Co-Axial Metrics on the Sphere and Algebraic Numbers

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Abstract. In this paper, we consider the following curvature equation

$$\Delta u + \mathrm{e}^u = 4\pi\left((\theta_0 - 1)\delta_0 + (\theta_1 - 1)\delta_1 + \sum_{j=1}^{n+m}\left(\theta_j' - 1\right)\delta_{t_j}\right) \qquad \text{in } \mathbb{R}^2,$$

$$u(x) = -2(1 + \theta_\infty)\ln|x| + O(1) \qquad \text{as } |x| \to \infty,$$

where θ_0, θ_1, θ_∞, and θ_j' are positive non-integers for $1 \le j \le n$, while $\theta_j' \in \mathbb{N}_{\ge 2}$ are integers for $n + 1 \le j \le n + m$. Geometrically, a solution u gives rise to a conical metric $\mathrm{d}s^2 = \frac{1}{2}\mathrm{e}^u|\mathrm{d}x|^2$ of curvature 1 on the sphere, with conical singularities at 0, 1, ∞, and t_j, $1 \le j \le n + m$, with angles $2\pi\theta_0$, $2\pi\theta_1$, $2\pi\theta_\infty$, and $2\pi\theta_j'$ at 0, 1, ∞, and t_j, respectively. The metric $\mathrm{d}s^2$ or the solution u is called *co-axial*, which was introduced by Mondello and Panov, if there is a developing map $h(x)$ of u such that the projective monodromy group is contained in the unit circle. The sufficient and necessary conditions in terms of angles for the existence of such metrics were obtained by Mondello–Panov (2016) and Eremenko (2020). In this paper, we fix the angles and study the locations of the singularities $t_1, \dots, t_{n+m}$. Let $A \subset \mathbb{C}^{n+m}$ be the set of those $(t_1, \dots, t_{n+m})$'s such that a co-axial metric exists, among other things we prove that (i) If $m = 1$, i.e., there is only one integer θ_{n+1}' among θ_j', then A is a finite set. Moreover, for the case $n = 0$, we obtain a sharp bound of the cardinality of the set A. We apply a result due to Eremenko, Gabrielov, and Tarasov (2016) and the monodromy of hypergeometric equations to obtain such a bound. (ii) If $m \ge 2$, then A is an algebraic set of dimension $\le m - 1$.

Key words: co-axial metric; location of singularities; algebraic set

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1 Introduction

Let $\left(\mathbb{S}^2, g_0\right)$ be the 2-dimensional sphere with the standard smooth metric g_0. In this paper, we consider the following classical problem in conformal geometry: Given $t_1, \dots, t_N \in \mathbb{S}^2$ and a set of positive real numbers $\theta_{t_1}, \dots, \theta_{t_N}$, is there a metric of constant curvature 1 conformal to g_0 such that the metric is smooth on $\mathbb{S}^2 \setminus \{t_1, \dots, t_N\}$ and has conical singularities at t_j with the angle $2\pi\theta_{t_j}$? This problem has been widely studied in the literature, and in particular, it was solved in [8, 9, 10] in the cases where there are at most 2 non-integer angles among $\{\theta_{t_1}, \dots, \theta_{t_N}\}$

(We simply say that the angle is non-integer if $\theta_j \notin \mathbb{Z}$). In this paper, we consider the case that there are at least 3 non-integer angles among $\{\theta_{t_1}, \ldots, \theta_{t_N}\}$. Without loss of generality, we may assume that three singularities with non-integer angles are 0, 1, and ∞. Then it is well known that this problem is equivalent to solving the following curvature equation

$$\Delta u + \mathrm{e}^u = 4\pi \left(\alpha_0 \delta_0 + \alpha_1 \delta_1 + \sum_{j=1}^{n+m} \alpha_{t_j} \delta_{t_j} \right) \qquad \text{in } \mathbb{R}^2 \cong \mathbb{C},$$

$$u(x) = -2(2 + \alpha_\infty) \ln |x| + O(1) \qquad \text{as } |x| \to \infty, \tag{1.1}$$

where δ_p is the Dirac measure at p, $n \geq 0$, $m \geq 1$, $\alpha_0, \alpha_1, \alpha_\infty, \alpha_{t_j} \in \mathbb{R}_{>-1} \setminus \mathbb{Z}$ for all $1 \leq j \leq n$ and $\alpha_{t_j} \in \mathbb{N}$ for all $n+1 \leq j \leq m$. Geometrically, a solution $u(x)$ of (1.1) leads to a conformal metric $\mathrm{d}s^2 = \frac{1}{2} \mathrm{e}^u |\mathrm{d}x|^2$ of curvature 1 on $\mathbb{S}^2$ with angles $2\pi\theta_p$ at conical singularities $p \in \{0, 1, \infty, t_j\}$, where the angles are given by

$$\theta_p = \alpha_p + 1, \qquad p \in \{0, 1, \infty, t_1, \ldots, t_{n+m}\},$$

so

$$\theta_p \in \mathbb{N}_{\geq 2}, \quad p \in \{t_{n+1}, \ldots, t_{n+m}\}, \qquad \theta_p \in \mathbb{R}_{>0} \setminus \mathbb{Z}, \quad p \in \{0, 1, \infty, t_1, \ldots, t_n\}.$$

It is known that if (1.1) has a solution $u(x)$, then there is a developing map $h(x)$ such that $h(x)$ is a multi-valued meromorphic function on $\mathbb{C}$ satisfying

$$u(x) = \ln \frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2}, \qquad x \in \dot{\mathbb{C}} := \mathbb{C} \setminus \{0, 1, t_1, \ldots, t_{m+n}\}.$$

This is known as the Liouville formula and this $h(x)$ is multi-valued and has its projective monodromy group contained in PSU(2). More precisely, given any $\ell \in \pi_1(\dot{\mathbb{C}}, q_0)$ where $q_0 \in \dot{\mathbb{C}}$ is base point, the analytic continuation of $h(x)$, denoted by $\ell^* h(x)$, is also a developing map of $u(x)$ and so $\ell^* h(x) = \frac{ah(x)+b}{ch(x)+d}$ for some projective monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{PSU}(2)$.

The curvature equation (1.1) has been extensively studied in the past several decades. For the recent development of this subject and its application, we refer [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 16, 17, 18] to the interested reader. So far, the existence of solutions for (1.1) with general singular sources is challenging and seems to be out of reach. Mondello and Panov [17] studied a reduced problem: *To describe possible angles (i.e., to describe the data $\{\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_{n+m}}\}$) such that (1.1) has solutions for some singular set $\{t_1, \ldots, t_{n+m}\}$.* They introduced the measure $d_1\big(\mathbb{Z}_o^{n+m+3}, \boldsymbol{\theta} - \mathbf{1}\big) = d_1\big(\mathbb{Z}_o^{n+m+3}, \boldsymbol{\alpha}\big)$ on

$$\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_{n+m}}),$$

i.e.,

$$\boldsymbol{\theta} - \mathbf{1} = \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_\infty, \alpha_{t_1}, \ldots, \alpha_{t_{n+m}}),$$

where $\mathbb{Z}_o^{n+m+3}$ is the subset of $\mathbb{Z}^{n+m+3}$ consisting of vectors with odd sums of coordinates, and d_1 is the ℓ_1-distance. Then they proved the following interesting result.

Theorem 1.1 ([17]). *Suppose that the angles θ_0, θ_1, θ_∞ and θ_{t_i}, $1 \leq i \leq n+m$, are fixed. Then the following hold.*

1. *If (1.1) has a solution $u(x)$ for some singular set $\{t_1, \ldots, t_{n+m}\}$, then*

$$d_1\big(\mathbb{Z}_o^{n+m+3}, \boldsymbol{\theta} - \mathbf{1}\big) \geq 1.$$

Furthermore, if

$$d_1\left(\mathbb{Z}_o^{n+m+3}, \boldsymbol{\theta} - \mathbf{1}\right) = 1, \tag{1.2}$$

then there is a developing map $h(x)$ such that the projective monodromy group of $h(x)$ is contained in the unit circle, i.e., any projective monodromy matrix of $h(x)$ is diagonal.

2. *Conversely, if $d_1\left(\mathbb{Z}_o^{n+m+3}, \boldsymbol{\theta} - \mathbf{1}\right) > 1$, then there exists some singular set $\{t_1, \ldots, t_{n+m}\}$ such that (1.1) has solutions.*

Definition 1.2. In [17], the corresponding metric $\frac{1}{2}e^{u(x)}|dx|^2$ (if exists) is called *co-axial* if there is a developing map $h(x)$ such that the projective monodromy group of $h(x)$ is contained in the unit circle. In this paper, we also call this solution $u(x)$ *co-axial* for convenience.

For the case (1.2), Theorem 1.1 shows that solutions must be co-axial if exist. However, the existence of co-axial metrics does not necessarily imply (1.2). Thus, there arises naturally the following question: Are there sufficient conditions on angles that guarantee the existence of a singular set $\{t_1, \ldots, t_{n+m}\}$ such that (1.1) has a co-axial solution? This question was completely solved by Eremenko [9].

Theorem 1.3 ([9]). *If (1.1) has a co-axial solution $u(x)$ for some singular set $\{t_1, \ldots, t_{n+m}\}$, then there are $\epsilon_p \in \{\pm 1\}$ for $p \in \{0, 1, \infty, t_1, \ldots, t_n\}$ such that*

$$k' := \sum_{p \in \{0,1,\infty,t_1,\ldots,t_n\}} \epsilon_p \theta_p \in \mathbb{Z}_{\geq 0}, \tag{1.3}$$

$$k'' := \sum_{j=n+1}^{n+m} \theta_{t_j} - (n+m+3) - k' + 2 \in 2\mathbb{Z}_{\geq 0}. \tag{1.4}$$

Moreover, if there is $\eta \in \mathbb{R}_{>0}$ such that $\mathbf{c} = \eta\mathbf{b} = \eta(b_1, \ldots, b_q)$ with $b_j \in \mathbb{N}$, $\gcd(b_1, \ldots, b_q) = 1$, where

$$\boldsymbol{c} := (\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_n}, \underbrace{1, \ldots, 1}_{k'+k''}),$$

then we also have

$$2 \max_{n+1 \leq j \leq n+m} \theta_{t_j} \leq \sum_{j=1}^{q} b_j. \tag{1.5}$$

Conversely, if there is no such η such that $\mathbf{c} = \eta\mathbf{b}$, then (1.3) and (1.4) are also sufficient; if there is such η such that $\mathbf{c} = \eta\mathbf{b}$, then (1.3)–(1.5) are also sufficient.

As we have remarked earlier, the existence of co-axial solutions does not necessarily imply (1.2). Let us recall Eremenko's example [9]: $(\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \theta_{t_2}) = (\beta, \beta, 2\beta, 2\beta, 3)$ for some $\beta \in \mathbb{R}_{>0} \setminus \mathbb{N}$, for which it follows from Theorem 1.3 that co-axial solutions exist but it does not satisfy (1.2).

Here we have an interesting observation about Mondello–Panov's condition (1.2) and Eremenko's condition (1.5).

Theorem 1.4. *Suppose that (1.2) holds. Then (1.3) and (1.4) imply (1.5) automatically, namely (1.1) has (co-axial) solutions for some singular set $\{t_1, \ldots, t_{n+m}\}$ if and only if (1.3) and (1.4) hold for some $\epsilon_p \in \{\pm 1\}$.*

Theorem 1.4 will be proved in Section 4. Define

$$\triangle_{n+m} := \big\{(t_1, \ldots, t_{n+m}) \in \mathbb{C}^{n+m} \mid t_j = 0, 1 \text{ or } t_j = t_k \text{ for some } j \neq k\big\}.$$

Throughout the paper, we let θ_0, θ_1, θ_∞, and θ'_j, $j = 1, \ldots, n + m$, be fixed real numbers such that θ_0, θ_1, θ_∞, and θ'_j, $j = 1, \ldots, n$, are non-integers and θ'_j, $j = n + 1, \ldots, n + m$, are integers greater than 1. Let

$$A = A_\theta := \big\{(t_1, \ldots, t_{n+m}) \in \mathbb{C}^{n+m} \setminus \triangle_{n+m} \mid (1.1) \text{ with } \theta_{t_j} = \theta'_j,$$
$$j = 1, \ldots, n + m, \text{ has co-axial solutions}\big\}. \tag{1.6}$$

An $(n + m)$-tuple $(t_1, \ldots, t_{n+m})$ in A will be called admissible for (1.1). By Theorem 1.3, we know that $A \neq \varnothing$ if and only if (1.3)–(1.5) hold. In this case, a natural question that interest us is:

> *What are the geometric and algebaric properties of the set A, such as its dimension and so on?*

This paper aims to study this problem. Note that (1.4) implies $m \geq 1$, this is the reason why we assume $m \geq 1$ in this paper.

Let us consider the special case $m = 1$ first. Denote

$$\mathbb{Q}_\theta := \mathbb{Q}\big(\theta_0, \theta_1, \theta_\infty, \theta'_1, \ldots, \theta'_n\big) \subsetneqq \mathbb{R}.$$

Theorem 1.5. *Let $m = 1$ and (1.3)–(1.5) hold with $\theta_{t_j} = \theta'_j$, i.e., $A \neq \varnothing$. Then A is a finite set with $\#A \leq 2^{n+2} \times \big(\theta'_{n+1} - 1\big)!$. Furthermore, for each $\boldsymbol{t} = (t_1, \ldots, t_{n+1}) \in A$, all t_j's are algebraic over $\mathbb{Q}_\theta$ and the field*

$$\mathbb{Q}_\theta(A) := \mathbb{Q}_\theta\big(\{t_j \mid (t_1, \ldots, t_{n+1}) \in A\}\big)$$

is a Galois extension of $\mathbb{Q}_\theta$ and

$$[\mathbb{Q}_\theta(A) : \mathbb{Q}_\theta] \leq M < \infty,$$

where M is a constant depending on n and θ'_{n+1}.

In the special case when there are only three non-integer angles, i.e., when $n = 0$, we can get a much sharper bound for the cardinality of A. In the following, we will write t_1 and θ'_1 simply by t and θ.

Theorem 1.6. *Assume that $n = 0$ and $m = 1$. Assume that $\epsilon_0, \epsilon_1 \in \{\pm 1\}$ are signs such that $k := \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \in \mathbb{Z}$ and $\theta \equiv k \bmod 2$. Then the cardinality of the set A is*

$$\begin{cases} 0, & \text{if } \theta \leq |k|, \\ (\theta^2 - k^2)/4, & \text{if } \theta > |k|, \end{cases}$$

counted with multiplicities (see Section 5 for an explanation of multiplicities). Moreover, if $t \in A$, then the degree of t over $\mathbb{Q}_\theta$ is at most $(\theta^2 - k^2)/4$.

We remark that the condition $\theta \equiv k \bmod 2$ is necessary for $A \neq \varnothing$, by (1.4). Interestingly, a key ingredient in the proof of the theorem is the monodromy of the classical hypergeometric equations.

