal{O}_X \to P \xrightarrow{[1]} \mathcal{O}_C \to P|_C$$

given by s_2, ϕ and s_1. Then we use resolutions $\bigwedge^\bullet N^\vee \to \mathcal{O}_C$ and $\bigwedge^\bullet N^\vee \otimes P \to P|_C$. Thus, we have to calculate the following triple product in the category of twisted complexes:

$$\begin{array}{ccccccc}
\mathcal{O}_X & & & & & & \\
\downarrow{\scriptstyle s_2} & & & & & & \\
P & & & & & & \\
\downarrow{\scriptstyle \phi} & & & & & & \\
\bigwedge^{n-1} N^\vee[n-1] & \xrightarrow{\delta_{n-1}(F)} & \cdots & \xrightarrow{\delta_2(F)} & N^\vee[1] & \xrightarrow{\delta_1(F)} & \mathcal{O}_X \\
\downarrow{\scriptstyle s_1} & & & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} \\
\bigwedge^{n-1} N^\vee \otimes P[n-1] & \xrightarrow{\delta_{n-1}(F)} & \cdots & \xrightarrow{\delta_2(F)} & N^\vee \otimes P[1] & \xrightarrow{\delta_1(F)} & P,
\end{array}$$

where we view ϕ as a morphism of degree 1 from P to the twisted complex $\bigoplus \bigwedge^i N^\vee[i]$. Now, the result follows from the formula for m_3 on twisted complexes (see [5, Section 7.6]). ■

2.3 Conormal Lie algebra

Let $\mathcal{V}$ be a stable bundle of positive degree on an elliptic curve E, with a fixed trivialization of ω_E, and consider the corresponding FO bracket Π on the projective space $X = \mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}\operatorname{Ext}^1(\mathcal{V}, \mathcal{O})$. Recall that for every point x of a smooth Poisson variety (X, Π) there is a natural Lie algebra structure on

$$\mathfrak{g}_x := (\operatorname{im} \Pi_x)^\perp \subset T_x^* X,$$

where we consider Π_x as a map $T_x^*X \to T_xX$. We call $\mathfrak{g}_x$ the *conormal Lie algebra*. In the case when Π vanishes on x, we have $\mathfrak{g}_x = T_x^*$.

Let us consider a nontrivial extension

$$0 \to \mathcal{O} \xrightarrow{i} \widetilde{\mathcal{V}} \xrightarrow{p} \mathcal{V} \to 0$$

with the class $\phi \in \operatorname{Ext}^1(\mathcal{V}, \mathcal{O})$. By Serre duality, we have the corresponding hyperplane $\langle\phi\rangle^\perp \subset H^0(\mathcal{V})$, and we have an identification $\langle\phi\rangle^\perp \simeq T_\phi^*\mathbb{P}H^0(\mathcal{V})^*$.

Consider a natural map

$$\operatorname{End}\big(\widetilde{\mathcal{V}}\big)/\langle \mathrm{id}\rangle \to \langle\phi\rangle^\perp \simeq T_\phi^*\mathbb{P}H^0(\mathcal{V})^* : \ A \mapsto p \circ A \circ i. \tag{2.3}$$

The following result was proved in [2].

Theorem 2.3.1. *The above map induces an isomorphism of Lie algebras from* $\operatorname{End}\big(\widetilde{\mathcal{V}}\big)/\langle \mathrm{id}\rangle$ *to the conormal Lie algebra of* Π *at the point* ϕ.

Note that in particular, the subspace $(\operatorname{im}\Pi_x)^\perp \subset \langle\phi\rangle^\perp$ is equal to the image of the map (2.3).

3 FO brackets associated with elliptic curves in $G(2,5)$

3.1 Proof of Theorem A

Lemma 3.1.1. *The subset $Z \subset G\big(5, \bigwedge^2 V\big)$ of 5-dimensional subspaces $W \subset \bigwedge^2 V$ such that $\dim(\mathbb{P}W \cap G(2,V)) \geq 2$ has codimension > 1.*

Proof. Let us denote by F the variety of flags $L \subset W \subset \bigwedge^2 V$, where $\dim(L) = 3$, $\dim(W) = 5$, such that $\mathbb{P}L \cap G(2,V) \neq \varnothing$. We claim that F is irreducible of dimension ≤ 30. Note that we have a proper closed subset $\widetilde{Z} \subset F$ consisting of (L,W) such that $\dim(\mathbb{P}W \cap G(2,V)) \geq 2$ (as an example of a point in $F \setminus \widetilde{Z}$, we can take W such that $E_W = \mathbb{P}W \cap G(2,V)$ is an elliptic curve and pick $\mathbb{P}L \subset \mathbb{P}W$ intersecting E_W). Since $\widetilde{Z}$ fibers over Z with fibers $G(3,5)$, our claim would imply that $\dim\big(\widetilde{Z}\big) = \dim Z + 6 < 30$, i.e., $\dim Z < 24$, as required.

To estimate the dimension of F, we observe that we have a fibration $F \to Y$ with fibers $G(2,7)$, where $Y \subset G\big(3, \bigwedge^2 V\big)$ is the subvariety of 3-dimensional subspaces L such that $\mathbb{P}L \cap G(2,V) \neq \varnothing$. Thus, it is enough to prove that Y is irreducible of dimension ≤ 20. Now we use a surjective map $\widetilde{Y} \to Y$, where $\widetilde{Y}$ is the variety of flags $\ell \subset L \subset \bigwedge^2 V$, where $\dim(\ell) = 1$, $\dim(L) = 3$, such that $\ell \in G(2,V)$. We have a fibration $\widetilde{Y} \to G(2,V)$ with fibers $G(2,9)$, hence $\widetilde{Y}$ is irreducible of dimension $6 + 14 = 20$. Hence, Y is irreducible of dimension ≤ 20. ■

Proof of Theorem A. First, we can apply Proposition 2.2.2 to an elliptic curve $E_W \subset X = G(2,V)$. Namely, as a bundle P on X we take $\mathcal{U}^\vee$, the dual of the universal subbundle. We can view the embedding

$$R := W^\perp \to \bigwedge^2 V^* = H^0(X, \mathcal{O}(1)),$$

where $\mathcal{O}(1) = \det(\mathcal{U}^\vee)$, as a regular section $F \in H^0(X,N)$, where $N = R^* \otimes \mathcal{O}(1)$. It is easy to see that we have a $\mathrm{GL}(V)$-invariant identification

$$\omega_X \simeq \det(V)^{-2} \otimes \mathcal{O}(-5).$$

Thus, by adjunction we get an isomorphism

$$\omega_{E_W} \simeq \det(N) \otimes \omega_X|_{E_W} \simeq \det(R^*) \otimes \det(V)^{-2} \otimes \mathcal{O}_{E_W}.$$

Since $\det(R^*) \simeq \det\big(\bigwedge^2 V\big) \otimes \det(W^*) \simeq \det(V)^4 \otimes \det(W^*)$, we can rewrite this as

$$\omega_{E_W} \simeq \det(W^*) \otimes \det(V)^2 \otimes \mathcal{O}_{E_W}. \tag{3.1}$$

The vanishings (2.1) and (2.2) in this case follow from the well known vanishings

$$H^*\big(X, \mathcal{U}^\vee(-i)\big) = 0 \quad \text{for} \quad 1 \leq i \leq 5,$$
$$\mathrm{Ext}^*\big(\mathcal{U}^\vee, \mathcal{U}^\vee(-i)\big) = 0 \quad \text{for} \quad 1 \leq i \leq 3,$$
$$\mathrm{Ext}^{<6}\big(\mathcal{U}^\vee, \mathcal{U}^\vee(-4)\big) = \mathrm{Ext}^{<6}\big(\mathcal{U}^\vee, \mathcal{U}^\vee(-5)\big) = 0$$

(see [4]). Thus, Proposition 2.2.2 gives a formula for Π_W.

