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Infinitesimal Modular Group: q-Deformed $\mathfrak{sl}_2$ and Witt Algebra

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Abstract. We describe new q-deformations of the 3-dimensional Heisenberg algebra, the simple Lie algebra $\mathfrak{sl}_2$ and the Witt algebra. They are constructed through a realization as differential operators. These operators are related to the modular group and q-deformed rational numbers defined by Morier-Genoud and Ovsienko and lead to q-deformed Möbius transformations acting on the hyperbolic plane.

Key words: quantum algebra; Lie algebra deformations; q-Virasoro; Burau representation

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1 Introduction and results

The construction of q-deformed rational numbers by Morier-Genoud and Ovsienko [12] starts from the observation that rational numbers are generated by the image of zero under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. This group is generated by the translation $T(x) = x + 1$ and the inversion $S(x) = -1/x$. The only relations between these operations are $S^2 = \mathrm{id} = (ST)^3$.

The q-deformed integers $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, where $q \in \mathbb{C}^*$, satisfy the relation $[n + 1]_q = q[n]_q + 1$. It is natural to introduce as q-analog to the translation T the transformation $T_q(x) = qx + 1$. The map $S_q(x) = -1/(qx)$ satisfies $S_q^2 = \mathrm{id} = (S_qT_q)^3$. The q-*rational numbers* are then defined by the image of zero under the action by T_q and S_q using for example the continued fraction representation of a rational number. Since these operations are Möbius transformations, we can represent them in matrix form as follows:

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

This coincides with the *reduced Burau representation* of the braid group B_3 with parameter $t = -q$ [4]. Indeed the standard generators of B_3 are represented by $\sigma_1 = T_q$ and $\sigma_2 = S_qT_qS_q = \begin{pmatrix} 1 & 0 \\ -q & q \end{pmatrix}$. One easily checks the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The faithfulness of specializations of the Burau representation (where q is not a formal parameter, but a non-zero complex number) is an open question [3, Section 7]. It was studied for real values of q in [18]. In [14], a link to q-deformed rational numbers allows to partially solve the open question.

Using Taylor expansions of q-rational numbers, one can define q-real numbers [13] which are power series in q with integer coefficients. A natural question is *how to do analysis with these q-real numbers?* Basic functions on real numbers are monomials and the exponential function, which are eigenfunctions of the vector fields associated to $\mathfrak{sl}_2(\mathbb{R})$ (acting on the completed line $\mathbb{RP}^1$). The goal of our investigation is to q-deform these vector fields and to analyze their eigenfunctions.

Following a suggestion of Valentin Ovsienko, we can associate to T_q a differential operator $D_{-1}(q)$, which corresponds to the infinitesimal q-shift. For $q = 1$, we have $D_{-1}(1) = \partial = \mathrm{d}/\mathrm{d}x$. This operator is given by

$$D_{-1} := (1 + (q-1)x)\partial.$$

One can directly check that D_{-1} commutes with T_q, where T_q acts on the space of functions by precomposition. The starting point of the paper is the question whether there is a differential operator associated to S_q. This would allow to define in some sense a Lie algebra for the modular group $\mathrm{PSL}_2(\mathbb{Z})$, or an infinitesimal version of the Burau representation of B_3.

In the classical setting for $q = 1$, there is an operator which *anti-commutes* with S:

$$S \circ x\partial + x\partial \circ S = 0,$$

where S acts on the space of functions by precomposition. We introduce the differential operator D_0, a q-deformed version of $x\partial$, given by

$$D_0 := (1 + (x-1)q)D_{-1} = (1 + (x-1)q)(1 + (q-1)x)\partial.$$

We will see that D_0 anti-commutes with S_q. Together with $D_1 := S_q \circ D_{-1} \circ S_q$, a deformation of $x^2\partial$, we get three differential operators which are closed under the bracket (see Theorem 2.3):