Remark 1.7. Theorem 1.6 yields a sharp non-existence result for (1.1) when $n = 0$ and $m = 1$. That is, when $\theta \equiv k \bmod 2$, if $t \in \mathbb{C}$ is transcendental over $\mathbb{Q}_\theta$ or is algebraic of degree greater than $(\theta^2 - k^2)/4$ over $\mathbb{Q}_\theta$, then (1.1) has no solutions (not just co-axial solutions). Indeed, suppose (1.1) has a solution $u(z)$, since $\theta \equiv k \bmod 2$ easily yields $d_1(\mathbb{Z}_o^4, \boldsymbol{\theta} - \mathbf{1}) = 1$, it follows from Theorem 1.1 that $u(z)$ is co-axial, i.e., $t \in A$, a contradiction with Theorem 1.6. It would be an interesting problem to obtain a similar result for the case $n > 0$.

Let $\theta = 2$ in Theorem 1.6. Then the set A is non-empty if and only if $k = 0$, and in this case, we will prove in Example 5.7 that $A = \{-\epsilon_0 \theta_0 / \theta_\infty\}$, so we obtain the following corollary.

Corollary 1.8 (see Example 5.7). *Consider the curvature equation*

$$\Delta u + \mathrm{e}^u = 4\pi((\theta_0 - 1)\delta_0 + (\theta_1 - 1)\delta_1 + \delta_t) \qquad in\ \mathbb{R}^2 \cong \mathbb{C},$$
$$u(x) = -2(1 + \theta_\infty)\ln|x| + O(1) \qquad as\ |x| \to \infty, \tag{1.7}$$

with $\theta_0, \theta_1, \theta_\infty \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Then (1.7) has co-axial solutions for some $t \in \mathbb{C} \setminus \{0, 1\}$ if and only if there are $\epsilon_0, \epsilon_1 \in \{\pm 1\}$ such that $\theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 = 0$. In this case,

1) *if $t \neq -\epsilon_0 \theta_0 / \theta_\infty$, then (1.7) has no solutions;*
2) *if $t = -\epsilon_0 \theta_0 / \theta_\infty$, then (1.7) has co-axial solutions.*

Let $\theta = 3$ in Theorem 1.6. Then the set A is non-empty only when $|k| = 1$. In this case, to simplify notations, we will write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by θ_0 and θ_1. Then we will prove in Example 5.8 that $A = \left\{ \frac{\theta_0 \theta_\infty \pm \sqrt{-k\theta_0 \theta_1 \theta_\infty}}{(\theta_0 + \theta_1)\theta_\infty} \right\}$, so the following result holds.

Corollary 1.9 (see Example 5.8). *Consider the curvature equation*

$$\Delta u + \mathrm{e}^u = 4\pi((\theta_0 - 1)\delta_0 + (\theta_1 - 1)\delta_1 + 2\delta_t) \qquad in\ \mathbb{R}^2 \cong \mathbb{C},$$
$$u(x) = -2(1 + \theta_\infty)\ln|x| + O(1) \qquad as\ |x| \to \infty, \tag{1.8}$$

with $\theta_0, \theta_1, \theta_\infty \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Then (1.8) has co-axial solutions for some $t \in \mathbb{C} \setminus \{0, 1\}$ if and only if there are $\epsilon_0, \epsilon_1 \in \{\pm 1\}$ such that $k := \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 = \pm 1$. In this case (we write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by θ_0 and θ_1 for convenience),

1) *if $t \notin \left\{ \frac{\theta_0 \theta_\infty \pm \sqrt{-k\theta_0 \theta_1 \theta_\infty}}{(\theta_0 + \theta_1)\theta_\infty} \right\}$, then (1.8) has no solutions;*
2) *if $t \in \left\{ \frac{\theta_0 \theta_\infty \pm \sqrt{-k\theta_0 \theta_1 \theta_\infty}}{(\theta_0 + \theta_1)\theta_\infty} \right\}$, then (1.8) has co-axial solutions.*

For general cases $m \geq 2$, we can not expect that A is a finite set. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$.

Theorem 1.10. *Let $m \geq 2$ and (1.3)–(1.5) hold, i.e., $A \neq \varnothing$. Then A is an algebraic set of dimension $\leq m - 1$. Furthermore, if we further assume*

$$\theta_0, \theta_1, \theta_\infty, \theta_1', \ldots, \theta_n' \in \overline{\mathbb{Q}}_{>0} \setminus \mathbb{Z}, \tag{1.9}$$

then for any $\boldsymbol{t} = (t_1, \ldots, t_{n+m}) \in A$, the transcendence degree of $\overline{\mathbb{Q}}(\boldsymbol{t})$ over $\overline{\mathbb{Q}}$ is $\leq m - 1$.

One of the main ideas in our proofs of all the main results is the integrability of the curvature equation. This integrability property allows us to connect the partial differential equation with a second-order Fuchsian ordinary differential equation. In Section 2, we will derive those ODE's. Applying integrability in terms of the developing maps, we will prove an equivalent condition for the existence of co-axial solutions. Based on this equivalent condition, we will prove Theorems 1.5 and 1.10 in Section 3. In Section 4, we give a proof of Theorem 1.4. In the final section, we present a proof of Theorem 1.6 using hypergeometric equations.

2   The integrability of curvature equations and complex ODEs

In this section, we briefly review how to connect the integrability of the curvature equation (1.1) with a class of second order ODEs in complex variables; see, e.g., [2, 9]. Given a solution $u(x)$ of (1.1), we define

$$Q(x) := -\frac{1}{2}\left(u_{xx}(x) - \frac{1}{2}u_x(x)^2 \right), \qquad x \in \mathbb{C}.$$

Then it is easy to see $Q_{\bar{x}}(x) \equiv 0$, so $Q(x)$ is a meromorphic function in $\mathbb{C}$ with poles belonging to

$$\mathcal{I} := \{0, 1, t_1, \ldots, t_{n+m}\}.$$

More precisely, at each $p \in \mathcal{I}$, since $\Delta(\ln|x - p|) = 2\pi\delta_p$, by (1.1) and a standard elliptic regularity argument (cf. [1]), we obtain that $u(x) - 2\alpha_p \ln|x - p|$ is bounded near p, i.e.,

$$u(x) = 2\alpha_p \ln|x - p| + O(1) \qquad \text{for } x \text{ near } p.$$

Consequently,

$$Q(x) = \frac{\alpha_p(\alpha_p + 2)}{4}(x - p)^{-2} + O\big((x - p)^{-1}\big) \qquad \text{for } x \text{ near } p,$$

and

$$Q(x) = \frac{\alpha_\infty(\alpha_\infty + 2)}{4}x^{-2} + O\big(x^{-3}\big) \qquad \text{for } x \to \infty.$$

Thus,

$$Q(x) = \sum_{p \in \mathcal{I}}\left[\frac{\beta_p}{(x - p)^2} + \frac{d_p}{(x - p)} \right], \tag{2.1}$$

where

$$\beta_p := \frac{\alpha_p(\alpha_p + 2)}{4}, \qquad p \in \{0, 1, t_1, \ldots, t_{n+m}, \infty\}, \tag{2.2}$$

and d_p's are some constants satisfying

$$\sum_{p \in \mathcal{I}} d_p = 0, \qquad \sum_{p \in \mathcal{I}}(\beta_p + pd_p) = \beta_\infty. \tag{2.3}$$

Thus, d_0 and d_1 can be uniquely determined by

$$\boldsymbol{d} := (d_{t_1}, \ldots, d_{t_{n+m}}), \qquad \boldsymbol{t} := (t_1, \ldots, t_{n+m})$$

as

$$d_1 = \beta_\infty - \sum_{p \in \mathcal{I}\setminus\{0,1\}}(\beta_p + pd_p), \qquad d_0 = \sum_{p \in \mathcal{I}\setminus\{0,1\}}(\beta_p + (p-1)d_p) - \beta_\infty,$$

while $\boldsymbol{d}$ could be considered as $n + m$ unknown coefficients.

On the other hand, as mentioned in the introduction, the Liouville theorem (see, e.g., [2]) says that for a solution $u(x)$ of (1.1), there is a developing map $h(x)$ such that

$$u(x) = \ln\frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2}, \qquad x \in \dot{\mathbb{C}} := \mathbb{C} \setminus \mathcal{I}. \tag{2.4}$$

The developing map $h(x)$ is multi-valued in $\mathbb{C}$ and due to (2.4), *the projective monodromy group of $h(x)$ is contained in* $\mathrm{PSU}(2)$. More precisely, given any $\ell \in \pi_1(\dot{\mathbb{C}}, q_0)$ where $q_0 \in \dot{\mathbb{C}}$ is a base point, the analytic continuation of $h(x)$, denoted by $\ell^* h(x)$, is also a developing map of $u(x)$ and so $\ell^* h(x) = \frac{ah(x)+b}{ch(x)+d}$ for some projective monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSU}(2)$.

To connect $Q(x)$ and $h(x)$, we note that (2.4) implies

$$\{h(x), x\} := \left(\frac{h''(x)}{h'(x)}\right)' - \frac{1}{2}\left(\frac{h''(x)}{h'(x)}\right)^2 = u_{xx}(x) - \frac{1}{2}u_x(x)^2 = -2Q(x), \tag{2.5}$$

where $\{h(x), x\}$ is the Schwarz derivative. We refer the reader to [2] for details of computation.

Consider the following second order linear ODE with the complex variable x:

$$y''(x) = Q(x)y(x), \qquad x \in \mathbb{C}. \tag{2.6}$$

Then a classical result (see [19]) says that for any basis $(\hat{y}_1, \hat{y}_2)$ of (2.6), the Schwarz derivative $\{\hat{y}_2/\hat{y}_1, x\}$ always satisfies

$$\{\hat{y}_2/\hat{y}_1, x\} = -2Q(x).$$

From this and (2.5), we conclude that if (2.6) is derived from a solution of (1.1), then there is a basis (y_1, y_2) of (2.6) such that

$$h(x) = y_2(x)/y_1(x). \tag{2.7}$$

Furthermore, the projective monodromy group of (2.6) with respect to y_1 and y_2 is the same as that of $h(x)$, i.e., is contained in $\mathrm{PSU}(2)$.

Conversely, given $Q(x)$ via (2.1)–(2.3), if the corresponding ODE (2.6) has a basis (y_1, y_2) such that the projective monodromy group of the ratio $h(x) = y_2(x)/y_1(x)$ is contained in $\mathrm{PSU}(2)$, then $u(x) := \ln \frac{8|h'(x)|^2}{(1+|h(x)|^2)^2}$ is a solution of (1.1).

For the reader's convenience, we now briefly recall some basic notions of ODEs like (2.6). We refer the reader to [14, 19] for a comprehensive introduction. Generally, (2.6) is called *Fuchsian* because the pole order of $Q(x)$ is at most 2 at any singularities. Note that the Riemann scheme (see [14, p. 11] for the definition of the Riemann scheme) of (2.6) is given by

$$\begin{pmatrix} 0 & 1 & t_1 & \cdots & t_{n+m} & \infty \\ -\frac{\alpha_0}{2} & -\frac{\alpha_1}{2} & -\frac{\alpha_{t_1}}{2} & \cdots & -\frac{\alpha_{t_{n+m}}}{2} & -(\frac{\alpha_\infty}{2}+1) \\ \frac{\alpha_0}{2}+1 & \frac{\alpha_1}{2}+1 & \frac{\alpha_{t_1}}{2}+1 & \cdots & \frac{\alpha_{t_{n+m}}}{2}+1 & \frac{\alpha_\infty}{2} \end{pmatrix}. \tag{2.8}$$

Observe that the exponent difference at t_j is an integer for $n+1 \le j \le n+m$. Since $Q(x)$ comes from a solution $u(x)$ of (1.1), it is well known that (2.6) is apparent at t_j (i.e., no logarithmic singularity near t_j) and so the local monodromy matrix of (2.6) at t_j is $(-1)^{\alpha_{t_j}} I_2$ for all $n+1 \le j \le n+m$. In this paper, we will also consider a Fuchsian differential equation (2.6) which may not come from a solution of (1.1). In this case, unless otherwise specified, *we always assume that* (2.6) *with the Riemann scheme* (2.8) *is apparent at t_j for all $n+1 \le j \le n+m$.*

It is well known that the necessary and sufficient condition of (2.6) being apparent at t_j for all $n+1 \le j \le n+m$ can be derived from the Frobenius method.

Theorem 2.1. *There are polynomials*

$$\hat{\mathcal{P}}_j(\boldsymbol{d}, \boldsymbol{t}) \in \mathbb{Q}_\theta(\boldsymbol{t})[\boldsymbol{d}] = d_{t_j}^{\theta_{t_j}} + l.o.t. \text{ in } \boldsymbol{d} \in \mathbb{Q}_\theta(\boldsymbol{t})[\boldsymbol{d}]$$

of total degree θ_{t_j} in $\boldsymbol{d}$, $j = n+1, \ldots, n+m$, such that (2.6) is apparent at t_j for all $n+1 \le j \le n+m$ if and only if $\boldsymbol{d}$ and $\boldsymbol{t}$ satisfy $\hat{P}_j(\boldsymbol{d}, \boldsymbol{t}) = 0$ for all $j = n+1, \ldots, n+m$.

Furthermore, $\hat{\mathcal{P}}_j(\boldsymbol{d}, \boldsymbol{t})$ can be written as

$$\hat{\mathcal{P}}_j(\boldsymbol{d}, \boldsymbol{t}) = \frac{\mathcal{P}_j(\boldsymbol{d}, \boldsymbol{t})}{\prod_{p \in \mathcal{I} \setminus \{t_j\}} (p - t_j)^{N_j}} \tag{2.9}$$

for some $N_j \in \mathbb{Z}_{\geq 0}$ and $\mathcal{P}_j(\boldsymbol{d}, \boldsymbol{t}) \in \mathbb{Q}_\theta[\boldsymbol{d}, \boldsymbol{t}]$ such that $\mathcal{P}_j(\boldsymbol{d}, \boldsymbol{t})$ and $\prod_{p \in \mathcal{I} \setminus \{t_j\}} (p - t_j)^{N_j}$ are coprime. Thus, (2.6) is apparent at t_j for all $n + 1 \leq j \leq n + m$ if and only if $\mathcal{P}_j(\boldsymbol{d}, \boldsymbol{t}) = 0$ for all $j = n + 1, \ldots, n + m$.