This shows that the association $W \mapsto \Pi_W$ gives a regular morphism

$$f\colon\ G\Big(5, \bigwedge^2 V\Big) \to \mathbb{P}H^0\Big(\mathbb{P}V, \bigwedge^2 T\Big).$$

Furthermore, we claim that

$$f^*\mathcal{O}(1) \simeq \mathcal{O}_{G(5,\bigwedge^2 V)}(1) \otimes \det(V)^2.$$

Indeed, we have a family of Gorenstein curves $\pi\colon \mathcal{C} \to B = G\big(5, \bigwedge^2 V\big) \setminus Z$ (with $\mathcal{C}_W = E_W$), where Z was defined in Lemma 3.1.1, such that

$$\omega_{\mathcal{C}/B} \simeq \pi^*\big(\mathcal{O}(1) \otimes \det(V)^2\big).$$

Indeed, this is implied by the argument leading to (3.1), which works for any curve (not necessarily smooth) cut out by $\mathbb{P}W$ in $G(2,V)$. This family of curves is equipped with a family of vector bundles $\mathcal{V}$ (the pull-back of $\mathcal{U}^\vee$ on $G(2,V)$), so that $\mathbb{P}H^0(\mathcal{C}_W, \mathcal{V}_W)^\vee = \mathbb{P}V$. As explained in [3, Section 4.2], we can view the corresponding constant family of projective spaces $\mathbb{P}V \times B$ as the coarse moduli space of a substack in the relative moduli of complexes on $\mathcal{C}$. Now [3, Proposition 4.1] implies that the relation $f^*\mathcal{O}(1) = \mathcal{O}(1) \otimes \det(V)^2$ holds over $B = G\big(5, \bigwedge^2 V\big) \setminus Z$. Since Z has codimension ≥ 1, it holds over the entire $G\big(5, \bigwedge^2 V\big)$.

Next, since $H^0\big(G\big(5, \bigwedge^2 V\big), \mathcal{O}(1)\big) \simeq \bigwedge^5\big(\bigwedge^2 V\big)^*$, the map f is given by a $\mathrm{GL}(V)$-invariant linear map

$$\bigwedge^5 \Big(\bigwedge^2 V\Big) \to H^0\Big(\mathbb{P}V, \bigwedge^2 T\Big) \otimes \det(V)^2.$$

To show that this map coincides with $\pi_{5,2}$, up to a constant factor, it remains to show that the space $\mathrm{Hom}_{\mathrm{GL}(V)}\big(\bigwedge 5\big(\bigwedge^2 V\big), H^0\big(\mathbb{P}V, \bigwedge^2 T\big) \otimes \det(V)^2\big)$ is 1-dimensional.

The representation of $\mathrm{GL}(V)$ on $H^0\big(\mathbb{P}V, \bigwedge^2 T\big)$ is easy to identify due to the exact sequence

$$0 \to \mathbf{k} \to V \otimes V^* \otimes \bigwedge^2 V \otimes S^2 V^* \to H^0\Big(\mathbb{P}V, \bigwedge^2 T\Big) \to 0.$$

Using the Littlewood–Richardson rule, we deduce

$$H^0\Big(\mathbb{P}V, \bigwedge^2 T\Big) \otimes \det(V^*) \simeq \Sigma^{3,1,1}(V^*),$$

where Σ^λ denotes the Schur functor associated with a partition λ. It follows that

$$H^0\Big(\mathbb{P}V, \bigwedge^2 T\Big) \otimes \det(V)^2 \simeq \Sigma^{3,3,2,2}(V).$$

On the other hand, the decomposition of the plethysm $e_5 \circ e_2$ (see [6, Section I.8, Example 6, p. 138]) shows that $\Sigma^{3,3,2,2}(V)$ appears with multiplicity 1 in the $\mathrm{GL}(V)$-representation $\bigwedge^5\big(\bigwedge^2 V\big)$. This implies the claimed assertion about $\mathrm{GL}(V)$-maps. ■

3.2 Rank stratification for a bracket of type $q_{5,2}$

Let E be an elliptic curve, $\mathcal{V}$ be a stable vector bundle of rank 2 and degree 5. We consider the FO bracket Π on the projective space $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{V}, \mathcal{O}) \simeq \mathbb{P}H^0(\mathcal{V})^*$. We want to describe the corresponding rank stratification of $\mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}^4$. More precisely, Π is generically nondegenerate, and we are going to determine the degeneration locus $\mathcal{D}_E \subset \mathbb{P}^4$ (where $\mathrm{rk}\,\Pi \leq 2$) and the zero locus S_E of Π.

For every point $p \in E$, we consider the subspace $\Lambda_p := \mathcal{V}|_p^* \subset H^0(\mathcal{V})^*$ and the corresponding projective line $\mathbb{P}\Lambda_p \subset \mathbb{P}H^0(\mathcal{V})^*$. Recall that the rank of Π at a point corresponding to an extension $\widetilde{\mathcal{V}}$ is equal to $5 - \dim \mathrm{End}\big(\widetilde{V}\big)$ (see [3, Proposition 2.3]).

Lemma 3.2.1.

(*i*) *The bracket* Π *vanishes at the point of* $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{V}, \mathcal{O})$ *corresponding to an extension*

$$0 \to \mathcal{O} \to \widetilde{\mathcal{V}} \to \mathcal{V} \to 0$$

if and only if this extension splits under $\mathcal{O} \to \mathcal{O}(p)$ *for some point* $p \in E$*, which happens if and only if* $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus \mathcal{V}'$*, where* $\mathcal{V}'$ *is semistable of rank* 2 *and degree* 4*. Furthermore, in this case* $\dim \mathrm{End}(\mathcal{V}') = 2$*, so* $\mathcal{V}'$ *is either indecomposable, or* $\mathcal{V}' \simeq L_1 \oplus L_2$*, where* L_1 *and* L_2 *are nonisomorphic line bundles of degree* 2*.*

(ii) *The bracket* Π *has rank* ≤ 2 *if and only the corresponding extension* $\widetilde{\mathcal{V}}$ *is unstable, or equivalently, there exists a line bundle* L_2 *of degree* 2 *such that the extension splits over the unique embedding* $L_2 \hookrightarrow \mathcal{V}$. *In other words, the extension class comes from a subspace of the form*

$$W_{L_2} := H^0(L_2)^\perp \subset H^0(\mathcal{V})^* = V, \tag{3.2}$$

where we use the unique embedding $L_2 \to \mathcal{V}$ *and consider the induced embedding* $H^0(L_2) \hookrightarrow H^0(\mathcal{V})$.

(iii) *Each plane* $\mathbb{P}W_{L_2} \subset \mathbb{P}V$ *is a Poisson subvariety, and there is an embedding of the curve* E *into* $\mathbb{P}W_{L_2}$ *by a degree* 3 *linear system, so that* $\mathbb{P}W_{L_2} \setminus E$ *is a symplectic leaf.*

Proof. (i) Suppose a nontrivial extension

$$0 \to \mathcal{O} \to \widetilde{\mathcal{V}} \to \mathcal{V} \to 0$$

splits under $\mathcal{O} \to \mathcal{O}(p)$. Then $\widetilde{\mathcal{V}}$ is an extension of $\mathcal{O}(p)$ by $\mathcal{V}'$ where $\mathcal{V}' \subset \mathcal{V}$ is the kernel of the corresponding surjective map $\mathcal{V} \to \mathcal{O}_p$. Hence, $\mathcal{V}'$ is semistable of slope 2, which implies that

$$\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus \mathcal{V}'.$$

It follows that $\dim \operatorname{End}(\mathcal{V}') \ge 2$, and so

$$\dim \operatorname{End}\left(\widetilde{\mathcal{V}}\right) = 3 + \dim \operatorname{End}(\mathcal{V}') \ge 5.$$

Hence, Π_E vanishes on the points of the line $\mathbb{P}\Lambda_p \subset \mathbb{P}V$, and we have $\dim \operatorname{End}(\mathcal{V}') = 2$, which means that either $\mathcal{V}'$ is indecomposable or $\mathcal{V}' \simeq L_1 \oplus L_2$, for two nonisomorphic line bundles L_1, L_2 of degree 2.

Conversely, assume Π vanishes at the point corresponding to $\widetilde{\mathcal{V}}$, so $\dim \operatorname{End}\left(\widetilde{\mathcal{V}}\right) = 5$. Then HN-components of $\widetilde{\mathcal{V}}$ cannot be three line bundles (since they would have to have different positive degrees that add up to 5), so $\widetilde{\mathcal{V}} = L \oplus \mathcal{V}'$ where L is a line bundle and $\mathcal{V}'$ is semistable of rank 2, $\deg(L) > 0$, $0 < \deg(\mathcal{V}')$, $\deg(L) + \deg(\mathcal{V}') = 5$.

The case $\deg(L) = 1$ leads to the locus discussed above. If $\deg(L) = 2$ and $\deg(\mathcal{V}') = 3$ then $\dim \operatorname{Hom}(\mathcal{V}', L) = 1$, so we get $\dim \operatorname{End}(\mathcal{V}') = 3$ which is impossible. If $\deg(L) \ge 3$, then $\deg(\mathcal{V}') \le 2$ and $\dim \operatorname{Hom}(\mathcal{V}', L) \ge 4$, so $\dim \operatorname{End}(\mathcal{V}) > 5$, a contradiction.