Theorem 1.1. *The operators D_{-1}, D_0 and D_1 form a Lie algebra with brackets*

$$[D_0, D_1] = \big(q^2 - q + 1\big)D_1 + (1-q)D_0, \qquad [D_0, D_{-1}] = -\big(q^2 - q + 1\big)D_{-1} + (1-q)D_0,$$
$$[D_{-1}, D_1] = 2D_0 + (1-q)(D_1 - D_{-1}).$$

The theorem tells us that the module over $\mathbb{R}[q]$ generated by D_{-1}, D_0 and D_1 is a deformation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ which we recover for $q = 1$. The Lie algebra $\mathfrak{sl}_2$ being simple, it does not allow for non-trivial deformations. Hence our deformation is isomorphic to $\mathfrak{sl}_2$ as a Lie algebra, but they are different as $\mathbb{Z}[q]$-modules. This is similar to quantum groups.

A fundamental role is played by the Möbius transformation

$$g_q(x) = \frac{1 + (x-1)q}{1 + (q-1)x},$$

which is a deformation of the identity. It is the eigenfunction of D_0 with eigenvalue $q^2 - q + 1$ and normalization $g_q(0) = 1 - q$. We call it the *q-rational transition map* since it makes a passage between two different q-deformations of rational numbers studied in [2]. More precisely (see Theorem 2.7):

Theorem 1.2. *The two q-deformations of rational numbers defined in* [2, Definition 2.6] *are linked via*

$$g_q\left(\left[\frac{r}{s}\right]^{\sharp}_{q}\right) = \left[\frac{r}{s}\right]^{\flat}_{q^{-1}}.$$

This theorem comes from the interplay between g_q, T_q and S_q given by $g_q \circ T_q = T_{q^{-1}} \circ g_q$ and $g_q \circ S_q = S_{q^{-1}} \circ g_q$ (see Proposition 2.6). The q-rational transition map also satisfies a sort of duality between q and x:

$$g_q(x)g_x(q) = 1.$$

The map g_q, as well as its multiplicative inverse $g_q^{-1} = 1/g_q$, behave very well with the three operators D_{-1}, D_0 and D_1 (see Propositions 2.11 and 3.1):

Proposition 1.3. *The q-rational transition map g_q and the differential operators D_{-1}, D_0, and D_1 interact in the following way:*

(1) $D_0(g_q) = (q^2 - q + 1)g_q$, $D_{-1}(g_q) = q + (1-q)g_q$, $D_1(g_q) = (q-1)g_q + g_q^2$,

(2) $g_q D_0 = (1-q)D_0 + (q^2 - q + 1)D_1$, $g_q D_{-1} = D_0 + (1-q)D_1$,

(3) $qg_q^{-1}D_0 = (q-1)D_0 + (q^2 - q + 1)D_{-1}$, $qg_q^{-1}D_1 = D_0 + (q-1)D_{-1}$.

These relations allow a deformation of the Witt algebra, the complexification of the Lie algebra of polynomial vector fields on the circle. The Witt algebra is described by a vector space basis $(\ell_n)_{n\in\mathbb{Z}}$ with bracket given by

$$[\ell_n, \ell_m] = (m-n)\ell_{n+m}.$$

This algebra can be realized as differential operators (or equivalently as vector fields) via $\ell_n = x^{n-1}\partial$. Putting for $n > 1$:

$$D_n = g_q^{n-1}D_1 \qquad \text{and} \qquad D_{-n} = \left(qg_q^{-1}\right)^{n-1}D_{-1},$$

we get a deformation of the Witt algebra (see Theorem 3.2):