Proof. Let us take t_{n+m} for example and in the following proof, we write $t_{n+m} = t$ for convenience. By the Frobenius method, (2.6) is apparent at the singularity t if and only if it has a solution of the form

$$y(x) = \sum_{j=0}^{\infty} c_j u^{j - \alpha_t/2}, \qquad u = x - t, \quad c_0 = 1.$$

Observe that

$$Q(x) = \sum_{p \in \mathcal{I}} \left[\frac{\beta_p}{(x-p)^2} + \frac{d_p}{(x-p)} \right] = \frac{\beta_t}{u^2} + \frac{d_t}{u} + \sum_{p \in \mathcal{I} \setminus \{t\}} \left[\frac{\beta_p}{(u+t-p)^2} + \frac{d_p}{u+t-p} \right]$$

$$= \frac{\beta_t}{u^2} + \frac{d_t}{u} + \sum_{j=0}^{\infty} (-1)^j \sum_{p \in \mathcal{I} \setminus \{t\}} \frac{\beta_p(j+1) + d_p(t-p)}{(t-p)^{j+2}} u^j = \sum_{j=0}^{\infty} \eta_j u^{j-2},$$

where

$$\eta_0 = \beta_t, \qquad \eta_1 = d_t, \qquad \eta_j = (-1)^j \sum_{p \in \mathcal{I} \setminus \{t\}} \frac{\beta_p(j-1) + d_p(t-p)}{(t-p)^j}, \qquad j \geq 2,$$

i.e., $\prod_{p \in \mathcal{I} \setminus \{t\}} (t-p)^j \eta_j \in \mathbb{Q}_\theta[\boldsymbol{d}, \boldsymbol{t}]$.

Consequently, $y''(x) = Q(x)y(x)$ if and only if

$$j(j - \theta_t)c_j = \sum_{k=0}^{j-1} \eta_{j-k} c_k, \qquad \forall j \geq 0, \tag{2.10}$$

where $(j - \alpha_t/2)(j - \alpha_t/2 - 1) - \eta_0 = j(j - \alpha_t - 1) = j(j - \theta_t)$ is used. Clearly, (2.10) holds automatically for $j = 0$, and (2.10) with $j = 1$ leads to $c_1 = \frac{d_t}{1 - \theta_t}$. By an induction argument, for any $2 \leq j \leq \theta_t - 1$, c_j can be uniquely solved by (2.10) as

$$c_j = r_j \left[d_t^j + \text{l.o.t. in } \boldsymbol{d} \right] \in \mathbb{Q}_\theta(\boldsymbol{t})[\boldsymbol{d}]$$

with total degree j in $\boldsymbol{d}$, where $r_j \in \mathbb{Q}_\theta \setminus \{0\}$. Furthermore,

$$\prod_{p \in \mathcal{I} \setminus \{t\}} (t-p)^j c_j \in \mathbb{Q}_\theta[\boldsymbol{d}, \boldsymbol{t}].$$

Consequently, (2.10) with $j = \theta_t$ leads to the existence of $r_{\theta_t} \in \mathbb{Q}_\theta \setminus \{0\}$ and a polynomial

$$\hat{P}(\boldsymbol{d}, \boldsymbol{t}) = d_t^{\theta_t} + \text{l.o.t. in } \boldsymbol{d} \in \mathbb{Q}_\theta(\boldsymbol{t})[\boldsymbol{d}]$$

with total degree θ_t in $\boldsymbol{d}$ such that

$$\sum_{k=0}^{\theta_t - 1} \eta_{j-k} c_k = r_{\theta_t} \hat{P}(\boldsymbol{d}, \boldsymbol{t}).$$

Then it is standard by the Frobenius theory that (2.6) is apparent at the singularity t if and only if $\hat{P}(\boldsymbol{d}, \boldsymbol{t}) = 0$. Furthermore, the above argument also implies the existence of an integer $0 \leq N \leq \theta_t$ such that

$$P(\boldsymbol{d}, \boldsymbol{t}) := \hat{P}(\boldsymbol{d}, \boldsymbol{t}) \prod_{p \in \mathcal{I} \setminus \{t\}} (t - p)^N \in \mathbb{Q}_\theta[\boldsymbol{d}, \boldsymbol{t}]$$

and $P(\boldsymbol{d}, \boldsymbol{t})$, $\prod_{p \in \mathcal{I} \setminus \{t\}} (t - p)^N$ are coprime. ∎

The case $n = 0$ is special and we plan to study the solvability of (1.1) for this case in another paper. In this case, the ODE (2.6) can be transformed into a Heun equation and this Heun equation has been well studied in [8, 12]. In Section 5, we will use some results from [8] and hypergeometric equations to prove Theorem 1.6.

3 Proofs of Theorems 1.5 and 1.10

This section is devoted to the proofs of Theorems 1.5 and 1.10. In the following we use notations

$$\mathcal{I} := \{0, 1, t_1, \ldots, t_{n+m}\}, \qquad \mathcal{I}_1 := \{0, 1, t_1, \ldots, t_n\}.$$

3.1 General setting

As mentioned in the introduction, following [17], we call the metric $\frac{1}{2} e^{u(x)} |\mathrm{d}x|^2$ and also the solution $u(x)$ of (1.1) to be *co-axial* if there is a developing map $h(x)$ of $u(x)$ such that the projective monodromy group of $h(x)$ is contained in the unit circle, that is, for any $\gamma \in \pi_1(\dot{\mathbb{C}}, q_0)$, the analytic continuation of $h(x)$ along γ, denoted by $\gamma^* h(x)$, satisfies $\gamma^* h(x) = \lambda(\gamma) h(x)$ for some $\lambda(\gamma) \in \mathbb{C}$ satisfying $|\lambda(\gamma)| = 1$. Clearly, this is equivalent to that the monodromy group of the associated ODE (2.6) with $Q(x) = -\frac{1}{2} \left(u_{xx}(x) - \frac{1}{2} u_x(x)^2 \right)$ is *commutative*. In Section 2, we have discussed the equivalence between the existence of a solution $u(x)$ of (1.1) and a Fuchsian ODE (2.6) with $Q(x)$ given by (2.1) satisfying $\mathcal{P}_j(\boldsymbol{d}, \boldsymbol{t}) = 0$ for all $n + 1 \leq j \leq n + m$ (see Theorem 2.1) such that the monodromy group is conjugate to a subgroup of SU(2). This equivalence can be further strengthened as follows when the solution $u(x)$ is co-axial.

In fact, given such an ODE (2.6), take the basis of local solutions near ∞ as follows

$$y_+(x) = x^{-\frac{\alpha_\infty}{2}} \sum_{j=0}^{\infty} c_{+,j} x^{-j}, \qquad c_{+,0} = 1,$$

$$y_-(x) = x^{\frac{\alpha_\infty}{2}+1} \sum_{j=0}^{\infty} c_{-,j} x^{-j}, \qquad c_{-,0} = 1. \tag{3.1}$$

Suppose that (2.6) comes from a co-axial metric. By (2.7), we have $h(x) = y_2(x)/y_1(x)$ for some solutions $y_i(x)$ of (2.6). Consider a large circle $|x| = R$ and let $h(e^{2\pi i}x)$ be the analytic continuation of $h(x)$ along the circle. Then by the co-axial condition, one has

$$h(e^{2\pi i}x) = \lambda h(x)$$

for some $\lambda \neq 0$. On the other hand, there is a matrix $M_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{C})$ such that

$$\begin{pmatrix} y_1(e^{2\pi i}x) \\ y_2(e^{2\pi i}x) \end{pmatrix} = M_\infty \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

These two identities together imply that $b = c = 0$. Thus, after multiplying by some nonzero constants, we may assume that

$$h(x) := y_+(x)/y_-(x), \tag{3.2}$$

which satisfies

$$h(x) = x^{-1-\alpha_\infty}\left(1 + O\left(x^{-1}\right)\right) = x^{-\theta_\infty}\left(1 + O\left(x^{-1}\right)\right) \qquad \text{as } x \to \infty. \tag{3.3}$$

Then under the basis $(y_+(x), y_-(x))$, we have $M_\infty = \begin{pmatrix} \mathrm{e}^{-\pi \mathrm{i}\alpha_\infty} & 0 \\ 0 & \mathrm{e}^{\pi \mathrm{i}\alpha_\infty} \end{pmatrix}$ with $\mathrm{e}^{-\pi \mathrm{i}\alpha_\infty} \neq \mathrm{e}^{\pi \mathrm{i}\alpha_\infty}$. Here for any $p \in \mathcal{I} \cup \{\infty\}$, we use $M_p \in \mathrm{SL}(2,\mathbb{C})$ to denote the monodromy matrix of ODE (2.6) with respect to a simple loop in $\pi_1(\dot{\mathbb{C}}, q_0)$ that encircles p once. Now suppose that the monodromy group of (2.6) is commutative (this holds if (2.6) comes from a co-axial solution $u(x)$), then it follows that all the monodromy matrices under the basis $(y_+(x), y_-(x))$ are diagonal, i.e., $M_p = \begin{pmatrix} \mathrm{e}^{\pm \pi \mathrm{i}\alpha_p} & 0 \\ 0 & \mathrm{e}^{\mp \pi \mathrm{i}\alpha_p} \end{pmatrix}$ for $p \in \mathcal{I}_1 = \{0, 1, t_1, \ldots, t_n\}$. This implies that after analytic continuation,

$$y_\pm(x) = (x-p)^{\rho_p^\pm}\left(c_p^\pm + O(|x-p|)\right) \qquad \text{near } p \in \mathcal{I}_1,$$

where $\{\rho_p^+, \rho_p^-\} = \left\{-\frac{\alpha_p}{2}, \frac{\alpha_p}{2} + 1\right\}$ are the local exponents of (2.6) at p and $c_p^\pm \neq 0$. Thus there exists $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{I}_1$ such that

$$h(x) = (x-p)^{\epsilon_p \theta_p}(c_p + O(x-p)) \qquad \text{as } x \to p \in \mathcal{I}_1, \quad c_p \neq 0.$$

From here and (3.3), we see that $h(x)$ can be written as

$$h(x) = \hat{h}(x) \prod_{p \in \mathcal{I}_1} (x-p)^{\epsilon_p \theta_p}, \tag{3.4}$$

where $\hat{h}(x)$ is *meromorphic* in $\mathbb{C}$ and satisfies $\hat{h}(x) = x^{-\theta_\infty - \epsilon_0 \theta_0 - \epsilon_1 \theta_1 - \sum_{j=1}^n \epsilon_{t_j} \theta_{t_j}}\left(1 + O\left(x^{-1}\right)\right)$ as $x \to \infty$, so

$$\hat{h}(x) \text{ is a rational function.} \tag{3.5}$$

Conversely, given an ODE (2.6) with (2.8) which might not come from a solution of (1.1), we consider $h(x) = y_+(x)/y_-(x)$ as in (3.1) and (3.2). If there are $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{I}_1$ such that $h(x)$ has the expression (3.4) with $\hat{h}(x)$ being a rational function, then for any $\gamma \in \pi_1(\dot{\mathbb{C}}, q_0)$, $\gamma^* h(x) = \lambda(\gamma) h(x)$ with $|\lambda(\gamma)| = 1$ (i.e., the projective monodromy group of $h(x)$ is a subgroup of the unit circle, or equivalently the monodromy matrices of (2.6) are all diagonal under the basis $(y_+(x), y_-(x))$). Thus

$$u(x) := \ln \frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2}$$

is well defined in $\mathbb{C}$. Then a direct computation shows that $u(x)$ is a co-axial solution of (1.1) with $h(x)$ being a developing map. Therefore, the above argument proves the following result.

Lemma 3.1. *The equation* (1.1) *has co-axial solutions if and only if there is an ODE* (2.6) *with the Riemann scheme* (2.8) *such that* $h(x) = y_+(x)/y_-(x)$ *has the expression* (3.4) *for some* $\epsilon_p \in \{\pm 1\}$ *for* $p \in \mathcal{I}_1$ *and some rational function* $\hat{h}(x)$.

Remark that $h(x)$ or equivalently $\hat{h}(x)$ might have zeros or poles at t_{n+j} for some $j \geq 1$. More precisely, the Riemann scheme (2.8) indicates that the exponent difference is $\alpha_{t_j} + 1 = \theta_{t_j}$ at t_j, so near t_j with $n + 1 \leq j \leq n + m$, we have

$$h(x) = \frac{y_+(x)}{y_-(x)} = (x - t_j)^\beta \left(c_{t_j} + O(x - t_j)\right) \qquad \text{for some } \beta \in \left\{0, \pm \theta_{t_j}\right\}, \, c_{t_j} \neq 0,$$

and so does $\hat{h}(x)$. From here and (3.5), there is $\mathcal{I}_1 \subset \mathcal{J}_1 \subset \mathcal{I}$ and $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J}_1 \setminus \mathcal{I}_1$ such that

$$h(x) = \prod_{p \in \mathcal{J}_1} (x-p)^{\epsilon_p \theta_p} \frac{\prod_{j=1}^{m_1} (x-a_j)}{\prod_{j=1}^{m_2} (x-b_j)}, \tag{3.6}$$

where a_j's and b_k's satisfy

$$a_j, b_k \in \mathbb{C} \setminus \mathcal{I}, \qquad \{a_1, \ldots, a_{m_1}\} \cap \{b_1, \ldots, b_{m_2}\} = \varnothing.$$

In other words,

$$\mathcal{J}_1 \setminus \mathcal{I}_1 = \{t_{n+j} \mid h(t_{n+j}) \in \{0, \infty\}, 1 \leq j \leq m\}.$$

Then by $h(x) = y_+(x)/y_-(x)$, we see that each a_j (resp. b_j) is a zero of $y_+(x)$ (resp. $y_-(x)$) and must be simple, namely each a_j is a simple zero of $h(x)$ and each b_j is a simple pole of $h(x)$, so $a_j \neq a_k$ and $b_j \neq b_k$ for any $j \neq k$. Therefore,

Any two elements of the collection $[p \in \mathcal{I}; a_j, 1 \leq j \leq m_1; b_k, 1 \leq k \leq m_2]$ are distinct.

Clearly, (3.3) implies

$$\sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p + m_1 - m_2 = -\theta_\infty. \tag{3.7}$$

By (3.6), we have

$$h'(x) = h(x) \left[\sum_{p \in \mathcal{J}_1} \frac{\epsilon_p \theta_p}{x-p} + \sum_{j=1}^{m_1} \frac{1}{x-a_j} - \sum_{j=1}^{m_2} \frac{1}{x-b_j} \right]$$

$$= \frac{h(x)G(x)}{\prod_{p \in \mathcal{J}_1}(x-p) \prod_{j=1}^{m_1}(x-a_j) \prod_{j=1}^{m_2}(x-b_j)}$$

$$= \prod_{p \in \mathcal{J}_1} (x-p)^{\epsilon_p \theta_p - 1} \frac{G(x)}{\prod_{j=1}^{m_2}(x-b_j)^2}, \tag{3.8}$$

where $G(x)$ is a polynomial defined by

$$G(x) := \prod_{j=1}^{m_1}(x-a_j) \prod_{j=1}^{m_2}(x-b_j) \sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p \prod_{q \in \mathcal{J}_1 \setminus \{p\}} (x-q)$$

$$+ \prod_{p \in \mathcal{J}_1}(x-p) \prod_{j=1}^{m_2}(x-b_j) \sum_{k=1}^{m_1} \prod_{j=1, \neq k}^{m_1} (x-a_j)$$

$$- \prod_{p \in \mathcal{J}_1}(x-p) \prod_{j=1}^{m_1}(x-a_j) \sum_{k=1}^{m_2} \prod_{j=1, \neq k}^{m_2} (x-b_j)$$

$$= \left(\sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p + m_1 - m_2 \right) x^{m_1+m_2+|\mathcal{J}_1|-1} + \text{l.o.t.} = -\theta_\infty x^L + \text{l.o.t.}, \tag{3.9}$$

where $|\mathcal{J}_1| := \#\mathcal{J}_1 \geq n+2$ and

$$L := m_1 + m_2 + |\mathcal{J}_1| - 1.$$

Note that the last identity is due to (3.7).