(ii) The rank of Π is ≤ 2 at $\widetilde{\mathcal{V}}$ if and only if $\dim \operatorname{End}\left(\widetilde{\mathcal{V}}\right) \ge 3$. Clearly, such $\widetilde{\mathcal{V}}$ has to be unstable. Conversely, any unstable $\widetilde{\mathcal{V}}$ would have form $L \oplus \mathcal{V}'$ with either $\operatorname{Hom}(L, \mathcal{V}') \ne 0$ or $\operatorname{Hom}(\mathcal{V}', L) \ne 0$, hence $\dim \operatorname{End}\left(\widetilde{\mathcal{V}}\right) \ge 3$.

Note that $\mu\left(\widetilde{\mathcal{V}}\right) = 5/3$. Hence, if the extension splits over some $L_2 \subset \mathcal{V}$, then $\widetilde{\mathcal{V}}$ is unstable. Conversely, if $\widetilde{\mathcal{V}}$ is unstable then either it has a line subbundle of degree 2, or a semistable subbundle $\mathcal{V}'$ of rank 2 and degree ≥ 4. But any such $\mathcal{V}'$ has a line subbundle of degree ≥ 2.

(iii) We can identify $H^0(L_2)^\perp$ with $H^0(L_3)^* \subset H^0(\mathcal{V})^*$, where $L_3 := \mathcal{V}/L_2$. It is easy to see that the intersection of $\mathbb{P}W_{L_2}$ with the zero locus of Π is exactly the image of E under the map given by $|L_3|$.

Given an extension $\widetilde{\mathcal{V}} \to \mathcal{V}$, split over $L_2 \subset \mathcal{V}$, the splitting $L_2 \to \widetilde{\mathcal{V}}$ is unique, and the quotient $\widetilde{\mathcal{V}}/L_2$ is an extension of $L_3 = \mathcal{V}/L_2$ by $\mathcal{O}$. It is well known that for points of $\mathbb{P}W_{L_2} \setminus E$ the latter extension is stable, so $\mathcal{V}_{L_3} = \widetilde{\mathcal{V}}/L_2$ is a stable bundle of rank 2 with determinant L_3. Since $\operatorname{Ext}^1(\mathcal{V}_{L_3}, L_2) = 0$, we deduce that $\widetilde{\mathcal{V}} = \mathcal{V}_{L_3} \oplus L_2$. Now we can calculate the image of the map (2.3). The space $\operatorname{End}\left(\widetilde{\mathcal{V}}\right)/\langle \mathrm{id} \rangle$ has a basis $\langle \mathrm{id}_{L_2}, e \rangle$, where e is a generator of $\operatorname{Hom}(\mathcal{V}_{L_3}, L_2)$. Their images under (2.3) both factor through $L_2 \to E$, hence the image of (2.3) (which is 2-dimensional) is $H^0(L_2) \subset H^0(\mathcal{V})$. But this is exactly the conormal subspace to the projective plane $\mathbb{P}W_{L_2}$. This shows that $\mathbb{P}W_{L_2} \setminus E$ (and hence $\mathbb{P}W_{L_2}$) is a Poisson subvariety. Since the rank of Π on $\mathbb{P}W_{L_2} \setminus E$ is equal to 2 and $\Pi|_E = 0$, we deduce that $\mathbb{P}W_{L_2} \setminus E$ is a symplectic leaf. ∎

By Lemma 3.2.1 (i), the vanishing locus of Π corresponds to extensions $\mathcal{V}$ by $\mathcal{O}$, which split over $\mathcal{O}(p)$. This is the union S_E of the lines $\mathbb{P}\Lambda_p$, where $\Lambda_p = \mathcal{V}|_p^* \subset \mathbb{P}H^0(\mathcal{V})^*$, over $p \in E$. The surface S_E is the image of the natural map $\mathbb{P}(\mathcal{V}^\vee) \to \mathbb{P}(V)$, associated with the embedding of bundles $\mathcal{V}^\vee \to V \otimes \mathcal{O}_E$. We will prove (see Lemma 3.2.3 below) that in fact this map induces an isomorphism of the projective bundle $\mathbb{P}(\mathcal{V}^\vee)$ with S_E.

Lemma 3.2.2. *Let $\mathcal{E}$ be a vector bundle over a smooth curve C and let $W \to H^0(C,\mathcal{E})$ be a linear map from a vector space W, such that for any $x \in C$ the composition $p_x\colon W \to H^0(C,\mathcal{E}) \to \mathcal{E}|_x$ is surjective, so that we have a morphism $f\colon\ \mathbb{P}(\mathcal{E}^\vee) \to \mathbb{P}(W^*)$. Assume that we have a closed subset $Z \subset \mathbb{P}(\mathcal{E}^\vee)$ with the following properties.*

- *For every $x, y \in C$, $x \neq y$, consider $p_x(\ker(p_y)) \subset \mathcal{E}|_x$. Then any $\ell \in \mathbb{P}(\mathcal{E}^\vee|_x)$, which is orthogonal to $p_x(\ker(p_y))$, is contained in Z.*
- *For every $x \in C$, consider the map $W \to H^0(\mathcal{E}|_{2x})$ and the induced map*
$$K_x := \ker(W \to \mathcal{E}|_x) \to T_x^*C \otimes \mathcal{E}|_x$$
(where we use the identification $T_x^*C \otimes \mathcal{E}|_x = \ker\big(H^0(\mathcal{E}|_{2x}) \to \mathcal{E}|_x)\big)$. Then any $\ell \in \mathbb{P}(\mathcal{E}^\vee|_x)$, which is orthogonal to the image of $K_x \otimes T_xC$, is contained in Z.

Then the map $\mathbb{P}(\mathcal{E}^\vee) \setminus Z \to \mathbb{P}(W^*)$ is a locally closed embedding.

Proof. Assume that for $x \neq y$, we have two nonzero functionals $\phi_x\colon \mathcal{E}|_x \to k$, $\phi_y\colon \mathcal{E}|_y \to k$ such that $\phi_x \circ p_x = \phi_y \circ p_y$. Then $(\phi_x \circ p_x)|_{\ker(\phi_y)} = 0$. Hence, ϕ_x vanishes on $p_x(\ker(p_y))$. By assumption, this can happen only when ϕ_x is in Z. Thus, the map from $\mathbb{P}(\mathcal{E}^\vee) \setminus Z$ is set-theoretically one-to-one.

Next, we need to check that our map is injective on tangent spaces. The tangent space to $\mathbb{P}(\mathcal{E}^\vee)$ at a point corresponding to $\ell \subset \mathcal{E}^\vee|_x$ can be described as follows. Consider the canonical extension

$$0 \to T_x^*C \otimes \mathcal{E}|_x \to H^0(\mathcal{E}|_{2x}) \to \mathcal{E}|_x \to 0.$$

Passing to the dual extension of $T_xC \otimes \mathcal{E}^\vee|_x$ by $\mathcal{E}^\vee|_x$, and restricting it to $T_xC \otimes \ell \subset T_xC \otimes \mathcal{E}^\vee|_x$, we get an extension

$$0 \to \mathcal{E}^\vee|_x \to H_\ell \to T_xC \otimes \ell \to 0.$$

Now the quotient $\big(\ell^{-1} \otimes H_\ell\big)/\mathbf{k}$, where we use the natural embedding

$$k = \ell^{-1} \otimes \ell \to \ell^{-1} \otimes \mathcal{E}^\vee|_x \to \ell^{-1} \otimes H_\ell,$$

is identified with the tangent space $T_\ell \mathbb{P}(\mathcal{E}^\vee)$.

The restriction of the map $H^0(\mathcal{E}|_{2x})^\vee \to W^*$, dual to the natural map $W \to H^0(\mathcal{E}|_{2x})$, to H_ℓ, induces a map

$$\big(\ell^{-1} \otimes H_\ell\big)/\mathbf{k} \to W^*/\ell,$$

which is exactly the tangent map to f. It is injective if and only if the map $H_\ell \to W^*$ is injective. Equivalently, the dual map $W \to H_\ell^*$ should be surjective. The latter map is compatible with (surjective) projections to $\mathcal{E}|_x$, so this is equivalent to surjectivity of the map

$$K_x = \ker(W \to \mathcal{E}|_x) \to \ker(H_\ell^* \to \mathcal{E}|_x) = T_x^*C \otimes \ell^{-1}.$$

The latter map factors as a composition

$$K_x \to T_x^*C \otimes \mathcal{E}|_x \to T_x^*C \otimes \ell^{-1},$$

so it is surjective (equivalently, nonzero) if and only if ℓ is not orthogonal to the image of $K_x \to T_x^*C \otimes \mathcal{E}|_x$. By assumption, this never happens for points of $\mathbb{P}(\mathcal{E}^\vee) \setminus Z$. ∎