Theorem 1.4. *The $(D_n)_{n\in\mathbb{Z}}$ form a Lie algebra with bracket given by (where $n, r > 0$):*

$$\begin{aligned}
&[D_0, D_n] = n(q^2 - q + 1)D_n + (q^2 - q + 1)\sum_{k=1}^{n-1}(1-q)^k D_{n-k} + (1-q)^n D_0,\\
&[D_n, D_{n+r}] = rD_{2n+r} + (q-1)rD_{2n+r-1},\\
&[D_{-n}, D_n] = 2nq^{n-1}D_0 + (2n-1)q^{n-1}(q-1)(D_{-1} - D_1),\\
&[D_{n+r}, D_{-n}] = (q-1)q^{n-1}(2n+r-1)D_{r+1} - \left(q^2 + (2n+r-2)q + 1\right)q^{n-1}D_r,\\
&\qquad\qquad - q^{n-1}(q^2 - q + 1)\sum_{k=1}^{r-1}(1-q)^k D_{r-k} - (1-q)^r q^{n-1}D_0.
\end{aligned}$$

The remaining brackets $[D_0, D_{-n}]$, $[D_{-n}, D_{-n-r}]$ and $[D_n, D_{-n-r}]$ obey similar formulas.

Integrating the vector fields associated to D_{-1}, D_0 and D_1 on the hyperbolic plane, we get Möbius transformations. We speculate about a q-deformed hyperbolic plane on which these transformations naturally act. The boundary of this deformed hyperbolic plane should be the q-deformed real numbers. Other interesting open questions include the link between our q-deformed $\mathfrak{sl}_2$ and the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, or the existence of a central extension of our deformed Witt algebra, which would give a deformed Virasoro algebra.

Deformations of rational numbers were introduced in [12], extended to real numbers in [13] and to Gaussian integers in [17]. Many different deformations of the Witt algebra or its central extension, the Virasoro algebra, have been introduced in the past: first in [6] and then in [5] deforming the matrix Lie bracket to $[A, B]_q = qAB - q^{-1}BA$. This also deforms the Jacobi identity. A similar construction was done in [9] viewing the Witt algebra as space of derivations of $\mathbb{C}[x^{\pm 1}]$ and using the q-differential $\partial_q(f) = \frac{f(qx)-f(x)}{qx-x}$. This was generalized in [8] to more general σ-derivatives. Deforming the cocycle gives a q-Virasoro algebra in [10], developed into a theory of q-deformed pseudo-differential operators in [11]. A deformation as Lie algebra in terms of an operator product expansion is given in [19]. A similar proposal can be found in [7, equation (1.3)], using a q-deformed Miura transformation. In [15, equation (38)], the deformation

$$[T_m(q), T_n(q)] = ([-n]_q - [-m]_q)(T_{n+m}(q^2) - T_{n+m}(q))$$

is studied. Yet another proposal from [16, formula (3.18)] gives operators $D_n(q)$ for $n \in \mathbb{Z}$ with commutator $[D_n(q), D_m(q)] = \big(q - q^{-1}\big)[n-m]_q D_{n+m}\big(q^2\big)$ (removing the central extension). Finally, in [1], a two-dimensional deformation using elliptic algebras is studied. All these approaches are different from ours.

Structure of the paper. In Section 2, we introduce and study the deformation of $\mathfrak{sl}_2$, the Heisenberg algebra and the q-rational transition map. This is broadened in Section 3 to a deformed Witt algebra. In the final Section 4, we study the Möbius transformations associated to these deformations.

2 Deformed $\mathfrak{sl}_2$ and Heisenberg algebra

The group SL_2 acts naturally on the projective line $\mathbb{P}^1$. We will work over $\mathbb{R}$ or $\mathbb{C}$. Differentiating this action at the identity gives a realization of the Lie algebra $\mathfrak{sl}_2$ as vector fields on $\mathbb{P}^1$. Using the two standard charts of $\mathbb{P}^1$ with transition function $x \mapsto 1/x$, the image of $\mathfrak{sl}_2 \to \operatorname{Vect}\big(\mathbb{P}^1\big)$ is generated by ∂, $x\partial$ and $x^2\partial$ written in the first chart, where we use the notation $\partial = \mathrm{d}/\mathrm{d}x$. One readily checks that these expressions are well-defined over the second chart.