Lemma 3.2. *Suppose $h(x)$ given by* (3.6) *is a developing map of a co-axial solution $u(x)$ of* (1.1)*, and let $G(x)$ be defined by* (3.9)*. Then*

$$L = m_1 + m_2 + |\mathcal{J}_1| - 1 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \alpha_p, \tag{3.10}$$

$$G(x) = -\theta_\infty \prod_{p \in \mathcal{I} \setminus \mathcal{J}_1} (x - p)^{\alpha_p}. \tag{3.11}$$

Proof. Since $h(x)$ is a developing map of $u(x)$, it follows from (2.5) that

$$-2Q(x) = \left(\frac{h''(x)}{h'(x)}\right)' - \frac{1}{2}\left(\frac{h''(x)}{h'(x)}\right)^2. \tag{3.12}$$

Let $G(p) = 0$, first we claim that $p \in \mathcal{I} \setminus \mathcal{J}_1$. Suppose not. Then

$$p \notin \{0, 1, t_1, \ldots, t_{n+m}, a_1, \ldots, a_{m_1}, b_1, \ldots, b_m\}.$$

Denote by $\gamma \geq 1$ to be the zero order of $G(x)$ at p, then $h'(x) = c(x - p)^\gamma (1 + O(x - p))$ with $c \neq 0$ and so a direct computation via (3.12) implies

$$-2Q(x) = \frac{-\gamma(\gamma + 2)}{2(x - p)^2} + O\big((x - p)^{-1}\big), \tag{3.13}$$

a contradiction with the fact that $Q(x)$ is holomorphic in $\mathbb{C} \setminus \{0, 1, t_1, \ldots, t_{n+m}\}$. This proves $p \in \mathcal{I} \setminus \mathcal{J}_1$.

Now for each $p \in \mathcal{I} \setminus \mathcal{J}_1$, we claim that p is a zero of $G(x)$ with order α_p. Suppose $h'(x) = c(x - p)^\gamma (1 + O(x - p))$ with $c \neq 0$ and $\gamma \in \mathbb{Z}_{\geq 0}$. Again a direct computation via (3.12) implies (3.13), so $\gamma(\gamma + 2)/4 = \beta_p = \alpha_p(\alpha_p + 2)/4$, namely $\gamma = \alpha_p$. This proves (3.10) and (3.11). ∎

Corollary 3.3. *Suppose that $h(x)$ given by* (3.6) *is a developing map of a co-axial solution $u(x)$ of* (1.1)*. Then $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ are determined by*

$$2m_1 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_p - \sum_{p \in \mathcal{J}} \epsilon_p \theta_p - (n + m + 1), \tag{3.14}$$

$$2m_2 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_p + \sum_{p \in \mathcal{J}} \epsilon_p \theta_p - (n + m + 1), \tag{3.15}$$

where $\mathcal{J} := \mathcal{J}_1 \cup \{\infty\}$ and $\epsilon_\infty = 1$. In particular, $\mathcal{I} \setminus \mathcal{J}_1 \neq \varnothing$.

Proof. Note that (3.7) and (3.10) can be written as

$$m_1 - m_2 = -\sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p - \theta_\infty = -\sum_{p \in \mathcal{J}} \epsilon_p \theta_p,$$

$$m_1 + m_2 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \alpha_p - |\mathcal{J}_1| + 1 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_{t_p} - |\mathcal{I} \setminus \mathcal{J}_1| - |\mathcal{J}_1| + 1$$

$$= \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_{t_p} - (n + m + 1),$$

so (3.14) and (3.15) hold, which also imply $\mathcal{I} \setminus \mathcal{J}_1 \neq \varnothing$. ∎

We now do some preparatory work for the proofs of Theorems 1.5 and 1.10. Denote $\boldsymbol{t} = (t_1, \ldots, t_{n+m})$, $\boldsymbol{a} = (a_1, \ldots, a_{m_1})$ and $\boldsymbol{b} = (b_1, \ldots, b_{m_2})$. Recall that $\mathcal{I}_1 \subset \mathcal{J}_1 \subsetneqq \mathcal{I}$, we denote $\boldsymbol{t}_1 = (t_j)_{t_j \in \mathcal{J}_1}$ and $\boldsymbol{t}_2 = (t_j)_{t_j \in \mathcal{I} \setminus \mathcal{J}_1}$ for convenience. For example, if

$$\mathcal{J}_1 = \{0, 1, t_1, \ldots, t_n, t_{n+1} \ldots, t_{n+i}\} \qquad \text{for some } 0 \le i \le m - 1,$$

then $\boldsymbol{t}_1 = (t_1, \ldots, t_{n+i})$ and $\boldsymbol{t}_2 = (t_{n+i+1}, \ldots, t_{n+m})$. Recall

$$\mathbb{Q}_\theta := \mathbb{Q}(\theta_0, \theta_1, \theta_\infty, \theta_1', \ldots, \theta_n') \subsetneqq \mathbb{R}.$$

Then it follows from (3.9) that

$$G(x) = -\theta_\infty \left(x^L + \sum_{j=1}^{L} R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) x^{L-j} \right), \tag{3.16}$$

where $R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) \in \mathbb{Q}_\theta[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1]$ is of total degree j for any $j \le L - 1$, while $R_L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) \in \mathbb{Q}_\theta[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1]$ is of total degree $L - 1$ due to the non-homogenous term $x - 1$ in the expression of $G(x)$.

On the other hand,

$$\prod_{p \in \mathcal{I} \setminus \mathcal{J}_1} (x - p)^{\alpha_p} = \prod_{t_j \in \mathcal{I} \setminus \mathcal{J}_1} (x - t_j)^{\alpha_{t_j}} = x^L + \sum_{j=1}^{L} C_j(\boldsymbol{t}_2) x^{L-j}, \tag{3.17}$$

where we use (3.10) and $C_j(\boldsymbol{t}_2) \in \mathbb{Q}[\boldsymbol{t}_2]$ are symmetric polynomials of $\boldsymbol{t}_2$ of degree j. Comparing (3.16) and (3.17), we conclude that (3.11) holds if and only if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ is a common zero of the polynomials

$$B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) := R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) - C_j(\boldsymbol{t}_2), \qquad 1 \le j \le L. \tag{3.18}$$

Remark that $B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ is of total degree j.

Definition 3.4. We say that a common zero $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) \in \mathbb{C}^{m_1 + m_2 + n + m}$ of the polynomials B_j's in (3.18) is *admissible* if any two elements of $(\boldsymbol{a}, \boldsymbol{b}, t_1, \ldots, t_{n+m})$ do not equal and none of them equals to 0, 1.

Then the above argument shows that once (2.6) comes from a co-axial solution $u(x)$ of (1.1), then there exist $\mathcal{I}_1 \subset \mathcal{J}_1 \subsetneqq \mathcal{I}$ and $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J}_1$ such that m_1, m_2 defined by (3.14) and (3.15) are non-negative integers, (3.10) holds and the corresponding polynomials B_j's in (3.18) has an admissible zero $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$. The following result shows that the converse statement also holds.

Lemma 3.5. *Suppose there exist $\mathcal{I}_1 \subset \mathcal{J}_1 \subsetneqq \mathcal{I}$ and $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J}_1$ such that m_1, m_2 defined by (3.14) and (3.15) are non-negative integers, i.e., (3.10) holds and the corresponding polynomials B_j's in (3.18) are well defined. If B_j's have an admissible zero $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$, then (1.1) has a co-axial solution $u(x)$ with its developing map given by (3.6).*

Proof. Under the above assumptions, we consider the function $h(x)$ given by (3.6). Then $h'(x)$ is given by (3.8) with $G(x)$ given by (3.9). Since $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ is an admissible zero of polynomials B_j's, we see that $G(x)$ satisfies (3.11). Inserting (3.11) into (3.8), we obtain

$$h'(x) = -\theta_\infty \frac{\prod_{p \in \mathcal{J}_1} (x - p)^{\epsilon_p \theta_p - 1} \prod_{p \in \mathcal{I} \setminus \mathcal{J}_1} (x - p)^{\alpha_p}}{\prod_{j=1}^{m_2} (x - b_j)^2}. \tag{3.19}$$

Furthermore, (3.14) and (3.15) imply (3.7), so

$$h(x) = x^{-\theta_\infty}\big(1 + O\big(x^{-1}\big)\big) \qquad \text{as } x \to \infty. \tag{3.20}$$

It follows that

$$h'(x) = -\theta_\infty x^{-\theta_\infty - 1}\big(1 + O\big(x^{-1}\big)\big) \qquad \text{as } x \to \infty. \tag{3.21}$$

Define $Q(x)$ via this $h(x)$ by (3.12), i.e., $Q(x) = -\frac{1}{2}\{h(x), x\}$. Then by (3.19)–(3.21), a direct computation shows that

$$Q(x) = \frac{\theta_p^2 - 1}{4(x - p)^2} + \frac{d_p}{x - p} + O(1) \qquad \text{near } p \in \mathcal{I} = \{0, 1, t_1, \ldots, t_{m+n}\}, \tag{3.22}$$

$$Q(x) = \frac{\theta_\infty^2 - 1}{4x^2} + O\big(x^{-3}\big) \qquad \text{near } x = \infty, \tag{3.23}$$

$$Q(x) = \frac{d_{b_j}}{x - b_j} + O(1) \qquad \text{near } x = b_j, \tag{3.24}$$

for some $d_p \in \mathbb{C}$, and $Q(x)$ is holomorphic at elsewhere. We need to prove $d_{b_j} = 0$.

Consider the corresponding ODE (2.6) with this $Q(x)$. Again by $Q(x) = -\frac{1}{2}\{h(x), x\}$, a classical result says that there is a basis $(y_1(x), y_2(x))$ of solutions of (2.6) such that $h(x) = y_1(x)/y_2(x)$. From here and the expression (3.6) of $h(x)$, we conclude that any solution of (2.6) has no logarithmic singularities at any singularities. On the other hand, by (3.24) we know that the local exponents of (2.6) at b_j are 0, 1, so there is a local solution of the following form

$$\tilde{y}(x) = 1 + c_1\big(x - b_j\big) + c_2\big(x - b_j\big)^2 + \cdots \qquad \text{near } b_j.$$

Inserting this and (3.24) into $\tilde{y}''(x) = Q(x)\tilde{y}(x)$ we immediately obtain $d_{b_j} = 0$, i.e., $Q(x)$ is holomorphic at any b_j. Thus it follows from (3.22) and (3.23) that the Riemann scheme of (2.6) is (2.8), and (2.6) is apparent at t_j for any $n + 1 \le j \le n + m$.

Let $(y_+(x), y_-(x))$ be the basis of local solutions of (2.6) given by (3.1). Then $(y_1(x), y_2(x)) = (y_+(x), y_-(x))\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $ad - bc \ne 0$, so $h(x) = \frac{ay_+(x) + cy_-(x)}{by_+(x) + dy_-(x)}$. Since under the basis $(y_+(x), y_-(x))$, we have $M_\infty = \left(\begin{smallmatrix} e^{-\pi i \alpha_\infty} & 0 \\ 0 & e^{\pi i \alpha_\infty} \end{smallmatrix}\right)$, so we see from (3.20) that

$$e^{-2\pi i \alpha_\infty}\frac{ay_+(x) + cy_-(x)}{by_+(x) + dy_-(x)} = e^{-2\pi i \theta_\infty}h(x) = \frac{ae^{-2\pi i \alpha_\infty}y_+(x) + cy_-(x)}{be^{-2\pi i \alpha_\infty}y_+(x) + dy_-(x)}.$$

From here and $e^{-2\pi i \alpha_\infty} \ne 1$, we easily obtain $b = c = 0$ and so $h(x) = ay_+(x)/dy_-(x) = y_+(x)/y_-(x)$, where $a/d = 1$ follows from (3.20) and (3.1). Then by Lemma 3.1, we conclude that (1.1) has a co-axial solution $u(x)$ with $h(x)$ being its developing map. ∎

Therefore, the problem turns to study the admissible zeros of the polynomials B_j's. The number of the polynomials is $L = m_1 + m_2 + |\mathcal{J}_1| - 1$ with $|\mathcal{J}_1| \le |\mathcal{I}| - 1 = n + m + 1$, and the number of variables is $m_1 + m_2 + n + m \ge L$. Thus there are two different cases: (i) $m_1 + m_2 + n + m = L$ and (ii) $m_1 + m_2 + n + m > L$. For the case (i), we may expect that the number of common zeros of the polynomials B_j's is finite, but we can not expect the validity of this assertion for the case (ii).

Therefore, let us study the case $m_1 + m_2 + n + m = L$ first. In this case, $|\mathcal{J}_1| = |\mathcal{I}| - 1 = n + m + 1$, so $\mathcal{I} \setminus \mathcal{J}_1$ consists of a single point belonging to $\{t_{n+j} \mid 1 \le j \le m\}$. Without loss of generality, we may assume

$$\mathcal{J}_1 = \{0, 1, t_1, \ldots, t_{n+m-1}\}, \qquad \mathcal{I} \setminus \mathcal{J}_1 = \{t_{n+m}\}. \tag{3.25}$$

Then $\boldsymbol{t}_1 = (t_1, \ldots, t_{n+m-1})$, (3.17) becomes

$$\prod_{p \in \mathcal{I} \setminus \mathcal{J}_1} (x - p)^{\alpha_p} = (x - t_{n+m})^{\alpha_{t_{n+m}}} = x^L + \sum_{j=1}^{L} (-1)^j \binom{L}{j} t_{n+m}^j x^{L-j},$$

with

$$L = m_1 + m_2 + n + m = \alpha_{t_{n+m}} \tag{3.26}$$

and so the polynomials B_j's become

$$B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) := R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) - \sum_{j=1}^{L} (-1)^j \binom{L}{j} t_{n+m}^j, \qquad 1 \leq j \leq L. \tag{3.27}$$

Define

$$\mathcal{A} = \mathcal{A}_{(\epsilon_p)} := \left\{ \boldsymbol{t} \in \mathbb{C}^{n+m} \,\middle|\, \begin{array}{l} \text{there is } \boldsymbol{a}, \boldsymbol{b} \text{ such that } (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) \text{ is} \\ \text{an admissible zero of the polynomials } B_j\text{'s in (3.27)} \end{array} \right\}.$$

Then Lemma 3.5 shows $\mathcal{A} \subset A$, where A is defined by (1.6). The first main result is as follows.