Lemma 3.2.3. *The map $\mathbb{P}(\mathcal{V}^\vee) \to S_E$ is an isomorphism.*

Proof. We will check the conditions of Lemma 3.2.2. It suffices to check surjectivity of the maps $H^0(\mathcal{V}) \to \mathcal{V}|_x \oplus \mathcal{V}|_y$ for $x \neq y$ and of $H^0(\mathcal{V}) \to H^0(\mathcal{V}|_{2x})$. But this follows from the exact sequence

$$0 \to \mathcal{V}(-D) \to \mathcal{V} \to \mathcal{V}|_D \to 0$$

for any effective divisor D of degree 2 and from the vanishing of $H^1(\mathcal{V}(-D))$ by stability of $\mathcal{V}$. ∎

By Lemma 3.2.1 (ii), the degeneracy locus $\mathcal{D}_E$ of our Poisson bracket (which is a quintic hypersurface) is the union of planes $\mathbb{P}W_{L_2} \subset \mathbb{P}V$ over $L_2 \in \mathrm{Pic}^2(E)$ (see (3.2)). Let us consider the vector bundle $\mathcal{W}$ over $\widetilde{E} := \mathrm{Pic}^2(E)$, such that the fiber of $\mathcal{W}$ over L_2 is W_{L_2}. Note that we have a natural identification $\widetilde{E} \simeq \mathrm{Pic}^3(E)\colon L_2 \mapsto L_3 := \det(\mathcal{V}) \otimes L_2^{-1}$. In terms of L_3 we have $W_{L_2} = H^0(L_3)^* \subset H^0(\mathcal{V})^*$, where we use a surjection $\mathcal{V} \to L_3$. To define the vector bundle $\mathcal{W}$ precisely, we consider the universal line bundle $\mathcal{L}_3$ of degree 3 over $E \times \widetilde{E} \simeq E \times \mathrm{Pic}^3(E)$, normalized so that the line bundle $p_{2*}\underline{\mathrm{Hom}}(p_1^*\mathcal{V}, \mathcal{L}_3)$ is trivial. We set $\mathcal{W} := p_{2*}(\mathcal{L}_3)^\vee$. Note that applying p_{2*} to the natural surjection $p_1^*\mathcal{V} \to \mathcal{L}_3$ we get a surjection $H^0(\mathcal{V}) \otimes \mathcal{O} \to p_{2*}(\mathcal{L}_3)$. Passing to the dual, we get a morphism $\mathbb{P}(\mathcal{W}) \to \mathbb{P}V$, whose image is $\mathcal{D}_E$.

Lemma 3.2.4. *The morphism $\mathbb{P}(\mathcal{W}) \to \mathcal{D}_E$ is an isomorphism over $\mathcal{D}_E \setminus S_E$.*

Proof. We need to check two conditions of Lemma 3.2.2 for the morphism $H^0(\mathcal{V}) \otimes \mathcal{O} \to \mathcal{W}^\vee$ over $\widetilde{E}$, with $Z \subset \mathbb{P}(\mathcal{W})$ being the preimage of S_E. Note that the intersection of Z with each plane $\mathbb{P}H^0(L_3)^* \subset H^0(\mathcal{V})^*$ is the elliptic curve E embedded by the linear system $|L_3|$.

To check the first condition, we use the exact sequence

$$0 \to H^0(L_2) \to H^0(\mathcal{V}) \to H^0(L_3) \to 0,$$

where $L_2 \otimes L_3 \simeq \det(\mathcal{V})$. If L_3' is different from L_3 then the composed map $L_2 \to \mathcal{V} \to L_3'$ is nonzero, hence, it identifies L_2 with the subsheaf $L_3'(-p)$ for some point $p \in E$. Hence, the image of $H^0(L_2)$ is precisely the plane $H^0(L_3'(-p)) \subset H^0(L_3')$. Hence, the only point of $\mathbb{P}H^0(L_3')^*$ orthogonal to this plane is the point $p \in E \subset \mathbb{P}H^0(L_3')^*$, which lies in Z.

To check the second condition, we need to understand the map $H^0(\mathcal{V}) \to H^0(\mathcal{W}^\vee|_{2x})$ for $x \in \widetilde{E} \simeq \mathrm{Pic}^3(E)$. For this we observe that this map is equal to the composition

$$H^0(\mathcal{V}) \to H^0(E \times \{2x\}, p_1^*\mathcal{V}|_{E\times\{2x\}}) \to H^0(E \times \{2x\}, \mathcal{L}_3|_{E\times\{2x\}}),$$

which is the map induced on H^0 by the morphism of sheaves on E,

$$\alpha\colon\ \mathcal{V} \to \mathcal{V} \otimes H^0(\mathcal{O}_{2x}) = p_{1*}(p_1^*\mathcal{V}|_{E\times\{2x\}}) \to p_{1*}(\mathcal{L}_3|_{E\times\{2x\}}).$$

Note that for $x = L_3$, the bundle $F_x := p_{1*}(\mathcal{L}_3|_{E\times\{2x\}})$ on E is an extension of L_3 by $T_x^*\widetilde{E} \otimes L_3$, which gives the Kodaira–Spencer map for the family $\mathcal{L}_3$, so this extension is nontrivial. The composition

$$\mathcal{V} \xrightarrow{\alpha} F_x \to L_3$$

is the canonical surjection with the kernel $L_2 \subset \mathcal{V}$. Hence, α fits into a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_2 & \longrightarrow & \mathcal{V} & \longrightarrow & L_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha|_{L_2}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & T_x^*\widetilde{E} \otimes L_3 & \longrightarrow & F_x & \longrightarrow & L_3 & \longrightarrow & 0.
\end{array}$$

Note that the map $\alpha|_{L_2}$ is nonzero, since otherwise we would get a splitting of the extension $F_x \to L_3$.

Now the kernel of the map $H^0(\mathcal{V}) \to \mathcal{W}^\vee|_x = H^0(L_3)$ is identified with $H^0(L_2)$, and the induced map $H^0(L_2) \to T_x^*\widetilde{E} \otimes H^0(L_3)$ is given by a nonzero map

$$\alpha|_{L_2}\colon\ L_2 \to T_x^*\widetilde{E} \otimes L_3 \simeq L_3.$$

Hence, its image is the subspace of the form $H^0(L_3(-p))$, and we again deduce that any point of $\mathbb{P}H^0(L_3)^*$ orthogonal to it lies in Z. ■

Corollary 3.2.5.

(*i*) *There is a regular map $\mathcal{D}_E \setminus S_E \to \widetilde{E}$ such that the fiber over L_2 is the symplectic leaf $\mathbb{P}W_{L_2} \setminus E$.*

(*ii*) *Any line contained in $\mathcal{D}_E$ is either contained in S_E (and so has form $\mathbb{P}\Lambda_p$ for some $p \in E$) or in some plane $\mathbb{P}W_{L_2}$, where $L_2 \in \operatorname{Pic}^2(E)$.*

Proof. For (ii) we observe that given a line $L \subset \mathcal{D}_E$ not contained in S_E, the restriction of the map $\mathcal{D}_E \setminus S \to \widetilde{E}$ to $L \setminus S_E \to \widetilde{E}$ is necessarily constant. Hence, L is contained in some plane $\mathbb{P}W_{L_2}$. Similarly, we have a fibration $S_E \to E$ with fibers $\mathbb{P}\Lambda_p$, so any line contained in S_E is one of the fibers. ■

3.3 Two-dimensional distribution on $G(2,5)$ associated with the elliptic curve

Let $E = E_W \subset G(2,V)$ be the elliptic curve obtained as the intersection with a linear subspace $\mathbb{P}W \subset \mathbb{P}\big(\bigwedge^2 V\big)$ in the Plücker embedding, where $\dim W = 5$. Equivalently, E is cut out by the linear subspace of sections $W^\perp \subset \bigwedge^2 V^* \simeq H^0(G(2,V),\mathcal{O}(1))$. As before, we denote by $\mathcal{V}$ the restriction of $\mathcal{U}^\vee$, the dual of the universal bundle. Then $\bigwedge^2(\mathcal{V})$ is the restriction of $\mathcal{O}(1)$, and we have an exact sequence

$$0 \to W^\perp \to \bigwedge^2 V^* \to H^0\left(E, \bigwedge^2(\mathcal{V})\right) \to 0.$$

In other words, we can identify the dual map to the embedding $W \hookrightarrow \bigwedge^2 V$ with the natural map

$$\bigwedge^2 H^0(\mathcal{V}) \to H^0\left(\bigwedge^2 \mathcal{V}\right).$$

We have a regular map $f\colon\ G(2,V) \setminus E \to \mathbb{P}^4$ given by the linear system $|W^\perp| \subset |\mathcal{O}(1)|$.

Definition 3.3.1. For every point $\Lambda \in G(2,V) \setminus E$, we define the subspace

$$D_\Lambda = D_{E,\Lambda} \subset T_\Lambda G(2,V)$$

as the kernel of the tangent map to f at Λ.