We construct a deformation of these three differential operators. They come as a realization of a Lie algebra which itself deforms $\mathfrak{sl}_2$. Together with a q-deformed identity map, we deform the 3-dimensional Heisenberg algebra.

2.1 Deformed $\mathfrak{sl}_2$

On $\mathbb{P}^1$, consider the Möbius transformations

$$T_q(x) = qx + 1 \qquad \text{and} \qquad S_q(x) = -\frac{1}{qx},$$

where $q \in \mathbb{C}^*$ is fixed or seen as a formal parameter. They deform the translation $x \mapsto x+1$ and the inversion $x \mapsto -1/x$. These transformations act on the space of functions on $\mathbb{P}^1$ by precomposition.

Consider the differential operator D_{-1} on $\mathbb{P}^1$ which is defined in the first chart by

$$D_{-1} := (1 + (q-1)x)\partial.$$

Proposition 2.1. *The operators D_{-1} and T_q commute, where T_q acts on the space of functions by precomposition.*

Proof. For a function $f(x)$, we have on the one side

$$D_{-1} \circ T_q(f(x)) = D_{-1}(f(qx+1)) = (1 + (q-1)x)qf'(qx+1).$$

On the other side,

$$T_q \circ D_{-1}(f(x)) = T_q((1 + (q-1)x)f'(x)) = (1 + (q-1)(qx+1))f'(qx+1).$$

Both expressions coincide. ■

The unique eigenfunction E_q of D_{-1} with eigenvalue 1 and normalization $E_q(0) = 1$ is a q-deformation of the exponential function, called the *Tsallis exponential* [20]. This was first observed by Valentin Ovsienko and Emmanuel Pedon.[1] To find E_q, one has to solve $f = D_{-1}f = (1 + (q-1)x)f'$, i.e., $(\ln f)' = \frac{1}{1+(q-1)x}$. The solution is given by

$$E_q(x) = (1 + (q-1)x)^{\frac{1}{q-1}}.$$

[1] Unpublished, private communication.

It satisfies $E_q(qx+1) = E_q(1)E_q(x)$ since $E_q(qx+1) = T_qE_q$ is also an eigenfunction of D_{-1} with eigenvalue 1.

The main new operator we introduce is the following:

$$D_0 := (1+(x-1)q)D_{-1} = (1+(x-1)q)(1+(q-1)x)\partial.$$

Proposition 2.2. *The operators D_0 and S_q anti-commute, where S_q acts on the space of functions by precomposition.*

The proof is a direct verification, similar to the proof of Proposition 2.1. An equivalent statement is $S_q \circ D_0 \circ S_q = -D_0$.

Proof. For a function $f(x)$, we have on the one side

$$D_0 \circ S_q(f(x)) = D_0 f\left(-\frac{1}{qx}\right) = (1+(x-1)q)(1+(q-1)x)f'\left(-\frac{1}{qx}\right)\frac{1}{qx^2}.$$

On the other hand,

$$\begin{aligned} S_q \circ D_0(f(x)) &= S_q((1+(x-1)q)(1+(q-1)x)f'(x)) \\ &= \left(1+q\left(-\frac{1}{qx}-1\right)\right)\left(1-\frac{1}{qx}(q-1)\right)f'\left(-\frac{1}{qx}\right) \\ &= -\frac{1}{qx^2}(1+(x-1)q)(1+(q-1)x)f'\left(-\frac{1}{qx}\right). \end{aligned}$$ ■

More generally, we can find all operators D of the form $p(x)\partial$ which anti-commute with S_q. The relation $\{D, S_q\} = 0$ gives

$$p(x) = -qx^2 p\left(-\frac{1}{qx}\right).$$

Adding as constraint that p has to be polynomial, it is clear that it is of degree at most 2. Plugging in $p(x) = p_0 + p_1x + p_2x^2$ gives a solution for any p_1 and $p_2 = -qp_0$. In other words, the two fundamental solutions are $p(x) = x$ and $p(x) = 1 - qx^2$. Note in particular that the undeformed operator $x\partial$ still anticommutes with S_q. The particular choice above for D_0 is $p_1 = -1 + 3q - q^2$ and $p_0 = 1 - q$. We will see below why this is the simplest choice.