Theorem 3.6. *Suppose that for $\mathcal{J}_1$ given by (3.25), there exist $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J}_1$ such that m_1, m_2 defined by (3.14) and (3.15) are non-negative integers, i.e., (3.26) holds and the corresponding polynomials B_j's in (3.27) are well defined. Suppose that $\mathcal{A} \neq \varnothing$. Then $\mathcal{A}$ is a finite set with $\#\mathcal{A} \leq \frac{(\theta_{t_{n+m}} - 1)!}{m_1! m_2!}$.*
Furthermore, for each $\boldsymbol{t} = (t_1, \ldots, t_{n+m}) \in \mathcal{A}$, all t_j's are algebraic over $\mathbb{Q}_\theta$, and the field

$$\mathbb{Q}_\theta(\mathcal{A}) := \mathbb{Q}_\theta(\{t_j \mid (t_1, \ldots, t_{n+m}) \in \mathcal{A}\})$$

is a Galois extension of $\mathbb{Q}_\theta$ and

$$[\mathbb{Q}_\theta(\mathcal{A}) : \mathbb{Q}_\theta] \leq M < \infty,$$

where M is a constant depending on the integer angles θ_{t_j}'s.

Proof. Given any $\boldsymbol{t}_0 = (t_{0,1}, \ldots, t_{0,n+m}) \in \mathcal{A}$. Then the polynomials B_j's in (3.27) have an admissible zero $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ such that $\boldsymbol{t} = \boldsymbol{t}_0$. We claim that

The polynomial system $B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) = 0$ with $1 \leq j \leq L$
has at most $L! = (\theta_{t_{n+m}} - 1)!$ solutions. $\tag{3.28}$

Once this claim is proved, we can conclude that $\mathcal{A}$ is a finite set with $\#\mathcal{A} \leq \frac{(\theta_{t_{n+m}} - 1)!}{m_1! m_2!}$ due to the fact that B_j is invariant under any permutation of $a_1, \ldots, a_{m_1}$ and any permutation of $b_1, \ldots, b_{m_2}$ for all j.

To prove the claim (3.28), we consider the homogenization of this polynomial system

$$\widetilde{B}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}, \varepsilon) := \varepsilon^j B_j(\boldsymbol{a}/\varepsilon, \boldsymbol{b}/\varepsilon, \boldsymbol{t}/\varepsilon) = 0, \qquad j = 1, \ldots, L. \tag{3.29}$$

We show that the solution of (3.29) with $\varepsilon = 0$ must be $\boldsymbol{0}$, i.e., $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) = \boldsymbol{0}$.

First of all, since $0, 1 \in \mathcal{J}_1$, we see from (3.9) and (3.27) that

$$B_L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) = \frac{\epsilon_0 \theta_0}{\theta_\infty} \prod_{j=1}^{n+m-1} t_j \prod_{j=1}^{m_1} a_j \prod_{j=1}^{m_2} b_j - (-1)^L t_{n+m}^L,$$

and so

$$\widetilde{B}_L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}, \varepsilon) = \frac{\epsilon_0 \theta_0}{\theta_\infty} \varepsilon \prod_{j=1}^{n+m-1} t_j \prod_{j=1}^{m_1} a_j \prod_{j=1}^{m_2} b_j - (-1)^L t_{n+m}^L.$$

Thus $\widetilde{B}_L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}, 0) = 0$ implies $t_{n+m} = 0$.

To prove that $(\boldsymbol{a}, \boldsymbol{b}, t_1, \ldots, t_{n+m-1}) = \boldsymbol{0}$, we define

$$\widetilde{h}(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1} \prod_{j=1}^{n+m-1} (x - t_j)^{\epsilon_{t_j} \theta_{t_j}} \frac{\prod_{j=1}^{m_1} (x - a_j)}{\prod_{j=1}^{m_2} (x - b_j)},$$

i.e., replace the non-homogenous factor $x - 1$ of $h(x)$ in (3.6) with x. Then

$$\widetilde{h}'(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 - 2} \prod_{j=1}^{n+m-1} (x - t_j)^{\epsilon_{t_j} \theta_{t_j} - 1} \frac{\widetilde{G}(x)}{\prod_{j=1}^{m_2} (x - b_j)^2},$$

where the expression of $\widetilde{G}(x)$ is obtained from that of $G(x)$ in (3.9) by replacing the term $x - 1$ with x. In other words,

$$\widetilde{G}(x) = -\theta_\infty \left(x^L + \sum_{j=1}^{L} \widetilde{R}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) x^{L-j} \right),$$

where $\widetilde{R}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1)$ is the homogeneous part of degree j of $R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$, and so

$$\widetilde{B}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, t_{n+m}, 0) = \widetilde{R}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) - (-1)^j \binom{L}{j} t_{n+m}^j, \qquad 1 \le j \le L.$$

Thus $\widetilde{B}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, 0, 0) = 0$ for any j yields $\widetilde{R}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) = 0$ for any j and so

$$\widetilde{G}(x) = -\theta_\infty x^L = -\theta_\infty x^{m_1 + m_2 + n + m}.$$

Suppose the number of those a_j's being 0 is n_1, the number of those b_j's being 0 is n_2 and denote

$$I_2 = \{ j \in \{1, \ldots, n + m - 1\} \mid t_j = 0 \}.$$

Then

$$\widetilde{h}(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_{t_j} \theta_{t_j} + n_1 - n_2} (c + o(x))$$

and so

$$\widetilde{h}'(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_{t_j} \theta_{t_j} + n_1 - n_2 - 1} (c' + o(x))$$

for some $c, c' \neq 0$. Therefore,

$$\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_{t_j} \theta_{t_j} + n_1 - n_2 - 1$$

$$= \epsilon_0 \theta_0 + \epsilon_1 \theta_1 - 2 + \sum_{j \in I_2} (\epsilon_{t_j} \theta_{t_j} - 1) + m_1 + m_2 + n + m - 2n_2,$$

namely $n_1 + n_2 + \#I_2 = m_1 + m_2 + n + m - 1$. Since $n_j \le m_j$ and $\#I_2 \le n + m - 1$, we obtain $n_1 = m_1$, $n_2 = m_2$ and $\#I_2 = n + m - 1$, i.e., $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) = \boldsymbol{0}$.

Therefore, the homogenized system (3.29) has no solutions at infinity. Then by the Bezout theorem, the polynomial system $B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) = 0$ with $1 \leq j \leq L$ has exactly $L!$ solutions by counting multiplicity. This proves the claim (3.28). Furthermore, it follows from Lemma 3.7 below that for any solution $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$, every element of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ is algebraic over $\mathbb{Q}_\theta$, i.e., every element belongs to $\overline{\mathbb{Q}_\theta}$, where $\overline{\mathbb{Q}_\theta}$ denotes the algebraic closure of $\mathbb{Q}_\theta$.

Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ be an admissible zero of the polynomials B_j's. Let $\sigma \colon \overline{\mathbb{Q}_\theta} \to \overline{\mathbb{Q}_\theta}$ be any automorphism of $\overline{\mathbb{Q}_\theta}$ such that $\sigma(x) = x$ for any $x \in \mathbb{Q}_\theta$. Then it follows from $B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) \in \mathbb{Q}_\theta[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}]$ that

$$(\sigma(a_1), \ldots, \sigma(a_{m_1}), \sigma(b_1), \ldots, \sigma(b_{m_2}), \sigma(t_1), \ldots, \sigma(t_{n+m}))$$

is also an admissible zero of the polynomials B_j's, namely $(\sigma(t_1), \ldots, \sigma(t_{n+m})) \in \mathcal{A}$ as long as $(t_1, \ldots, t_{n+m}) \in \mathcal{A}$. This proves that $\mathbb{Q}_\theta(\mathcal{A})$ is a Galois extension of $\mathbb{Q}_\theta$, and $[\mathbb{Q}_\theta(\mathcal{A}) : \mathbb{Q}_\theta]$ is bounded by a constant depending on the integer angles θ_{t_j}'s, because the degree of the minimal polynomial of each t_j is bounded by $(\theta_{t_{n+m}} - 1)!$ and $\#\mathcal{A} \leq (\theta_{t_{n+m}} - 1)!$. The proof is complete. ∎

Lemma 3.7. *Given complex numbers $y_1, \ldots, y_k$, let K be the algebraic closure of the field $\mathbb{Q}(y_1, \ldots, y_k)$. Let $H_j(x_1, \ldots, x_\ell) \in K[x_1, \ldots, x_\ell]$ for $j = 1, \ldots, \ell$. Suppose that the polynomial system*

$$H_j(x_1, \ldots, x_\ell) = 0, \qquad 1 \leq j \leq \ell, \tag{3.30}$$

has only finitely many solutions. Then the coordinates of any solution $(t_1, \ldots, t_\ell)$ of (3.30) are algebraic over $\mathbb{Q}(y_1, \ldots, y_\ell)$.

Proof. Let I be the ideal of $K[x_1, \ldots, x_\ell]$ generated by the polynomials H_j. For $i = 1, \ldots, \ell$, consider the elimination ideal $I_i = I \cap K[x_i]$. Under the assumption that (3.30) has finitely many solutions, I_i is generated by some polynomial $f_i(x_i) \in K[x_i]$ whose roots are precisely those t_i such that $(t_1, \ldots, t_\ell)$ is a solution of (3.30) for some $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_\ell)$, by the closure theorem in the elimination theory (see [7, Theorem 3, Section 3.3]). Since all roots of f_i are in K, this shows that the coordinates of any solution $(t_1, \ldots, t_\ell)$ of (3.30) are algebraic over $\mathbb{Q}(y_1, \ldots, y_\ell)$. ∎

3.2 The case $m = 1$

This section is devoted to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $m = 1$ and (1.3)–(1.5) with $\theta_{t_j} = \theta'_j$ hold, i.e., the set A in (1.6) is non-empty. Given any $\boldsymbol{t}_0 = (t_{0,1}, \ldots, t_{0,n+1}) \in A$. Then there exist $\mathcal{I}_1 \subset \mathcal{J}_1 \subsetneq \mathcal{I}$ and $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J}_1$ such that m_1, m_2 defined by (3.14) and (3.15) are non-negative integers, (3.26) holds and the corresponding polynomials B_j's in (3.18) has an admissible zero $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ with $\boldsymbol{t} = \boldsymbol{t}_0$. Since $m = 1$ implies $\mathcal{I}_1 \cup \{t_{n+m}\} = \mathcal{I}$, we always have

$$\mathcal{J}_1 = \mathcal{I}_1 = \{0, 1, t_1, \ldots, t_n\} \qquad \text{and} \qquad \mathcal{I} \setminus \mathcal{I}_1 = \{t_{n+1}\},$$

i.e., it satisfies (3.25) and so the polynomials B_j's become (3.27). Then by Theorem 3.6, we have $\boldsymbol{t}_0 \in \mathcal{A}_{(\epsilon_p)}$ and $\#\mathcal{A}_{(\epsilon_p)} \leq \frac{(\theta_{t_{n+1}} - 1)!}{m_1! m_2!} = \frac{(\theta'_{n+1} - 1)!}{m_1! m_2!}$.

Remark that in general, the set

$$\Lambda := \left\{ (\epsilon_p)_{p \in \mathcal{I}_1} \;\middle|\; \begin{array}{l} \epsilon_p \in \{\pm 1\}, \ m_1, \ m_2 \text{ defined by (3.14) and (3.15) are} \\ \text{non-negative integers and so (3.26) holds} \end{array} \right\}$$

might contain multiple elements. Of course $\#\Lambda \leq 2^{n+2}$. Since the above argument implies

$$A = \bigcup_{(\epsilon_p) \in \Lambda} \mathcal{A}_{(\epsilon_p)},$$

we conclude that A is a finite set with $\#A \leq 2^{n+2}(\theta'_{n+1} - 1)!$ and $\mathbb{Q}_\theta(A)$ is a finite Galois extension of $\mathbb{Q}_\theta$ with $[\mathbb{Q}_\theta(A) : \mathbb{Q}_\theta] \leq M'$, where M' is a constant depending on n and the integer angle θ'_{n+1}. The proof is complete. ∎

3.3 The general case $m \geq 2$

Since $n + 2 \leq |\mathcal{J}_1| \leq n + m + 1$, we have $m_1 + m_2 + n + m \geq L = m_1 + m_2 + |\mathcal{J}_1| - 1$, and in particular, the case $m_1 + m_2 + n + m > L$ appears generally, so we can not expect that the corresponding polynomial system

$$B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}) = R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) - C_j(\boldsymbol{t}_2) = 0, \qquad 1 \leq j \leq L \tag{3.31}$$

has only finitely many solutions. This is the difference from the $m = 1$ case. Our key observation is following.

Lemma 3.8. *The solution set W of the polynomial system* (3.31) *in $\mathbb{C}^{m_1+m_2+n+m}$ is an affine algebraic set of dimension $n + m + 1 - |\mathcal{J}_1| \leq m - 1$.*

Consequently, the dimension of the set of admissible zeros of the polynomials B_j's is $\leq m - 1$.

Proof. Note that

$$W = \left\{ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, \boldsymbol{t}_2) \in \mathbb{C}^{m_1+m_2+n+m} \mid B_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, \boldsymbol{t}_2) = 0, \; \forall j \right\},$$

and

$$\widetilde{W} := W \setminus \left\{ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, \boldsymbol{t}_2) \in \mathbb{C}^{m_1+m_2+n+m} \mid t_j = 0 \text{ for some } t_j \text{ in } \boldsymbol{t}_2 \right\}$$

is an open subset of W. Fix any $(n + m + 2 - |\mathcal{J}_1|)$-tupe

$$\boldsymbol{t}_{2,0} = (t_{j,0}) \in \left\{ \boldsymbol{t}_2 \in \{1\} \times \mathbb{C}^{n+m+1-|\mathcal{J}_1|} \mid t_j \neq 0 \text{ for any } t_j \text{ in } \boldsymbol{t}_2 \right\},$$

(i.e., the first component of $\boldsymbol{t}_{2,0}$ is 1.) we claim that

there are $L!$ points (*by counting multiplicities*) in W satisfying $\boldsymbol{t}_2 = t\boldsymbol{t}_{2,0}$ for some $t \in \mathbb{C}$. $\tag{3.32}$

Once this claim is proved, then the dimension of W is $n + m + 1 - |\mathcal{J}_1| \leq m - 1$.