Note that for generic Λ, one has $\dim D_\Lambda = 2$. We have the following characterization of D_Λ.

Lemma 3.3.2. *Let $\Lambda \subset V$ be a 2-dimensional subspace corresponding to a point of $G(2,V) \setminus E$.*

(*i*) *Under the identification $T_\Lambda G(2,V) \otimes \det(\Lambda) \simeq \Lambda \otimes V/\Lambda$, we have*

$$D_\Lambda \otimes \det(\Lambda) = W \cap (\Lambda \wedge V) = W \cap (\Lambda \otimes V/\Lambda),$$

where the second intersection is taken in $\bigwedge^2 V/\bigwedge^2 \Lambda$.

(*ii*) *For each* $v \in \Lambda$, *let us denote by* $\pi_v\colon T_\Lambda G(2,V) \to V/\Lambda$ *the natural projection. Assume that* $\Pi_{E,v}$ *has rank* 4, *for some nonzero* $v \in \Lambda$. *Then* D_Λ *is* 2*-dimensional, and* $\pi_v(D_\Lambda)$ *is the* 2*-dimensional subspace of* V/Λ *given as follows:*

$$\pi_v(D_\Lambda) = \{x \in V/\Lambda \mid x \wedge \Pi^{\mathrm{norm}}_{E,v} = 0\},$$

where $\Pi^{\mathrm{norm}}_{E,v} \in \bigwedge^2(V/\Lambda)$ *is the image of* $\Pi_{E,v} \in \bigwedge^2(V/v)$.

Proof. (i) The map f is the composition of the Plücker embedding $G(2,V) \to \mathbb{P}(\bigwedge^2 V)$ with the linear projection

$$\mathbb{P}\left(\bigwedge^2 V\right) \setminus \mathbb{P}(W) \to \mathbb{P}\left(\bigwedge^2 V/W\right).$$

Thus, the tangent map to f at $\Lambda \subset W$ is the composition

$$\mathrm{Hom}(\Lambda, V/\Lambda) \xrightarrow{\alpha} \mathrm{Hom}\left(\bigwedge^2\Lambda, \bigwedge^2 V/\bigwedge^2\Lambda\right) \to \mathrm{Hom}\left(\bigwedge^2\Lambda, \bigwedge^2 V/\left(\bigwedge^2\Lambda + W\right)\right),$$

where $\alpha(A)(l_1 \wedge l_2) = Al_1 \wedge l_2 + l_1 \wedge Al_2 \bmod \bigwedge^2\Lambda$. Equivalently, the map α is the natural map

$$\mathrm{Hom}(\Lambda, V/\Lambda) \simeq \Lambda^* \otimes V/\Lambda \simeq \det{}^{-1}(\Lambda) \otimes \Lambda \otimes V/\Lambda \to \det{}^{-1}(\Lambda) \otimes \bigwedge^2 V/\bigwedge^2\Lambda,$$

given by $l \otimes (v \bmod \Lambda) \mapsto l \wedge v \bmod \bigwedge^2\Lambda$.

Now the assertion follows from the identification

$$W = \ker\left(\bigwedge^2 V/\bigwedge^2\Lambda \to \bigwedge^2 V/\left(\bigwedge^2\Lambda + W\right)\right).$$

(ii) Our identification of Π_W from Theorem A implies the following property of the bivector $\Pi_{W,v} \in \bigwedge^2(V/v)$. Consider the natural map $\phi_v\colon W \to \bigwedge^2(V/v)$. Recall that $S = S_E \subset \mathbb{P}V$ denotes the surface, obtained as the union of lines corresponding to $E \subset G(2,V)$. We claim that the map ϕ_v is injective if and only if $\langle v\rangle$ is not in S. Indeed, an element in the kernel of ϕ_v is an element $v \wedge v'$ contained in W, so the plane $\langle v, v'\rangle$ corresponds to a point of E. Hence, this is true when $\Pi_{W,v}$ is nonzero.

Now assume the rank of $\Pi_{W,v}$ is 4. We have a nondegenerate symmetric pairing on $\bigwedge^2(V/v)$ with values in $\det(V/v)$, given by the exterior product. Now our description of Π_W implies that for $\langle v\rangle \notin S$, $\Pi_{W,v}$ is nonzero and

$$\phi_v(W) = \langle \Pi_{W,v}\rangle^\perp.$$

Since $\Pi_{W,v}$ has maximal rank, the skew-symmetric form $(x_1, x_2) = x_1 \wedge x_2 \wedge \Pi_{W,v}$ on V/v is nondegenerate. Hence, the subspace $(\Lambda/\langle v\rangle) \otimes (V/\Lambda)$ cannot be contained in $\langle \Pi_{W,v}\rangle^\perp$ (this would mean that $\Lambda/\langle v\rangle$ lies in the kernel of $(\cdot,\cdot)$). Hence, the intersection

$$I := (\Lambda/\langle v\rangle) \otimes (V/\Lambda) \cap \langle \Pi_{W,v}\rangle^\perp$$

is 2-dimensional. Since the subspace $\phi_v(W \cap (\Lambda \wedge V))$ is contained in I, we deduce that its dimension is ≤ 2, and so $\dim D_\Lambda \leq 2$. But we also know that $\dim D_\Lambda \geq 2$, hence in fact, we have $\dim D_\Lambda = 2$ and $\phi_v(W \cap (\Lambda \wedge V)) = I$.

The last assertion follows from the fact that under trivialization of $\Lambda/\langle v\rangle$, the subspace $I \subset V/\Lambda$ coincides with $\pi_v(D_\Lambda)$. ∎

Definition 3.3.3. We define $\Sigma_E \subset G(2,V)$ as the closed locus of points $\Lambda \in G(2,V)$ such that $\dim W \cap (\Lambda \wedge V) \geq 3$.

Lemma 3.3.4. *One has $\Sigma_E \subset G(2,V) \setminus E$.*

Proof. We have to prove that $\dim W \cap (\Lambda_p \wedge V) \le 2$, where $\Lambda_p = H^0(\mathcal{V}|_p)^* \subset H^0(\mathcal{V})^* = V$ for some $p \in E$. We have, $\Lambda_p^\perp = H^0(\mathcal{V}(-p)) \subset H^0(\mathcal{V})$ and so, $V/\Lambda_p \simeq H^0(\mathcal{V}(-p))^*$.

The intersection $W \cap (\Lambda_p \wedge V)$ is the kernel of the composed map

$$W \hookrightarrow \bigwedge^2 V \to \bigwedge^2 (V/\Lambda_p).$$

The dual map can be identified with the composition

$$\bigwedge^2 H^0(\mathcal{V}(-p)) \to \bigwedge^2 H^0(\mathcal{V}) \to H^0(\det \mathcal{V}),$$

which also factors as the composition

$$\bigwedge^2 H^0(\mathcal{V}(-p)) \to H^0\left(\bigwedge^2 (\mathcal{V}(-p))\right) = H^0((\det \mathcal{V})(-2p)) \subset H^0(\det \mathcal{V}).$$

We need to check that this map has corank 2, or equivalently the first arrow is an isomorphism.

Set $\mathcal{V}' = \mathcal{V}(-p)$. This is a stable bundle of rank 2 and degree 3. We need to check that the map

$$\bigwedge^2 H^0(\mathcal{V}') \to H^0(\det \mathcal{V}')$$

is surjective. For any point $p' \in E$, we have an exact sequence

$$0 \to H^0(\mathcal{O}(p')) \to H^0(\mathcal{V}') \to H^0((\det \mathcal{V}')(-p')) \to 0$$

and it is easy to see that the restriction of the above map to $H^0(\mathcal{O}(p')) \wedge H^0(\mathcal{V}')$ surjects onto the subspace $H^0((\det \mathcal{V}')(-p')) \subset H^0(\det \mathcal{V}')$. Varying the point $p' \in E$, we get the needed surjectivity. ∎

Thus, by Lemma 3.3.2 (i), Σ_E is exactly the set of points $\Lambda \in G(2,V) \setminus E$ where $\dim D_\Lambda \ge 3$. We have the following geometric description of Σ_E. Recall that we have a collection of 3-dimensional subspaces $W_q \subset V$, associated with points of $\widetilde{E} = \mathrm{Pic}^2(E)$ (see (3.2)).

Proposition 3.3.5. *For $\Lambda \in G(2,V)$, we have $\Lambda \in \Sigma_E$ if and only if the corresponding line $\mathbb{P}\Lambda$ is contained in some plane $\mathbb{P}W_q$, where $q \in \widetilde{E}$. In other words, $\Sigma_E = \bigcup_{q \in \widetilde{E}} G(2, W_q)$.*

Proof. Assume first that $\Lambda \in \Sigma_E$. As we have seen above, this means that $\Lambda \in G(2,V) \setminus E$ and $\dim D_\Lambda \ge 3$. By Lemma 3.3.2 (ii), this implies that the rank of the Poisson bracket Π_W on points of $\mathbb{P}\Lambda$ is ≤ 2. Hence, by Lemma 3.2.1 (ii), $\mathbb{P}\Lambda$ is contained in the quintic $\mathcal{D}_E$. By Corollary 3.2.5, this implies that $\mathbb{P}\Lambda$ is contained in some plane $\mathbb{P}W_q$.