Let us determine the eigenfunctions of D_0 with eigenvalue α. One has to solve $\alpha f = D_0 f$, i.e., $(\ln f)' = \frac{\alpha}{(1+(q-1)x)(1+(x-1)q)}$. The solutions are

$$\left(\frac{1+(x-1)q}{1+(q-1)x}\right)^{\frac{\alpha}{q^2-q+1}}.$$

We define the *q-rational transition map*

$$g_q(x) = \frac{1+(x-1)q}{1+(q-1)x}, \tag{2.1}$$

which is the unique eigenfunction of D_0 with eigenvalue q^2-q+1 and normalization $g_q(0) = 1-q$. We can think of g_q as a deformation of the identity map. We study this function more in detail below in Section 2.2.

Now we come back to the discussion why our D_0 is the simplest choice. Consider an operator $D = p(x)\partial$ anti-commuting with S_q, i.e., of the form $p(x) = p_0 + p_1x - qp_0x^2$ with arbitrary $p_0, p_1 \in \mathbb{Z}[q]$. We impose that D deforms $x\partial$, that is $p_0(1) = 0$ and $p_1(1) = 1$. We also impose the leading terms of p_0, p_1 to be ± 1. We wish that the eigenfunctions of D are

Möbius transformations in $\mathbb{Z}[q]$. This is only the case if the discriminant of $p_0 + p_1 x - q p_0 x^2$ is a square in $\mathbb{Z}[q]$. This leads to the equation $p_1(q)^2 + 4qp_0(q)^2 = R(q)^2$ for some $R \in \mathbb{Z}[q]$. This is equivalent to $4qp_0^2 = (R - p_1)(R + p_1)$. Excluding the case where $p_0 = 0$ which leads to the undeformed operator $x\partial$, the next simplest case is $p_0(q) = 1 - q$. By treating all possible factorizations of $4q(1-q)^2$, we see that the p_1 with lowest degree has to be $p_1(q) = -1 + 3q - q^2$ which is the case for our choice D_0.

We complete the operators D_{-1} and D_0 to a deformed $\mathfrak{sl}_2$. For that, we wish to deform $x^2\partial$. Note that $x^2\partial = S \circ \partial \circ S$. This motivates the following definition:

$$D_1 := S_q \circ D_{-1} \circ S_q = (1 + (x-1)q)x\partial.$$

By definition, D_1 commutes with $S_q T_q S_q$.

Our first result is that these three operators give a Lie algebra deforming $\mathfrak{sl}_2$:

Theorem 2.3. *The operators D_{-1}, D_0 and D_1 form a Lie algebra with brackets*

$$[D_0, D_1] = \big(q^2 - q + 1\big)D_1 + (1-q)D_0, \qquad [D_0, D_{-1}] = -\big(q^2 - q + 1\big)D_{-1} + (1-q)D_0,$$
$$[D_{-1}, D_1] = 2D_0 + (1-q)(D_1 - D_{-1}).$$

For $q = 1$, we get the Lie algebra $\mathfrak{sl}_2$.