To prove this claim, we insert $\boldsymbol{t}_2 = t\boldsymbol{t}_{2,0}$ into (3.17), we obtain

$$\prod_{t_j \in \mathcal{I} \setminus \mathcal{J}_1} (x - t_{j,0}t)^{\alpha_{t_j}} = x^L + \sum_{j=1}^{L} C_j(\boldsymbol{t}_{2,0}) t^j x^{L-j}.$$

Then the polynomial system (3.31) becomes

$$\widehat{B}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, t) := R_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1) - C_j(\boldsymbol{t}_{2,0}) t^j = 0, \qquad j = 1, \ldots, L. \tag{3.33}$$

The advantage of this new polynomial system (3.33) is that the numbers of unknowns and equations are the same. Note that $C_L(\boldsymbol{t}_{2,0}) = (-1)^L \prod t_{j,0}^{\alpha_{t_j}} \neq 0$. Then the same proof as (3.29) shows that the homogenized system of (3.33) has no solutions at infinity, so it follows from the Bezout theorem that the polynomial system (3.33) has exactly $L!$ solutions by counting multiplicity. This proves the claim (3.32). ∎

Proof of Theorem 1.10. Due to the finite choices of $\mathcal{J}_1$ and $(\epsilon_p)_{p \in \mathcal{J}_1}$, the assertion that the dimension of A is $\leq m - 1$ follows from Lemma 3.8.

Now we further assume (1.9). Then

$$\mathbb{Q}_\theta = \mathbb{Q}\big(\theta_0, \theta_1, \theta_\infty, \theta_1', \ldots, \theta_n'\big) \subsetneq \overline{\mathbb{Q}}.$$

Given any $\boldsymbol{t}_0 = \big(\tilde{t}_1, \ldots, \tilde{t}_{n+m}\big) \in A$, i.e., (1.1) has a co-axial solution for some $\boldsymbol{t}_0 = \big(\tilde{t}_1, \ldots, \tilde{t}_{n+m}\big)$ satisfying $\tilde{t}_j \neq 0, 1$ for any j and $\tilde{t}_j \neq \tilde{t}_k$ for any $j \neq k$. Without loss of generality, we may assume the corresponding

$$\mathcal{J}_1 = \{0, 1, t_1, \ldots, t_{n+i}\}, \qquad |\mathcal{J}_1| = n + i + 2,$$

$$\mathcal{I} \setminus \mathcal{J}_1 = \{t_{n+i+1}, \ldots, t_{n+m}\},$$

for some $0 \leq i \leq m - 1$. Denote

$$\boldsymbol{t}_{1,0} = \big(\tilde{t}_1, \ldots, \tilde{t}_{n+i}\big), \qquad \boldsymbol{t}_{2,0} = \Big(1, \tfrac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \tfrac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}\Big).$$

Then there is $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_1, t) = \big(\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{t}_{1,0}, \tilde{t}_{n+i+1}\big) \in \mathbb{C}^{m_1+m_2+|\mathcal{J}_1|-1}$ is a solution of (3.33). Since $\mathbb{Q}_\theta \subset \overline{\mathbb{Q}}$ implies

$$\widehat{B}_j(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_2, t) \in \overline{\mathbb{Q}}\Big[\tfrac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \tfrac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}\Big][\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}_2, t],$$

it follows from Lemma 3.7 that $\big(\boldsymbol{t}_{1,0}, \tilde{t}_{n+i+1}\big)$ is algebraic over $\overline{\mathbb{Q}}\Big[\tfrac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \tfrac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}\Big]$, namely, the transcendence degree of $\overline{\mathbb{Q}}\big(\tilde{t}_1, \ldots, \tilde{t}_{n+m}\big)$ over $\overline{\mathbb{Q}}$ is $\leq m - 1$. The proof is complete. ∎

4 Eremenko's theorem

Eremenko's Theorem 1.3 is a deep result, and in this section, we would like to make some discussions about it and prove Theorem 1.4. By using the notions in Section 3, we suppose that there exist $\mathcal{I}_1 \subset \mathcal{J}_1 \subsetneq \mathcal{I}$ and $\epsilon_p \in \{\pm 1\}$ for $p \in \mathcal{J} := \mathcal{J}_1 \cup \{\infty\}$ such that

$$k' := \sum_{p \in \mathcal{J}} \epsilon_p \theta_p \in \mathbb{Z}_{\geq 0}, \tag{4.1}$$

$$k'' := \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_p - k' - (n + m + 1) \in 2\mathbb{Z}_{\geq 0}. \tag{4.2}$$

Set

$$\boldsymbol{c} := (\underbrace{\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_p}_{p \text{ runs over } \mathcal{J}}, \underbrace{1, \ldots, 1}_{k'+k''}).$$

Suppose that

> *Either $\boldsymbol{c}$ is incommensurable or there is $\eta \neq 0$ such that $\boldsymbol{c} = \eta\mathbf{b} = \eta(b_1, \ldots, b_q)$ with $b_j \in \mathbb{N}$, $\gcd(b_1, \ldots, b_q) = 1$ and*
> $$2 \max_{j \in \mathcal{I} \setminus \mathcal{J}_1} \theta_j \leq \sum_{j=1}^{q} b_j. \tag{4.3}$$

Then the proof of Theorem 1.3 in [9] actually yields the following result, a more precise form of Theorem 1.3.

Theorem 4.1 ([9]). *Equation* (1.1) *admits a co-axial solution for some singular set* $\{t_1, \ldots,$ $t_{n+m}\}$ *with the developing map*

$$h(x) = \prod_{p \in \mathcal{J}_1} (x - p)^{\epsilon_p \theta_p} \frac{\prod_{j=1}^{m_1} (x - a_j)}{\prod_{j=1}^{m_2} (x - b_j)}$$

if and only if (4.1)–(4.3) *hold.*

It is not difficult to see that the conditions (1.3)–(1.5) in Theorem 1.3 imply (4.1)–(4.3) in Theorem 4.1. Theorem 4.1 shows that under the conditions (4.1)–(4.3), the polynomials B_j's in (3.18) always have an admissible zero. This is really a remarkable result.

Example 4.2. Consider the case $\{\theta_0, \theta_1, \theta_\infty, \theta_t\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \theta_t\}$ with $\theta_t \in \mathbb{N}_{\geq 2}$. If θ_t is small, then it is easy to calculate those t's such that (1.1) has co-axial solutions. For example, if $\theta_t = 2$, then $t = \frac{2}{3}$; if $\theta_t = 3$, then $t = \frac{2 \pm 2i}{3}$. When $\theta_t \geq 4$, the polynomial system $B_j = 0$ becomes very complicated. In Section 5, we will give more examples.

Example 4.3. Consider the case $\{\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \theta_{t_2}\} = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \theta_{t_2}\}$ with $\theta_{t_2} \in \mathbb{N}_{\geq 2}$. If $\theta_{t_2} = 3$, then $t_2 = \frac{1 \pm i\sqrt{3}}{2}$.

Now we turn back to the condition (1.5) in Theorem 1.3 if the vector $\boldsymbol{c}$ is commensurable. Under the conditions (1.3)–(1.4), it is not difficult to see that (1.5) holds automatically provided $k' + k'' \geq 1$ and $\theta_{t_j} > 1$ for all j. A more interesting thing is the assertion of Theorem 1.4, which says that (1.5) holds provided that Mondello–Panov's condition (1.2) holds.

Proof of Theorem 1.4. Here we prove a more general result than Theorem 1.4. To simplify the notations, we reformulate this problem. Given $n \geq 1$ positive non-integer numbers $\theta_1, \ldots, \theta_n$ and $m - n \geq 0$ positive integer numbers $\theta_{n+1}, \ldots, \theta_m \in \mathbb{N}_{\geq 2}$. Suppose

$$d_1\left(\mathbb{Z}_o^m, \boldsymbol{\theta} - \boldsymbol{1}\right) = 1, \tag{4.4}$$

and there are $\epsilon_j \in \{\pm 1\}$ for $1 \leq j \leq n$ such that

$$k' := \sum_{j=1}^n \epsilon_j \theta_j \in \mathbb{Z}_{\geq 0}, \tag{4.5}$$

$$k'' := \sum_{j=n+1}^m \theta_j - m - k' + 2 \in 2\mathbb{Z}_{\geq 0}. \tag{4.6}$$

Let

$$\boldsymbol{c} := (\theta_1, \ldots, \theta_n, \underbrace{1, \ldots, 1}_{k' + k''}),$$

and suppose $\boldsymbol{c}$ is commensurable, i.e., there is $\eta \in \mathbb{R}_{>0}$ such that $\boldsymbol{c} = \eta \boldsymbol{b} = \eta(b_1, \ldots, b_q)$ with $b_j \in \mathbb{N}$ and $\gcd(b_1, \ldots, b_q) = 1$. We want to prove that

$$2 \max_{n+1 \leq j \leq m} \theta_j \leq \sum_{j=1}^q b_j = \sum_{j=1}^n b_j + \left(k' + k''\right)\eta^{-1}. \tag{4.7}$$

Then Theorem 1.4 is equivalent to (4.7) with the case $n \geq 3$.

Clearly, (4.4) implies $n \geq 2$ and (4.5) and (4.6) imply $m - n \geq 1$. By renaming θ_j's if necessary, we may always assume

$$0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n, \qquad 2 \leq \theta_{n+1} \leq \theta_{n+2} \leq \cdots \leq \theta_m.$$

By (4.6), we have

$$k'' + k' = \sum_{j=n+1}^{m} \theta_j - m + 2 \geq \theta_m + 2(m - n - 1) - m + 2,$$

i.e.,

$$\max_{n+1 \leq j \leq m} \theta_j = \theta_m \leq k' + k'' - m + 2n \leq k' + k'' + n - 1,$$

so to prove (4.7), it suffices to prove

$$2n - 2 \leq \sum_{j=1}^{n} b_j + \left(\eta^{-1} - 2\right)\left(k' + k''\right). \tag{4.8}$$

On the other hand, since $\mathbf{c} = \eta\mathbf{b} = \eta(b_1, \ldots, b_q)$ with $b_j \in \mathbb{N}$, we have $\eta \neq 1$, which implies $\eta^{-1} \in \mathbb{N}_{\geq 2}$ if $k' + k'' \geq 1$. Thus we always have

$$\left(\eta^{-1} - 2\right)\left(k' + k''\right) \geq 0 \qquad \text{and} \qquad \sum_{j=1}^{n} b_j \geq n.$$

This implies that (4.8) holds if $n = 2$. Thus it remains to consider the case $n \geq 3$. We consider two cases separately.

Case 1. $k' + k'' \geq 1$. This is a simple case. Then $N := \eta^{-1} \in \mathbb{N}_{\geq 2}$ and for $1 \leq j \leq n$, $\theta_j = b_j/N \notin \mathbb{Z}$. Consequently, $\text{dist}(\theta_j, \mathbb{Z}) := \min_{k \in \mathbb{Z}} |\theta_j - k| \geq 1/N$ for all $1 \leq j \leq n$, so

$$1 = d_1\left(\mathbb{Z}_o^m, \boldsymbol{\theta} - \mathbf{1}\right) \geq \sum_{j=1}^{n} \text{dist}(\theta_j, \mathbb{Z}) \geq n/N,$$

i.e., $N \geq n$. Thus

$$\sum_{j=1}^{n} b_j + \left(\eta^{-1} - 2\right)\left(k' + k''\right) = \sum_{j=1}^{n} b_j + (N - 2)\left(k' + k''\right) \geq n + n - 2.$$

This proves (4.8).

Case 2. $k' + k'' = 0$. This case is not trivial. Note that (4.8) is equivalent to

$$2n - 2 \leq \sum_{j=1}^{n} b_j = \eta^{-1} \sum_{j=1}^{n} \theta_j. \tag{4.9}$$

Note that $1 \leq b_1 \leq b_2 \leq \cdots \leq b_n$. If $b_3 \geq 2$ then we have

$$\sum_{j=1}^{n} b_j \geq 1 + 1 + 2(n - 2) = 2n - 2,$$

i.e., (4.9) holds. So we only need to consider the remaining case that $b_1 = b_2 = b_3 = 1$, i.e., $\eta = \theta_1 = \theta_2 = \theta_3$. Let $\ell \geq 3$ such that

$$b_1 = \cdots = b_\ell = 1, \qquad b_{\ell+1} \geq 2. \tag{4.10}$$

Now we claim that $b_n \geq \ell$. Once this claim is proved, then

$$\sum_{j=1}^{n} b_j \geq \ell + 2(n - \ell - 1) + \ell = 2n - 2,$$

i.e., (4.9) holds and the proof is complete.

Assume by contradiction that $b_n \leq \ell - 1$. Since $k' = k'' = 0$, we have

$$\sum_{j=n+1}^{m} \theta_j = m - 2,$$

and there is $J_1 \subset \{1, \ldots, n\}$ such that $\sum_{j \in J_1} \theta_j = \sum_{j \in J_2} \theta_j$, where $J_2 = \{1, \ldots, n\} \setminus J_1$. This implies $\sum_{j \in J_1} b_j = \sum_{j \in J_2} b_j$, so

$$\sum_{j=1}^{n} b_j \qquad \text{is even.} \tag{4.11}$$

On the other hand, by (4.4) there is $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ such that $\sum_{j=1}^{m} k_j$ is odd and

$$\sum_{j=1}^{m} \left| \theta_j - 1 - k_j \right| = 1.$$

Since $\theta_j \in \mathbb{Z}$ for $j \geq n+1$, we have $k_j = \theta_j - 1$ for $j \geq n+1$ and so $\sum_{j=n+1}^{m} k_j = \sum_{j=n+1}^{m} (\theta_j - 1) = m - 2 - (m - n) = n - 2$. Denote $\tilde{k}_j = k_j + 1$, then we obtain

$$\sum_{j=1}^{n} \left| \theta_j - \tilde{k}_j \right| = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} \tilde{k}_j = \sum_{j=1}^{m} k_j + 2 \text{ is odd.}$$

Note that $\theta_j = b_j \eta = b_j \theta_1$. Let $\theta_1 = a + r$ with $a \in \mathbb{Z}$ and $r \in (0, 1)$.

Case 2.1. $0 < r \leq 1/2$. Then

$$1 = \sum_{j=1}^{n} \left| \theta_j - \tilde{k}_j \right| \geq \sum_{j=1}^{\ell} \left| \theta_1 - \tilde{k}_j \right| \geq \ell r,$$

so $r \leq 1/\ell \leq 1/3$. Consequently, for any $1 \leq j \leq n$, $b_j \leq b_n \leq \ell - 1$ implies $\theta_j = b_j \theta_1 = b_j a + b_j r$ with $b_j r \in (0, 1)$. From here and $1 = \sum_{j=1}^{n} \left| \theta_j - \tilde{k}_j \right|$, we claim that

$$\left| \theta_j - \tilde{k}_j \right| = b_j r \qquad \text{and so} \qquad \tilde{k}_j = b_j a, \qquad \forall j. \tag{4.12}$$

Indeed, we have $\left| \theta_j - \tilde{k}_j \right| \in \{b_j r, 1 - b_j r\}$ for all j. Recalling (4.10), if there is $1 \leq j_0 \leq \ell$ such that $\left| \theta_{j_0} - \tilde{k}_{j_0} \right| = 1 - r$, then

$$1 \geq \sum_{j=1}^{\ell} \left| \theta_1 - \tilde{k}_j \right| \geq (\ell - 1) r + 1 - r > 1,$$

a contradiction. Thus $\left| \theta_j - \tilde{k}_j \right| = b_j r = r$ for all $j \leq \ell$. Consequently, if $\left| \theta_{j_0} - \tilde{k}_{j_0} \right| = 1 - b_{j_0} r$ for some $\ell + 1 \leq j_0 \leq n$, then it follows from $b_{j_0} \leq \ell - 1$ that

$$1 = \sum_{j=1}^{n} \left| \theta_j - \tilde{k}_j \right| \geq \ell r + 1 - b_{j_0} r > 1,$$

a contradiction. This proves the claim (4.12) and so $a \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} \tilde{k}_j$ is odd, a contradiction with (4.11).