Conversely, assume that we have a 2-dimensional subspace $\Lambda \subset H^0(M)^* \subset H^0(\mathcal{V})^* = V$, where $\mathcal{V} \to M$ is a surjection to a degree 3 line bundle M. Then $\Lambda = \langle s \rangle^\perp \subset H^0(M)^*$ for some 1-dimensional subspace $\langle s \rangle \subset H^0(M)$. Set $P = \Lambda^\perp \subset H^0(\mathcal{V})$. Then P is the preimage of $\langle s \rangle \subset H^0(M)$ under the projection $H^0(\mathcal{V}) \to H^0(M)$.

By Lemma 3.3.2, the space D_Λ is isomorphic to the kernel of the composed map

$$W \to \bigwedge^2 V \to \bigwedge^2 (V/\Lambda).$$

Hence, $\dim(D_\Lambda)$ is equal to the corank of the dual map

$$\bigwedge^2 (P) \to \bigwedge^2 H^0(\mathcal{V}) \to H^0\left(\bigwedge^2 \mathcal{V}\right). \tag{3.3}$$

Let B denote the divisor of zeroes of s. We claim that the image of (3.3) is contained in the subspace $H^0(\bigwedge^2\mathcal{V}(-B)) \subset H^0(\bigwedge^2\mathcal{V})$. Indeed, we have an exact sequence

$$0 \to N \to \mathcal{V} \to M \to 0,$$

where N is a line bundle of degree 2. It is easy to see that the composed map

$$H^0(N) \wedge H^0(\mathcal{V}) \hookrightarrow \bigwedge^2 H^0(\mathcal{V}) \to H^0\left(\bigwedge^2\mathcal{V}\right)$$

coincides with the natural multiplication map

$$H^0(N) \wedge H^0(\mathcal{V})/\bigwedge^2 H^0(N) \simeq H^0(N) \otimes H^0(M) \to H^0(N \otimes M) \simeq H^0\left(\bigwedge^2\mathcal{V}\right).$$

The exact sequence

$$0 \to H^0(N) \to P \to \langle s\rangle \to 0$$

shows that $\bigwedge^2 P \subset H^0(N) \wedge H^0(\mathcal{V})$ and its image in $H^0(N) \otimes H^0(M)$ is contained in $H^0(N) \otimes \langle s\rangle$. This proves our claim about the image of the map (3.3). It follows that the corank of this map is ≥ 3, so $\Lambda \in \Sigma_E$. ■

Corollary 3.3.6. *The locus of lines in $\mathbb{P}^4$ contained in the degeneration locus $\mathcal{D}_E$ of Π_E corresponds to the union $E \sqcup \Sigma_E \subset G(2, V)$.*

Proof. Combine Proposition 3.3.5 with Corollary 3.2.5 (ii). The union is disjoint by Lemma 3.3.4. ■

Lemma 3.3.7. *Let $\Lambda \in G(2, V) \setminus E$.*

- (*i*) *For any 3-dimensional subspace $M \subset V$ containing Λ, one has $W \cap \bigwedge^2 M = \bigwedge^2\Lambda$.*
- (*ii*) *Assume that for generic $v \in \Lambda$, the rank of $\Pi_{E,v}$ is 4. Then the map $D_\Lambda \otimes \mathcal{O} \to V/\Lambda \otimes \mathcal{O}(1)$ over the projective line $\mathbb{P}\Lambda$ is an embedding of a rank 2 subbundle.*

Proof. (i) Since all elements of $\bigwedge^2 M$ are decomposable, the intersection $Q := W \cap \bigwedge^2 M$ is a linear subspace consisting of decomposable elements. But all decomposable elements of W are of the form $\bigwedge^2\Lambda_p$ for some point $p \in E$. Hence, we would get an embedding $\mathbb{P}(Q) \to E$, which implies that Q is 1-dimensional, so $Q = \bigwedge^2\Lambda$.

(ii) From part (i) and from Lemma 3.3.2 we get that for any 3-dimensional subspace $M \subset V$ containing Λ, one has $D_\Lambda \cap \Lambda \otimes M/\Lambda = 0$. Let us set $P = V/\Lambda$, and let us consider the exact sequence over $\mathbb{P}\Lambda$,

$$0 \to D_\Lambda \otimes \mathcal{O}(-1) \to P \otimes \mathcal{O} \to Q \to 0.$$

We want to prove that the rank 1 sheaf Q on $\mathbb{P}^1$ has no torsion. Since $\deg(Q) = 2$ and Q is generated by global sections, we only have to exclude the possibilities $Q \simeq \mathcal{O}_x \oplus \mathcal{O}(1)$ and $Q \simeq T \oplus \mathcal{O}$, where T is a torsion sheaf of length 2.

Assume first that $Q \simeq \mathcal{O}_x \oplus \mathcal{O}(1)$. Consider the composed surjection $f\colon P \otimes \mathcal{O} \to Q \to \mathcal{O}(1)$. It is induced by a surjection $P \to H^0(\mathcal{O}(1))$, which has 1-dimensional kernel $\langle v\rangle$. It follows that the inclusion of $D_\Lambda \otimes \mathcal{O}(-1)$ into $P \otimes \mathcal{O}$ factors as

$$D_\Lambda \otimes \mathcal{O}(-1) \to \langle v\rangle \otimes \mathcal{O} \oplus \mathcal{O}(-1) \to P \otimes \mathcal{O}.$$

Hence, D_Λ has a nontrivial intersection with $H^0(\mathcal{O}(1)) \otimes \langle v\rangle = \Lambda \otimes M/\Lambda \subset \Lambda \otimes V/\Lambda$, for some 3-dimensional $M \subset V$, containing Λ. This is a contradiction, as we proved that there could be no such M.

In the case $Q \simeq T \oplus \mathcal{O}$, we get that $D_\Lambda \otimes \mathcal{O}(-1)$ is contained in the kernel of a surjection $P \otimes \mathcal{O} \to \mathcal{O}$, i.e., $D_\Lambda \otimes \mathcal{O}(-1)$ is contained in $\mathcal{O}^2 \subset P \otimes \mathcal{O}$. But any embedding $\mathcal{O}(-1)^2 \to \mathcal{O}^2$ factors through some $\mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{O}^2$ (occurring as kernel of the surjection $\mathcal{O}^2 \to \mathcal{O}_x$, for some point x in the support of the quotient). Hence, we can finish again as in the previous case. ■

Remark 3.3.8. The rational map f from $G(2,V)$ to $\mathbb{P}^4$ has the following interpretation, which can be proved using projective duality. Start with a generic line $\ell \subset \mathbb{P}(V)$. Then the intersection $\ell \cap \mathcal{D}_E$ with the degeneration quintic of Π_E consists of 5 points. Taking the images of these points under the projection $\mathcal{D}_E \setminus S_E \to \widetilde{E}$ (see Corollary 3.2.5) we get a divisor D_ℓ of degree 5 on $\widetilde{E}$. All these divisors will belong to a certain linear system $\mathbb{P}^4$ of degree 5, and the map $\ell \mapsto D_\ell$ is exactly our map f.

3.4 Calculation of the Schouten bracket and proof of Theorem B

Lemma 3.4.1.

(i) Let $E \subset G(2,V)$ be the elliptic curve defined by $W \subset \bigwedge^2 V$. Then for each point $p \in E$, the bivector Π_E vanishes on the projective line $\mathbb{P}\Lambda_p \subset \mathbb{P}V$, where $\Lambda_p \subset V$ is the 2-dimensional subspace corresponding to p. For a generic point v of Λ_p, the Lie algebra $\mathfrak{g} = T_v^*\mathbb{P}V$ has a basis (h_1, h_2, e_1, e_2) such that

$$[h_1, h_2] = [e_1, e_2] = 0,$$
$$[h_i, e_i] = 2e_i, \qquad [h_j, e_i] = -e_i \qquad \text{for} \quad i \neq j.$$

Equivalently, the linearization of Π_E takes form

$$\Pi_E^{\text{lin}} = 2e_1\partial_{h_1} \wedge \partial_{e_1} - e_1\partial_{h_2} \wedge \partial_{e_1} + 2e_2\partial_{h_2} \wedge \partial_{e_2} - e_2\partial_{h_1} \wedge \partial_{e_2}.$$

Furthermore, the conormal subspace $N^\vee_{\mathbb{P}\Lambda_p, v} \subset \mathfrak{g}^*$ is spanned by e_1, e_2, $h_1 + h_2$ (dually the tangent space to $T_{\mathbb{P}\Lambda_p}$ is spanned by $\partial_{h_1} - \partial_{h_2}$).