Proof. The proof is a straightforward computation. All D_i are of the form $g(x)\partial$ with g a polynomial of degree at most 2. This explains why we can express any bracket as linear combination of D_{-1}, D_0 and D_1. The non-trivial part is that the coefficients are in $\mathbb{Z}[q]$. Since $D_0 = (1 + (x-1)q)D_{-1}$, we get

$$[D_0, D_{-1}] = -D_{-1}(1 + (x-1)q)D_{-1} = -q(1 + (q-1)x)^2\partial.$$

Similarly, we have $D_0 = (1 + (q-1)x)x^{-1}D_1$, hence

$$[D_0, D_1] = -D_1\big(x^{-1} + q - 1\big)D_1 = (1 + (x-1)q)^2\partial.$$

The last bracket can be computed to be $[D_{-1}, D_1] = \big(1 - q + 2qx + q(q-1)x^2\big)\partial$. One explicitely checks that these three brackets coincide with results claimed in the theorem.

Finally, it is clear that these brackets satisfy the Jacobi identity since we know a representation of the operators D_i as differential operators. ∎

The Lie algebra $\mathfrak{sl}_2$ being simple, it does not allow any non-trivial deformations. Our q-deformation is indeed abstractly isomorphic to $\mathfrak{sl}_2$ when q and $q^2 - q + 1$ are invertible. To give an explicit isomorphism, denote by (f, h, e) the generators of $\mathfrak{sl}_2$ given by the differential operators $\big(\partial, x\partial, x^2\partial\big)$. They satisfy $[h, e] = e$, $[h, f] = -f$ and $[e, f] = -2h$. The following is an isomorphism of Lie algebras between (D_{-1}, D_0, D_1) and (f, h, e):

$$f = q^{-1/2}\left(D_{-1} + \frac{q-1}{q^2 - q + 1}D_0\right), \qquad h = \frac{D_0}{q^2 - q + 1},$$
$$e = q^{-1/2}\left(D_1 + \frac{1-q}{q^2 - q + 1}D_0\right).$$

Using this isomorphism to $\mathfrak{sl}_2$, we can describe a 2-dimensional representation of the deformed Lie algebra defined by (D_{-1}, D_0, D_1). Using the standard realization $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ and $e = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$, we get

$$D_{-1} = \begin{pmatrix} \frac{1-q}{2} & 0 \\ q^{1/2} & \frac{q-1}{2} \end{pmatrix}, \qquad D_0 = \begin{pmatrix} \frac{q^2-q+1}{2} & 0 \\ 0 & \frac{-q^2+q-1}{2} \end{pmatrix}, \qquad D_1 = \begin{pmatrix} \frac{q-1}{2} & -q^{1/2} \\ 0 & \frac{1-q}{2} \end{pmatrix}.$$

Note that this representation is not in $\mathfrak{sl}_2(\mathbb{Q}[q])$. A direct computation shows that there is no 2-dimensional representation of our q-deformed $\mathfrak{sl}_2$ into $\mathfrak{sl}_2(\mathbb{Q}[q])$. In dimension 3, there is of course the adjoint representation into $\mathfrak{sl}_3(\mathbb{Z}[q])$.

Remark 2.4. It is tempting to consider D_{-1}, D_1 and $\widehat{D}_0 := [D_{-1}, D_1]$. The operator $\widehat{D}_0$ still anti-commutes with S_q and the bracket relations are

$$[\widehat{D}_0, D_{\pm 1}] = \pm\big(q^2+1\big)D_{\pm 1} \pm (q-1)^2 D_{\mp 1}.$$

The main drawback of this choice is that the eigenfunctions of $\widehat{D}_0$ are Möbius transformations with coefficients not in $\mathbb{Z}[q]$.

Remark 2.5. A simpler and very similar Lie algebra deforming $\mathfrak{sl}_2$ is given by generators (d_{-1}, d_0, d_1) with brackets

$$[d_0, d_{-1}] = -qd_{-1} + (1-q)d_0, \qquad [d_0, d_1] = qd_1 + (1-q)d_0,$$
$$[d_{-1}, d_1] = 2d_0 + (1-q)(d_1 - d_{-1}).$$

It can be obtained as our deformation for a formal parameter q with relation $(q-1)^2 = 0$. Then $q^2 - q + 1 = q$. One checks that the Jacobi identity still holds.