Case 2.2. $1/2 < r < 1$. Then $\theta_1 = a + 1 - (1 - r)$ and

$$1 = \sum_{j=1}^{n} \left| \theta_j - \tilde{k}_j \right| \geq \sum_{j=1}^{\ell} \left| \theta_1 - \tilde{k}_j \right| \geq \ell(1 - r),$$

so $1 - r \leq 1/\ell \leq 1/3$. Consequently, for any $1 \leq j \leq n$, $b_j \leq b_n \leq \ell - 1$ implies $\theta_j = b_j\theta_1 = b_j(a+1) - b_j(1-r)$ with $b_j(1-r) \in (0,1)$. Then as in Case 2-1, we can prove that

$$\left| \theta_j - \tilde{k}_j \right| = b_j(1-r) \qquad \text{and so} \qquad \tilde{k}_j = b_j(a+1), \qquad \forall j,$$

which implies so $(a+1)\sum_{j=1}^n b_j = \sum_{j=1}^n \tilde{k}_j$ is odd, again a contradiction with (4.11).

The proof is complete. ∎

As mentioned in the introduction, the converse statement of Theorem 1.4 can not hold in general, i.e., (1.3)–(1.5) can not imply Mondello–Panov's condition (1.2). For example, let $\{\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \theta_{t_2}\} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \theta_{t_2}\}$ with $\theta_{t_3} \in 2\mathbb{N} + 1$. Then it does not satisfy (1.2) but satisfy (1.3)–(1.5), so co-axial solutions exist for some singular set $\{t_1, t_2\}$. But this is not the case for $n = 0$ (i.e., exactly three non-integer angles).

5 Proof of Theorem 1.6

The goal of this section is to prove Theorem 1.6. Write t_1 and θ_{t_1} simply by t and θ. The function $Q(x)$ in the differential equation (2.6) associated to the curvature equation (1.1) in the case under consideration is

$$Q(x) = \frac{\beta_0}{x^2} + \frac{d_0}{x} + \frac{\beta_1}{(x-1)^2} + \frac{d_1}{x-1} + \frac{\beta_t}{(x-t)^2} + \frac{d_t}{x-t}, \tag{5.1}$$

where $\beta_p = \alpha_p(\alpha_p + 2)/4 = (\theta_p^2 - 1)/4$ and d's satisfy (2.3), i.e.,

$$d_0 + d_1 + d_t = 0, \qquad \beta_0 + \beta_1 + \beta_t + d_1 + td_t = \beta_\infty. \tag{5.2}$$

Before we proceed further, we note that the set $\{\alpha_p/2 + 1, -\alpha_p/2\}$ is invariant under the substitution $\alpha_p \mapsto -2 - \alpha_p = -\theta_p - 1$. For simplicity of discussion later on, we assume that α_0 and α_1 satisfy

$$\alpha_0 = \epsilon_0\theta_0 - 1, \qquad \alpha_1 = \epsilon_1\theta_1 - 1, \tag{5.3}$$

where ϵ_0, ϵ_1 are given in the assumption of Theorem 1.6. Thus, $\alpha_0 + \alpha_1 + \alpha_\infty = \epsilon_0\theta_0 + \epsilon_1\theta_1 + \theta_\infty - 3$ is an integer.

In view of (5.2), we regard d_0 and d_1 as functions of $d := d_t$ and let $Q_{d,t}(x)$ denote the rational function $Q(t)$ in (5.1). By Theorem 2.1, there exists a polynomial $\mathcal{P}(d,t) \in \mathbb{Q}_\theta[d,t]$ of degree θ in d such that the differential equation

$$y''(x) = Q_{d,t}(x)y(x) \tag{5.4}$$

is apparent at t if and only if $\mathcal{P}(d,t) = 0$, where $\mathbb{Q}_\theta = \mathbb{Q}(\theta_0, \theta_1, \theta_\infty)$. The degree of $\mathcal{P}(d,t)$ in d is θ, but its total degree is in general strictly larger than θ (see (2.9)). In our proof of Theorem 1.6, we shall introduce another pair (λ, t) of unknowns in place of (d, t).

Let

$$\lambda := td_0 + t\frac{\alpha_0\alpha_1}{2} + \frac{\alpha_0\alpha_t}{2} = t(t-1)d_t + t(\beta_0 + \beta_1 + \beta_t - \beta_\infty) + t\frac{\alpha_0\alpha_1}{2} + \frac{\alpha_0\alpha_t}{2}, \tag{5.5}$$

where the second equality follows from (5.2). Indeed, the parameter λ is used as the accessary parameter of a certain Heun equation considered in [11], where for the simplicity of computation, the apparent singularity was put at $x = 0$ instead of $x = t$ in this paper, and the condition for apparentness at $x = 0$ was shown to be equivalent to the vanishing of the characteristic polynomial of a finite Jacobi matrix (see [11, Proposition 2.4 and equation (2.8)]). For the convenience of

the reader, we recall the equivalence, stated in our setting, as follows (see also [6, Lemma B.8]): (5.4) is apparent at t if and only if λ is an eigenvalue of the $\theta \times \theta$ matrix

$$
M = M(t) := \begin{pmatrix}
B_1 & A_1 & & & & \\
D_2 & B_2 & A_2 & & & \\
& D_3 & B_3 & A_3 & & \\
& & \ddots & \ddots & \ddots & \\
& & & D_{\theta-1} & B_{\theta-1} & A_{\theta-1} \\
& & & & D_\theta & B_\theta
\end{pmatrix} \tag{5.6}
$$

(undisplayed entries are all 0), where

$$
\begin{aligned}
A_j &= A_j(t) := t(t-1)j(j-\theta), \\
B_j &= B_j(t) := (2t-1)(j-1)(j-2) - (j-1)[(t-1)\alpha_0 + t\alpha_1 + (2t-1)\alpha_t] + t\alpha'\alpha'', \\
D_j &= D_j(t) := (j-2)(j-3) - (j-2)(\alpha_0 + \alpha_1 + \alpha_t) + \alpha'\alpha'',
\end{aligned}
$$

with

$$
\alpha' := -\frac{\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty}{2} - 1, \qquad \alpha'' := \frac{\alpha_\infty - \alpha_0 - \alpha_1 - \alpha_t}{2}.
$$

That is, (5.4) is apparent at t if and only if λ is an eigenvalue of $M(t)$, i.e., a zero of the polynomial

$$
P(\lambda, t) := \det(\lambda I_\theta - M(t)).
$$

The polynomial $P(\lambda, t)$ has the following properties.

Lemma 5.1.

(i) *The total degree and the degree in λ of $P(\lambda, t)$ are both θ.*

(ii) *The roots of $P(\lambda, 0)$ are $(j-1)(\alpha_0 + \alpha_t - j + 2)$, $j = 1, \ldots, \theta$.*

(iii) *Let $P_\infty(\lambda) := \lim_{t \to \infty} P(\lambda t, t)/t^\theta$. Then the roots of $P_\infty(\lambda)$ are*

$$
\begin{aligned}
\hat{\lambda}_j &:= (j-1)(\alpha_\infty + \alpha_t - j + 2) - \beta_\infty - \beta_t + \beta_0 + \beta_1 \\
&\quad + \frac{\alpha_0 \alpha_1}{2} - \frac{\alpha_t \alpha_\infty}{2}, \qquad j = 1, \ldots, \theta.
\end{aligned} \tag{5.7}
$$

Proof. Statement (i) follows easily from the observations that $\deg_t A_j(t) = 2$, $\deg_t B_j(t) = 1$, and $D_j(t)$ are constant polynomials.

The second property follows immediately from the fact that $A_j(0) = 0$ and hence the roots of $P(\lambda, 0)$ are simply $B_j(0)$. The third property is proved in [6, Theorem B.3]. Since the conclusions are stated in different ways, here we provide a sketch of proof.

Let $u = 1/x$. We check directly that $y(x)$ is a solution of (5.4) if and only if $\widetilde{y}(u) := uy(1/u)$ satisfies

$$
\frac{\mathrm{d}^2}{\mathrm{d}u^2} \widetilde{y}(u) = \widetilde{Q}(u)\widetilde{y}(u), \tag{5.8}
$$

where $\widetilde{Q}(u) = u^{-4}Q(1/u)$. Clearly, this is a Fuchsian differential equation with Riemann scheme

$$
\begin{pmatrix}
0 & 1 & 1/t & \infty \\
-\alpha_\infty/2 & -\alpha_1/2 & -\alpha_t/2 & -(\alpha_0/2 + 1) \\
\alpha_\infty/2 + 1 & \alpha_1/2 + 1 & \alpha_t/2 + 1 & \alpha_0/2
\end{pmatrix}.
$$

Thus,

$$\widetilde{Q}(u) = \frac{\beta_\infty}{u^2} + \frac{\widetilde{d}_0}{u} + \frac{\beta_1}{(u-1)^2} + \frac{\widetilde{d}_1}{u-1} + \frac{\beta_t}{(u-1/t)^2} + \frac{\widetilde{d}_{1/t}}{u-1/t}$$

for some complex numbers $\widetilde{d}_0$, $\widetilde{d}_1$, and $\widetilde{d}_{1/t}$ satisfying

$$\widetilde{d}_0 + \widetilde{d}_1 + \widetilde{d}_{1/t} = 0, \qquad \widetilde{d}_1 + \frac{\widetilde{d}_{1/t}}{t} + \beta_\infty + \beta_1 + \beta_t = \beta_0.$$

In fact, computing the partial fraction decomposition of $\widetilde{Q}(u)$, we find that

$$\widetilde{d}_0 = 2\beta_1 + 2t\beta_t + d_1 + t^2 d_t, \qquad \widetilde{d}_1 = -2\beta_1 - d_1, \qquad \widetilde{d}_{1/t} = -2t\beta_t - t^2 d_t.$$

We now apply parts (i) and (ii) to (5.8). Let

$$\widetilde{\lambda} = \frac{\widetilde{d}_0}{t} + \frac{\alpha_\infty \alpha_1}{2t} + \frac{\alpha_\infty \alpha_t}{2}.$$

By parts (i) and (ii), there is a polynomial $\widetilde{P}(\widetilde{\lambda}, 1/t)$ such that (5.8) is apparent at $1/t$ if and only if $\widetilde{P}(\widetilde{\lambda}, 1/t) = 0$ and the roots of $\widetilde{P}(\widetilde{\lambda}, 0)$ are

$$(j-1)(\alpha_\infty + \alpha_t - j + 2), \qquad j = 1, \ldots, \theta.$$

On the other hand, we see from (5.2) and (5.5) that

$$\begin{aligned}
\widetilde{d}_0 &= t(t-1)d_t + 2t\beta_t + \beta_\infty + \beta_1 - \beta_0 - \beta_t \\
&= \lambda + t(\beta_\infty + \beta_t - \beta_0 - \beta_1) - t\frac{\alpha_0 \alpha_1}{2} - \frac{\alpha_0 \alpha_t}{2} + \beta_\infty + \beta_1 - \beta_0 - \beta_t,
\end{aligned}$$

so $\widetilde{\lambda}$ and λ are related by

$$\begin{aligned}
\widetilde{\lambda} = {}& \frac{\lambda}{t} + \beta_\infty + \beta_t - \beta_0 - \beta_1 - \frac{\alpha_0 \alpha_1}{2} + \frac{\alpha_\infty \alpha_t}{2} \\
& + \frac{1}{t}(\beta_\infty + \beta_1 - \beta_0 - \beta_t) + \frac{1}{2t}(\alpha_1 \alpha_\infty - \alpha_0 \alpha_t).
\end{aligned}$$

Letting $t \to \infty$, we conclude that the roots of $P_\infty(\lambda)$ are given by (5.7).　　■

Corollary 5.2. *Let λ and $\hat{\lambda}_j$ be given by (5.5) and (5.7), respectively. Let $y_+(x; \lambda, t)$ and $y_-(x; \lambda, t)$ be solutions of (5.4) of the form*

$$y_+(x; \lambda, t) = x^{\alpha_0/2+1} \sum_{j=0}^{\infty} c_{+,j}(\lambda, t) x^j, \qquad c_{+,0}(\lambda, t) = 1,$$

$$y_-(x; \lambda, t) = x^{-\alpha_0/2} \sum_{j=0}^{\infty} c_{-,j}(\lambda, t) x^j, \qquad c_{-,0}(\lambda, t) = 1.$$

Then as $t \to \infty$ with $\lambda/t \to \hat{\lambda}_j$, $y_\pm(x; \lambda, t)$ converge to solutions of the Fuchsian differential equation with Riemann scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ -\alpha_0/2 & -\alpha_1/2 & -(\hat{\alpha}_\infty/2 + 1) \\ \alpha_0/2 + 1 & \alpha_1/2 + 1 & \hat{\alpha}_\infty/2 \end{pmatrix}, \tag{5.9}$$

where $\hat{\alpha}_\infty = \alpha_\infty + \alpha_t - 2j + 2$.