(ii) We have an identification

$$H^0(\mathbb{P}\Lambda_p, N_{\mathbb{P}\Lambda_p}) \simeq H^0(\mathbb{P}\Lambda_p, V/\Lambda_p \otimes \mathcal{O}(1)) \simeq \Lambda_p^* \otimes V/\Lambda_p \simeq T_pG(2,V).$$

Under this identification, the line $T_pE \subset T_pG(2,V)$ has the property that the corresponding global section of $N_{\mathbb{P}\Lambda_p}$ evaluated at generic $v \in \mathbb{P}\Lambda_p$ spans the line

$$\langle \partial_{h_1}, \partial_{h_2} \rangle / \langle \partial_{h_1} - \partial_{h_2} \rangle \subset N_{\mathbb{P}\Lambda_p, v} \simeq V/\Lambda_p.$$

Equivalently, the tangent space at v to the surface $S_E \subset \mathbb{P}V$ is $\langle \partial_{h_1}, \partial_{h_2} \rangle \subset T_v\mathbb{P}V$.

(iii) Let Π' be a Poisson bracket compatible with Π_E. Then for $p \in E$ and a generic $v \in \Lambda_p$, one has

$$\Pi'_v \in \langle (2\partial_{h_1} - \partial_{h_2}) \wedge \partial_{e_1}, (2\partial_{h_2} - \partial_{h_1}) \wedge \partial_{e_2}, \partial_{h_1} \wedge \partial_{h_2} \rangle. \tag{3.4}$$

Proof. (i) Extensions $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ by $\mathcal{O}$, corresponding to the line $\mathbb{P}\Lambda_p$, are exactly nontrivial extensions that split under $\mathcal{O} \to \mathcal{O}(p)$. We claim that for a generic point of $\mathbb{P}\Lambda_p$ we have $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus L_1 \oplus L_2$, where L_1 and L_2 are nonisomorphic line bundles of degree 2. Indeed, by Lemma 3.2.1 (ii), the only other possibility is $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus \mathcal{V}'$, where $\mathcal{V}'$ is a nontrivial extension of M by M, where $M^2 \simeq \det(\mathcal{V})(-p)$. Since the corresponding extension splits over the unique embedding $M \to \mathcal{V}$, this gives one point on the line $\mathbb{P}\Lambda_p$ for each of the four possible line bundles M.

We can compute the Lie algebra $\mathfrak{g}$ for the point corresponding to $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus L_1 \oplus L_2$ using the isomorphism of Theorem 2.3.1,

$$\operatorname{End}\big(\widetilde{\mathcal{V}}\big)/\langle \mathrm{id}\rangle \xrightarrow{\sim} \mathfrak{g} \subset H^0(\mathcal{V}). \tag{3.5}$$

We consider the following basis in $\operatorname{End}\big(\widetilde{\mathcal{V}}\big)/\langle \mathrm{id}\rangle$:

$$h_i = \mathrm{id}_{L_i} - \mathrm{id}_{\mathcal{O}(p)}, \qquad e_i \in \operatorname{Hom}(\mathcal{O}(p), L_i), \quad i = 1, 2.$$

Then it is easy to check the claimed commutator relations between these elements.

The conormal subspace to $\mathbb{P}\Lambda_p$ is identified with $\Lambda_p^\perp = H^0(\mathcal{V}(-p))$. The image of the subspace $\operatorname{Hom}(\mathcal{O}(p), L_1 \oplus L_2)$ under the map (3.5) will consist of compositions

$$\mathcal{O} \to \mathcal{O}(p) \to L_1 \oplus L_2 \to \mathcal{V},$$

which vanish at p, so they are contained in $H^0(\mathcal{V}(-p))$. We have

$$h_1 + h_2 = \mathrm{id}_{L_1} \oplus \mathrm{id}_{L_2} - 2\,\mathrm{id}_{\mathcal{O}(p)} \equiv -3\,\mathrm{id}_{\mathcal{O}(p)} \bmod \langle \mathrm{id}_{\widetilde{\mathcal{V}}}\rangle,$$

and the element $\mathrm{id}_{\mathcal{O}(p)}$ is mapped under (3.5) to the composition

$$\mathcal{O} \to \mathcal{O}(p) \to \mathcal{V},$$

which also vanishes at p. This proves our claim about the conormal subspace.

(ii) To identify the direction corresponding to T_pE, we first recall that the map $E \to G(2, V)$ is associated with the subbundle $\mathcal{V}^\vee \hookrightarrow V \otimes \mathcal{O}$ over E. We have an exact sequence

$$0 \to T_p^*E \otimes \mathcal{V}|_p \to H^0(\mathcal{V}|_{2p}) \to \mathcal{V}|_p \to 0.$$

The dual of the natural map $V^* \to H^0(\mathcal{V}|_{2p})$ fits into a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}^\vee|_p & \longrightarrow & H^0(\mathcal{V}|_{2p})^* & \longrightarrow & T_pE \otimes \mathcal{V}^\vee|_p & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\sim} & & \big\downarrow & & \big\downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & \Lambda_p & \longrightarrow & V & \longrightarrow & V/\Lambda_p & \longrightarrow & 0 \end{array}$$

and the map β corresponds to a map $T_pE \to \operatorname{Hom}(\mathcal{V}^\vee|_p, V/\Lambda_p) = \operatorname{Hom}(\Lambda_p, V/\Lambda_p)$ which is the tangent map to $E \to G(2, V)$. Note that the dual to β is the natural linear map

$$(V/\Lambda_p)^* = \ker\big(H^0(\mathcal{V}) \to \mathcal{V}|_p\big) \to \ker\big(H^0(\mathcal{V}|_{2p}) \to \mathcal{V}|_p\big) \simeq T_p^*E \otimes \mathcal{V}|_p. \tag{3.6}$$

Now, given a functional $v\colon \mathcal{V}|_p \to k$, the image of T_pE under $\pi_v\colon \Lambda_p^* \otimes V/\Lambda_p \to V/\Lambda_p$ corresponds to the composition of (3.6) with v. In other words, it is given by the composition

$$\Lambda_p^\perp = H^0(\mathcal{V}(-p)) \to \mathcal{V}(-p)|_v \simeq \mathcal{V}|_p \xrightarrow{v} k$$

(here we use a trivialization of T_pE).

Let $\widetilde{\mathcal{V}} \to \mathcal{V}$ be the extension corresponding to v. As we have seen in (i), for a generic v, we have $\widetilde{V} \simeq \mathcal{O}(p) \oplus L_1 \oplus L_2$, where L_i are as above. As we have seen in (i), under the isomorphism (3.5), $\Lambda_p^\perp = H^0(\mathcal{V}(-p))$ is the image of the subspace $\langle h_1 + h_2, e_1, e_2\rangle$.

Hence, it remains to check that the composition

$$\langle e_1, e_2\rangle \to H^0(\mathcal{V}(-p)) \to \mathcal{V}(-p)|_p \simeq \mathcal{V}|_p \xrightarrow{v} k,$$

is zero (where the first arrow is induced by (3.5)). Let us consider the element e_1 (the case of e_2 is similar). It maps to the element of $H^0(\mathcal{V}(-p))$ given by the embedding

$$\mathcal{O} \to L_1(-p) \to \mathcal{V}(-p),$$

where we use the composed map $L_1 \to \widetilde{\mathcal{V}} \to \mathcal{V}$. Thus, it is enough to check that the composition $L_1 \to \mathcal{V} \xrightarrow{v} \mathcal{O}_p$ is zero. To this end we use the fact that the extension $\widetilde{\mathcal{V}}$ is the pull-back of the standard extension $\mathcal{O}(p) \to \mathcal{O}_p$ via v. Hence, we have a commutative diagram with exact rows and columns,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(p) & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle v} & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \widetilde{\mathcal{V}} & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & L_1 \oplus L_2 & \xrightarrow{\mathrm{id}} & L_1 \oplus L_2 & &
\end{array}$$

which shows that the composition $L_1 \oplus L_2 \to \mathcal{V} \to \mathcal{O}_p$ is zero.