2.2 q-rational transition map

The map g_q defined in (2.1) plays a fundamental role, both for generalizing the q-deformation from $\mathfrak{sl}_2$ to the Witt algebra in Section 3, and in the theory of q-deformed rationals as we shall see now. It allows to pass between two different q-deformations of the rational numbers.

Recall that the q-rational transition map is defined by

$$g_q(x) = \frac{1 + (x-1)q}{1 + (q-1)x},$$

which is a deformation of the identity. It is the eigenfunction of D_0 with eigenvalue $q^2 - q + 1$ and normalization $g_q(0) = 1 - q$. Note that g_q is a Möbius transformation associated to the matrix

$$\begin{pmatrix} q & 1-q \\ q-1 & 1 \end{pmatrix},$$

which is of determinant $q^2 - q + 1$. For $q \neq 1$, g_q is an elliptic transformation since its normalized trace is given by

$$\frac{q+1}{\sqrt{q^2-q+1}} < 2.$$

The unique fixed point on $\mathbb{H}^2$ is $\frac{1+i\sqrt{3}}{2}$ which is independent of q.

From the definition of g_q, we see the following duality between q and x:

$$g_q(x)g_x(q) = 1.$$

Proposition 2.6. *The functions g_q, T_q and S_q, seen as 2×2 matrices satisfy:*

$$g_q T_q = qT_{q^{-1}} g_q \qquad \text{and} \qquad g_q S_q = qS_{q^{-1}} g_q.$$

Therefore, seen as Möbius transformations, we have $g_q \circ T_q = T_{q^{-1}} \circ g_q$ and $g_q \circ S_q = S_{q^{-1}} \circ g_q$.

Proof. Both assertions can be checked by a direct computation:

$$g_qT_q = \begin{pmatrix} q & 1-q \\ q-1 & 1 \end{pmatrix} \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q^2 & 1 \\ q^2-q & q \end{pmatrix} = qT_{q^{-1}}g_q,$$

and similarly

$$g_qS_q = \begin{pmatrix} q & 1-q \\ q-1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} q-q^2 & -q \\ q & 1-q \end{pmatrix} = qS_{q^{-1}}g_q.$$

The second identity can be derived also as follows: since $S_qD_0S_q = -D_0$, we see that both $g_q^{-1}(x)$ and $g_q(S_q(x))$ are eigenfunctions of D_0 with eigenvalue $-q^2+q-1$. Hence they have to be multiple of each other. The precise relation is given by $g_q(S_q(x)) = \frac{-q}{g_q(x)} = S_{q^{-1}}(g_q(x))$. ■

We describe now the main link to q-deformed rational numbers. In [13, Remark 3.2], the authors notice that the procedure for q-deformed irrational numbers gives two different answers when applied to rationals. This was further developed in [2], from which we borrow the notations. When one approaches a rational r/s from the right by a sequence of rationals strictly bigger than r/s, the procedure gives the so-called *right q-rational* $[r/s]_q^\sharp$. This is the deformation obtained from applying T_q and S_q to zero described at the beginning of the Introduction. When approaching r/s from the left, the limit gives another q-deformation of r/s, called *left q-rational* and denoted by $[r/s]_q^\flat$ [2, Theorem 2.11].

The precise formulas given in [2, Definition 2.6] can be written in our context as follows: consider $U = TST$, which is the function $U(x) = \frac{1}{1+1/x}$, and its q-analog $U_q = T_qS_qT_q$. For a rational $r/s \in \mathbb{Q}$, take the unique even continued fraction expression $r/s = [a_1, a_2, \dots, a_{2n}]$. This means that $r/s = T^{a_1}U^{a_2}T^{a_3}\cdots U^{a_{2n}}(\infty)$. By convention, we put $\infty = []$, the empty expression. Then

$$\left[\frac{r}{s}\right]_q^\sharp = T_q^{a_1}U_q^{a_2}T_q^{a_3}\cdots U_q^{a_{2n}}(\infty), \tag{2.2}$$

and

$$\left[\frac{r}{s}\right]_q^\flat = T_q^{a_1}U_q^{a_2}T_q^{a_3}\cdots U_q^{a_{2n}}\left(\frac{1}{1-q}\right). \tag{2.3}$$