Proof. Let $\hat{d}_0$, $\hat{d}_1$, $\hat{d}_t$ be the limits of d_0, d_1, and d_t as $t \to \infty$ and $\lambda/t \to \hat{\lambda}_j$. From (5.5), we know that

$$\hat{d}_0 = (j-1)(\alpha_\infty + \alpha_t - j + 2) - \beta_\infty - \beta_t + \beta_0 + \beta_1 - \frac{\alpha_t \alpha_\infty}{2}$$

and $(t-1)d_t$ converge to finite numbers as $t \to \infty$ and $\lambda/t \to \hat{\lambda}_j$. The latter implies that $\hat{d}_t = 0$. Thus, we have

$$\hat{Q}_j(x) := \lim_{t \to \infty, \, \lambda/t \to \hat{\lambda}_j} Q_{d,t}(x) = \frac{\beta_0}{x^2} + \frac{\hat{d}_0}{x} + \frac{\beta_1}{(x-1)^2} + \frac{\hat{d}_1}{x-1}.$$

From the relation $d_0 + d_1 + d_t = 0$, we see that $\hat{d}_1 = -\hat{d}_0$ and

$$\hat{d}_1 + \beta_0 + \beta_1 = -(j-1)(\alpha_\infty + \alpha_t - j + 2) + \beta_\infty + \beta_t + \frac{\alpha_t \alpha_\infty}{2}$$

$$= \frac{1}{4}(\alpha_\infty + \alpha_t - 2j + 2)(\alpha_\infty + \alpha_t - 2j + 4).$$

It follows that

$$y''(x) = \hat{Q}_j(x) y(x)$$

is a Fuchsian differential equation with Riemann scheme given by (5.9). The convergence of $Q_{d,t}(x)$ to $\hat{Q}_j(x)$ is clearly uniform on any compact subset of $\mathbb{C} \setminus \{0, 1\}$. Therefore, $y_\pm(x; \lambda, t)$ converge to solutions of $y''(x) = \hat{Q}_j(x) y(x)$. This completes the proof. ∎

Now for $p \in \{0, 1, \infty, t\}$, let $M_p(\lambda, t)$ be the monodromy matrices of (5.4) around $x = p$ with respect to the basis $(y_+(x; \lambda, t), y_-(x; \lambda, t))^t$. Recalling that the Riemann scheme of (5.4) is

$$\begin{pmatrix} 0 & 1 & t & \infty \\ -\alpha_0/2 & -\alpha_1/2 & -\alpha_t/2 & -(\alpha_\infty/2 + 1) \\ \alpha_0/2 + 1 & \alpha_1/2 + 1 & \alpha_t/2 + 1 & \alpha_\infty/2 \end{pmatrix},$$

we know that

$$M_0(\lambda, t) = \begin{pmatrix} e^{\pi i \alpha_0} & 0 \\ 0 & e^{-\pi i \alpha_0} \end{pmatrix}, \qquad M_t(\lambda, t) = (-1)^{\theta - 1} I_2.$$

Also, the monodromy matrices satisfy $M_0 M_1 M_t M_\infty = I_2$.

Lemma 5.3. *We have*

$$M_1(\lambda, t) = \begin{pmatrix} e^{\pi i \alpha_1} & b(\lambda, t) \\ c(\lambda, t) & e^{-\pi i \alpha_1} \end{pmatrix}$$

for some complex numbers $b(\lambda, t)$ and $c(\lambda, t)$ satisfying $b(\lambda, t)c(\lambda, t) = 0$.

Proof. Write $M_1(\lambda, t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the eigenvalues of M_1 are $e^{\pi i \alpha_1}$ and $e^{-\pi i \alpha_1}$ and those of M_∞ are $e^{\pi i \alpha_\infty}$ and $e^{-\pi i \alpha_\infty}$. Together with the relation $M_0 M_1 M_t M_\infty = I_2$, these informations yield

$$a + d = e^{\pi i \alpha_1} + e^{-\pi i \alpha_1},$$

$$a e^{\pi i \alpha_0} + d e^{-\pi i \alpha_0} = (-1)^{\theta - 1} \left(e^{\pi i \alpha_\infty} + e^{-\pi i \alpha_\infty} \right).$$

Eliminating d, we get

$$\left(e^{\pi i \alpha_0} - e^{-\pi i \alpha_0} \right) a = (-1)^{\theta - 1} \left(e^{\pi i \alpha_\infty} + e^{-\pi i \alpha_\infty} \right) - \left(e^{\pi i (\alpha_1 - \alpha_0)} + e^{\pi i (-\alpha_1 - \alpha_0)} \right).$$

Now we have $\alpha_0 + \alpha_1 + \alpha_\infty = \epsilon_0\theta_0 + \epsilon_1\theta_1 + \theta_\infty - 3 = k - 3$. Thus, the condition $\theta \equiv k \mod 2$ implies that

$$e^{\pi i(-\alpha_1-\alpha_0)} = e^{\pi i(3-k+\alpha_\infty)} = (-1)^{\theta-1}e^{\pi i\alpha_\infty}$$

and hence $(-1)^{\theta-1}e^{-\pi i\alpha_\infty} = e^{\pi i(\alpha_0+\alpha_1)}$. From these, we see that $a = e^{\pi i\alpha_1}$. It follows that $d = e^{-\pi i\alpha_1}$ and $bc = 0$. This proves the lemma. ∎

Let $P(\lambda, t) = P_1(\lambda, t) \cdots P_m(\lambda, t)$ be the factorization of $P(\lambda, t)$ into a product of irreducible polynomials over $\mathbb{C}$. It is easy to see that for each j, we have either $b(\lambda, t) \equiv 0$ or $c(\lambda, t) \equiv 0$ on $\mathcal{A}_j := \{(\lambda, t) \in \mathbb{C}^2 : P_j(\lambda, t) = 0\}$. Thus, if we set

$$P_b(\lambda, t) := \prod_{b(\lambda,t)\equiv 0 \text{ on } \mathcal{A}_j} P_j(\lambda, t), \qquad P_c(\lambda, t) := \prod_{c(\lambda,t)\equiv 0 \text{ on } \mathcal{A}_j} P_j(\lambda, t),$$

then the cardinality of the set A is equal to the number of intersections of the two curves $P_b(\lambda, t) = 0$ and $P_c(\lambda, t) = 0$ in the affine plane $\mathbb{C}^2$, with the multiplicity of a point in A defined to be that of the corresponding intersection point of P_b and P_c. Furthermore, by part (iii) of Lemma 5.1, the two curves $P_b(\lambda, t) = 0$ and $P_c(\lambda, t) = 0$ do not intersect at infinity. Therefore, by Bezout's theorem, the number of intersections of the two curves in the affine plane is $(\deg P_b)(\deg P_c)$. Hence, to prove Theorem 1.6, we only need to determine the degrees of P_b and P_c.

Proposition 5.4. *Let $k := \theta_\infty + \epsilon_0\theta_0 + \epsilon_1\theta_1$ be given as in the statement of Theorem 1.6. If $\theta \le |k|$, then*

$$\begin{cases} \deg P_b = 0 \text{ and } \deg P_c = \theta, & \text{if } k > 0, \\ \deg P_b = \theta \text{ and } \deg P_c = 0, & \text{if } k < 0. \end{cases}$$

If $\theta > |k|$, then $\deg P_b = (\theta - k)/2$ and $\deg P_c = (\theta + k)/2$.

To prove Proposition 5.4, we recall that, by Lemma 5.1 (i), the total degree and the degree in λ of $P(\lambda, t)$ are equal. It is easy to see that every factor of $P(\lambda, t)$ has the same property. It follows that the degree of $P_b(\lambda, t)$ is equal to the degree of the polynomial $P_{b,\infty}(\lambda) := \lim_{t\to\infty} P_b(\lambda t, t)/t^{\deg P_b}$, which in turn is equal to the number of $\hat{\lambda}_j$ that are roots of $P_{b,\infty}(\lambda)$, where $\hat{\lambda}_j$ are given by (5.7). In other words, to determine the degree of $P_b(\lambda, t)$, we shall count how many $\hat{\lambda}_j$ such that $\lim_{t\to\infty,\lambda/t\to\hat{\lambda}_j} c(\lambda, t) \ne 0$ there are.

Lemma 5.5. *Let $\hat{\lambda}_j$ be given by (5.7) and*

$$\hat{b} = \lim_{t\to\infty,\lambda/t\to\hat{\lambda}_j} b(r, t), \qquad \hat{c} = \lim_{t\to\infty,\lambda/t\to\hat{\lambda}_j} c(r, t).$$

Then

$$\begin{cases} \hat{b} = 0, \ \hat{c} \ne 0, & \text{if } j > (k+\theta)/2, \\ \hat{b} \ne 0, \ \hat{c} = 0, & \text{if } j \le (k+\theta)/2. \end{cases}$$

Proof. By Corollary 5.2, $y_\pm(x; \lambda, t)$ converge to solutions $\hat{y}_\pm(x)$ of the Fuchsian differential equation with Riemann scheme (5.9), where $\hat{y}_\pm(x)$ are local solutions near 0 of the form $\hat{y}_+(x) = x^{\alpha_0/2+1}(1 + O(x))$ and $\hat{y}_-(x) = x^{-\alpha_0/2}(1 + O(x))$. Then $w_{0,\pm}(x) := x^{\alpha_0/2}(x - 1)^{\alpha_1/2}\hat{y}_\pm(x)$ are solutions of the hypergeometric differential equation with Riemann scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix},$$

where

$$\gamma = -\alpha_0, \qquad \alpha = -\left(\frac{\alpha_0 + \alpha_1 + \hat{\alpha}_\infty}{2} + 1\right), \qquad \beta = \frac{\hat{\alpha}_\infty - \alpha_0 - \alpha_1}{2}.$$

Let $w_{1,\pm}(x)$ be the local solutions near $x = 1$ of the same hypergeometric differential equation of the form $w_{1,+}(x) = (x-1)^{\gamma-\alpha-\beta}(1 + O(x-1))$ and $w_{1,-}(x) = 1 + O(x-1)$. By [14, Chapter 2, Theorem 4.7.1], the connection matrix P such that $(w_{0,+}(x), w_{0,-}(x))^t = P(w_{1,+}(x), w_{1,-}(x))^t$ is

$$P = \begin{pmatrix} \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)} & \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} \\ \dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\alpha)\Gamma(\beta)} & \dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \end{pmatrix}.$$

Then the monodromy matrix of the hypergeometric differential equation around $x = 1$ with respect to the basis $(w_{0,+}(x), w_{0,-}(x))^t$ is

$$P \begin{pmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & 0 \\ 0 & 1 \end{pmatrix} P^{-1}.$$

Now according to our choice of α_0 and α_1 in (5.3), we have

$$\alpha = -\left(\frac{\alpha_0 + \alpha_1 + \hat{\alpha}_\infty}{2} + 1\right) = -\frac{(\epsilon_0\theta_0 - 1) + (\epsilon_1\theta_1 - 1) + (\theta_\infty - 1 + \theta - 1 - 2j + 2)}{2} - 1$$
$$= j - \frac{k+\theta}{2},$$

which is an integer by the assumption $\theta \equiv k \bmod 2$. Thus, P is upper-triangular when $j \leq (k+\theta)/2$ and is lower-triangular when $j > (k+\theta)/2$. This implies that $\hat{b} \neq 0$ and $\hat{c} = 0$ if $j \leq (k+\theta)/2$, and $\hat{b} = 0$ and $\hat{c} \neq 0$ if $j > (k+\theta)/2$. ∎

Proof of Proposition 5.4 and Theorem 1.6. Since $j = 1, 2, \ldots, \theta$, Proposition 5.4 follows from Lemma 5.5. Consequently, Theorem 1.6 holds. ∎

Remark 5.6. The entry D_j in (5.6) can be expressed as

$$D_j = \frac{1}{4}(2j - k - \theta - 2)(2j - k - \theta + 2\theta_\infty - 2).$$

Thus, one has $D_j = 0$ when $j = (k+\theta)/2 + 1$. It follows that when $|k| < \theta$, the matrix M in (5.6) is of the form

$$M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix},$$

where $M_1 = M_1(t)$ and $M_2 = M_2(t)$ are square matrices of size $(\theta + k)/2$ and $(\theta - k)/2$, respectively. Hence $P(\lambda, t) := \det(\lambda I - M) = \det(\lambda I - M_1)\det(\lambda I - M_2)$. One would expect that $P_c(\lambda, t) = \det(\lambda I - M_1)$ and $P_b(\lambda, t) = \det(\lambda I - M_2)$. Indeed, one can prove that this is the case by considering the roots of $P_b(\lambda, 0)$ and $P_c(\lambda, 0)$. We will not give the details here.

For small θ, it is possible to completely write down the set A. We give some examples below.

Example 5.7. Assume that $\theta = 2$. Then Theorem 1.6 implies that the set A is non-empty only when $k = 0$. In this case, we compute that

$$P(\lambda, t) = \lambda(\lambda - \theta_\infty t - \epsilon_0 \theta_0).$$

The intersection of the two lines $\lambda = 0$ and $\lambda - \theta_\infty t - \epsilon_0 \theta_0 = 0$ is $(\lambda, t) = (0, -\epsilon_0 \theta_0 / \theta_\infty)$. Therefore, the set A consists of a single point $-\epsilon_0 \theta_0 / \theta_\infty$.

Note that in [6, Example 5.5], we have shown that for the case $\theta_0 = 1/3$, $\theta_1 = 1/6$, and $\theta_\infty = 1/2$ with $\epsilon_0 = \epsilon_1 = -1$, the set A is $\{2/3\}$. This agrees with the general result above.

Example 5.8. Assume that $\theta = 3$. Then Theorem 1.6 implies that the set A is non-empty only when $|k| = 1$. In this case, to simplify notations, we will write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by θ_0 and θ_1. We compute that when $k = 1$,

$$\begin{aligned}
P(\lambda, t) = {}&(\lambda + (\theta_0 + \theta_1)t - 2\theta_0) \\
&\times \left(\lambda^2 - ((\theta_0 + \theta_1 - 2)t + \theta_0 + 1)\lambda + (\theta_0 - 1)(\theta_0 + \theta_1)t\right),
\end{aligned}$$

and when $k = -1$,

$$\begin{aligned}
P(\lambda, t) = {}&\lambda\big(\lambda^2 + ((3\theta_0 + 3\theta_1 + 2)t - 3\theta_0 - 1)\lambda \\
&+ 2(\theta_0 + \theta_1)(\theta_0 + \theta_1 + 1)t^2 - 4\theta_0(\theta_0 + \theta_1 + 1)t + \theta_0(\theta_0 + 1)\big).
\end{aligned}$$

We find that the set A consists of

$$\frac{\theta_0 \theta_\infty \pm \sqrt{-k\theta_0 \theta_1 \theta_\infty}}{(\theta_0 + \theta_1)\theta_\infty}.$$

Note that the case $\theta_0 = 1/3$, $\theta_1 = 1/6$, and $\theta_\infty = 1/2$ was considered in [6, Example 5.6], where we found that the set A is $\{2(1 \pm i)/3\}$, as the result above says.

Example 5.9. Let $\theta = 4$. Again, we write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by θ_0 and θ_1, respectively. When $k = 2$, we find that the set A consists of the three roots of

$$\theta_\infty(\theta_\infty - 1)(\theta_\infty - 2)t^3 + 3\theta_0 \theta_\infty(\theta_\infty - 1)t^2 + 3\theta_\infty \theta_0(\theta_0 - 1)t + \theta_0(\theta_0 - 1)(\theta_0 - 2).$$

When $k = -2$, they are the roots of

$$\theta_\infty(\theta_\infty + 1)(\theta_\infty + 2)t^3 + 3\theta_0 \theta_\infty(\theta_\infty + 1)t^2 + 3\theta_\infty \theta_0(\theta_0 + 1)t + \theta_0(\theta_0 + 1)(\theta_0 + 2).$$

When $k = 0$, they are the roots of

$$\theta_\infty^2(\theta_\infty^2 - 1)t^4 + 4\theta_0 \theta_\infty(\theta_\infty^2 - 1)t^3 + 6\theta_0 \theta_\infty(\theta_0 \theta_\infty + 1)t^2 + 4\theta_\infty \theta_0(\theta_0^2 - 1)t + \theta_0^2(\theta_0^2 - 1).$$

(The polynomials $P(\lambda, t)$ themselves are too complicated to be displayed here.)

Acknowledgements

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