(iii) This is obtained by a straightforward computation using the vanishing of $[\Pi_E, \Pi']$ and the formula for Π_E^{lin} from part (i). ■

Lemma 3.4.2. *Let $E, E' \subset G(2,V)$ be a pair of elliptic curves obtained as linear sections, such that $[\Pi_E, \Pi_{E'}] = 0$. Then E is not contained in $\Sigma_{E'} \subset G(2,V)$.*

Proof. Assume $E \subset \Sigma_{E'}$. Then, by the description of $\Sigma_{E'}$ in Proposition 3.3.5, for every $p \in E$ there exists a line bundle L_2 of degree 2 on E' such that the image of $H^0(\mathcal{V}|_p)^* \to H^0(E,\mathcal{V})^* = V$ is contained in $H^0(E', L_2)^\perp \subset H^0(E',\mathcal{V}')^* = V$. In other words, each line $\mathbb{P}\Lambda_p \subset \mathbb{P}V$, for $p \in E$, is contained in the projective plane $\mathbb{P}H^0(E', L_2)^\perp \subset \mathbb{P}V$. This plane intersects the zero locus of $\Pi_{E'}$ in a smooth cubic (see Lemma 3.2.1 (iii)), hence, for a generic point $v \in \Lambda_p$, the rank of $\Pi_{E'}|_v$ is 2.

Hence, $\Pi_{E'}|_v = w_1 \wedge w_2$, where $\langle w_1, w_2\rangle$ is the tangent plane to the leaf of $\Pi_{E'}$ (i.e., to the projective plane $\mathbb{P}H^0(E', L_2)^\perp$). Furthermore, the plane $\langle w_1, w_2\rangle$ contains the tangent line to $\mathbb{P}\Lambda_p$ at v. In the notation of Lemma 3.4.1 (i), the latter tangent line is spanned by $\partial_{h_1} - \partial_{h_2}$. So, $\Pi_{E'}|_v = (\partial_{h_1} - \partial_{h_2}) \wedge w$ for some tangent vector w. But we also know by Lemma 3.4.1 (iii) that $\Pi_{E'}|_v$ is a linear combination of $(2\partial_{h_1} - \partial_{h_2}) \wedge \partial_{e_1}$, $(2\partial_{h_2} - \partial_{h_1}) \wedge \partial_{e_2}$ and $\partial_{h_1} \wedge \partial_{h_2}$. This is possible only when $w \in \langle \partial_{h_1}, \partial_{h_2}\rangle$, which is the tangent plane to the surface S_E (see Lemma 3.4.1 (ii)).

This implies that S_E is tangent to the corresponding projective plane $\mathbb{P}H^0(E', L_2)^\perp \subset \mathcal{D}_{E'}$. Assume first that $S_E \not\subset S_{E'}$. Then we get that the regular morphism

$$S_E \setminus S_{E'} \to \mathcal{D}_{E'} \setminus S_{E'} \to \mathrm{Pic}^2(E')$$

(see Corollary 3.2.5) has zero tangent map at every point. Hence, S_E is contained in a projective plane, which is a contradiction (since the map $\mathbb{P}(\mathcal{V}^\vee) \to \mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}V$ induces an isomorphism on sections of $\mathcal{O}(1)$).

Finally, if $S_E \subset S_{E'}$ then $E = E' \subset G(2,V)$ and, we get a contradiction by Lemma 3.3.4. ■

Proof of Theorem B. (i) We can assume that $E \neq E'$. We will check that for a generic point $p \in E$, one has

$$T_pE \subset D_{E',p} \subset T_pG(2,V). \tag{3.7}$$

By Lemma 3.4.2, for a generic $p \in E$, we have $p \notin \Sigma_{E'}$, hence, by Corollary 3.3.6, the line $\mathbb{P}\Lambda_p$ is not contained in the degeneracy locus $\mathcal{D}_{E'}$ of $\Pi_{E'}$. Let us pick a generic point v of Λ_p, so that the rank of $\Pi_{E',v}$ is 4. We want to study the normal projection

$$\Pi^{\mathrm{norm}}_{E',v} \in \wedge^2(T_v\mathbb{P}V/T_v\mathbb{P}\Lambda_p) \simeq \wedge^2(V/\Lambda_p)$$

(see Lemma 3.3.2).

Recall that in the notation of Lemma 3.4.1, the tangent space to $\mathbb{P}\Lambda_p$ at v is spanned by $\partial_{h_1} - \partial_{h_2}$. Hence, the inclusion (3.4) implies that $\Pi^{\mathrm{norm}}_{E',v}$ is proportional to a bivector of the form $\partial_{h_1} \wedge \xi$. By Lemma 3.4.1 (ii), we can reformulate this as

$$\Pi^{\mathrm{norm}}_{E',v} \in \pi_v(T_pE) \wedge V/\Lambda_p \subset \wedge^2(V/\Lambda_p).$$

By Lemma 3.3.2 (ii), the subspace $\pi_v(D_{E',p}) \subset V/\Lambda_p$ consists of x such that $x \wedge \Pi^{\mathrm{norm}}_{E',v} = 0$. Thus, we deduce the inclusion

$$\pi_v(T_pE) \subset \pi_v(D_{E',p}) \subset V/\Lambda_p$$

for generic $v \in \Lambda_p$.

In other words, the section s generating

$$T_pE \subset T_{\Lambda_p}G(2,V) \simeq \mathrm{Hom}(\Lambda_p, V/\Lambda_p) \simeq H^0(\mathbb{P}\Lambda_p, V/\Lambda_p \otimes \mathcal{O}(1))$$

has the property that for generic point $v \in \mathbb{P}\Lambda_p$ the evaluation $s(v)$ belongs to the image of the evaluation at v of the embedding $D_{E',p} \otimes \mathcal{O} \to V/\Lambda_p \otimes \mathcal{O}(1)$. Since by Lemma 3.3.7 the latter is an embedding of a subbundle, this implies that in fact $s \in D_{E',p}$ as claimed.

This proves the inclusion (3.7) for a generic $p \in E$. But this implies that the composed map

$$E \setminus E' \to G(2,V) \setminus E' \to \mathbb{P}^4$$

has zero derivative everywhere, so it is constant. Hence, E is contained in a linear section of $\mathbb{P}U \cap G(2,V)$, for some 6-dimensional subspace $U \subset \bigwedge^2 V$ containing W'. Hence, $\dim(W + W') \leq 6$.

Conversely, assume W and W' are such that $U = W + W'$ is 6-dimensional. Then we claim that $[\Pi_W, \Pi_{W'}] = 0$. Indeed, since the space of such pairs (W, W') is irreducible, it is enough to consider the case when the surface $S = \mathbb{P}U \cap G(2,V)$ is smooth. Then E_W and $E_{W'}$ are anticanonical divisors on S, and we can apply [3, Theorem 4.4] to the bundle $\mathcal{V}_S := \mathcal{U}^\vee|_S$ on S. The fact that $(\mathcal{O}_S, \mathcal{V}_S)$ is an exceptional pair is easily checked using Koszul resolutions, as in Section 2.2.

(ii) It is well known that if a collection of k-dimensional subspaces in a vector space has the property that any two subspaces intersect in a $(k-1)$-dimensional space, then either all of them are contained in a fixed $(k+1)$-dimensional subspace, or they contain a fixed $(k-1)$-dimensional subspace. The statement immediately follows from (i) using this fact for $k = 5$ and the collection (W_i). ∎

Proof of Corollary C. By Theorem B (ii), the brackets (Π_{W_i}) are pairwise compatible when either there exists a 6-dimensional subspace $U \subset \bigwedge^2 V$, containing all W_i, or there is a 4-dimensional subspace $K \subset \bigwedge^2 V$, contained in all W_i. In the former case the corresponding tensors $\bigwedge^5 W_i$ are all contained in the 6-dimensional subspace

$$\bigwedge^5 U \subset \bigwedge^5 \left(\bigwedge^2 V\right).$$

In the latter case all the tensors $\bigwedge^5 W_i$ are contained in the 6-dimensional subspace

$$\bigwedge^4 K \otimes \left(\bigwedge^2 V/K\right) \simeq \left(\bigwedge^4 K\right) \wedge \left(\bigwedge^2 V\right) \subset \bigwedge^5 \left(\bigwedge^2 V\right).$$

Conversely, by [3, Theorem 4.4], if we take a smooth linear section $S = \mathbb{P}U \cap G(2,V)$, where $\dim U = 6$, we claim that we will get a 6-dimensional subspace of compatible Poisson brackets coming from anticanonical divisors of S. We just need to show that the corresponding linear map from $H^0\left(S, \omega_S^{-1}\right)$ to the space of Poisson bivectors on $\mathbb{P}(V)$ is injective. Suppose there exists an anticanonical divisor $E_0 \subset E$ such that the corresponding Poisson bivector is zero. Pick a generic anticanonical divisor E. Then all elliptic curves in the pencil $E + tE_0$ map to the same Poisson bivector. But this is impossible since we can recover $E \subset G(2,V)$ from the corresponding Poisson bracket Π_E on $\mathbb{P}(V)$, as the set of all lines lying in the zero locus S_E (see Section 3.2). ■

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