To give some examples, we have $[0]_q^\sharp = 0$ and $[0]_q^\flat = \frac{q-1}{q}$, $[1]_q^\sharp = 1$ and $[1]_q^\flat = q$, $[2]_q^\sharp = 1+q$ and $[2]_q^\flat = 1+q^2$, $[\infty]_q^\sharp = \infty$ and $[\infty]_q^\flat = \frac{1}{1-q}$.

It was noticed numerically by Valentin Ovsienko that g_q is a transition between these two q-deformations of rational numbers. This is made precise in the following:

Theorem 2.7. *The passage between the two q-deformations of rationals is given by*

$$g_q\left(\left[\frac{r}{s}\right]_q^\sharp\right) = \left[\frac{r}{s}\right]_{q^{-1}}^\flat.$$

Note that q gets inversed to q^{-1}. The proof is an application of Proposition 2.6.

Proof. Proposition 2.6 gives $g_qU_q = U_{q^{-1}}g_q$. Using equation (2.2) and again Proposition 2.6, we get

$$g_q\left(\left[\frac{r}{s}\right]_q^\sharp\right) = g_qT_q^{a_1}U_q^{a_2}T_q^{a_3}\cdots U_q^{a_{2n}}(\infty) = T_{q^{-1}}^{a_1}U_{q^{-1}}^{a_2}T_{q^{-1}}^{a_3}\cdots U_{q^{-1}}^{a_{2n}}g_q(\infty).$$

Now $g_q(\infty) = \frac{q}{q-1} = \frac{1}{1-q^{-1}}$. Hence we conclude by equation (2.3). ■

As an application, we can reprove the positivity property of left q-rationals, proven in [2, Appendix A.1] via an explicit combinatorial interpretation.

Corollary 2.8. *For $r/s > 1$, we have $R^\flat, S^\flat \in \mathbb{N}[q]$, where $[r/s]^\flat_q = R^\flat(q)/S^\flat(q)$.*

Proof. From [12, Proposition 1.3], we know that for $r/s \in \mathbb{Q}_{>1}$ the right q-rational $[r/s]^\sharp_q = R^\sharp(q)/S^\sharp(q)$ is a rational function in q with positive coefficients, i.e., $R^\sharp, S^\sharp \in \mathbb{N}[q]$. We also know from [12, Theorem 2] that if $r/s > r'/s'$, then $R^\sharp S'^\sharp - R'^\sharp S^\sharp \in \mathbb{N}[q]$. Since $r/s > 1$, we get $R^\sharp - S^\sharp \in \mathbb{N}[q]$. Since $r/s+1 > r/s$ and $[r/s+1]^\sharp_q = q[r/s]^\sharp_q+1$, we also get $(q-1)R^\sharp + S^\sharp \in \mathbb{N}[q]$. Finally, we deduce that

$$\left[\frac{r}{s}\right]^\flat_{q^{-1}} = g_q\left(\left[\frac{r}{s}\right]^\sharp_q\right) = \frac{q\big(R^\sharp - S^\sharp\big) + S^\sharp}{(q-1)R^\sharp + S^\sharp}$$

has positive coefficients. Multiplying both numerator and denominator with an appropriate power of q, we get the same for $[r/s]^\flat_q$. ∎

Finally, we can use the transition function $g_q(x)$ as reparametrization of $\mathbb{P}^1$. To emphasize the dependence of our differential operators, we will write here $D_{-1}(q,x) = (1+(q-1)x)\partial_x$ and similar for D_0 and D_1.

Proposition 2.9. *Reparametrizing $\mathbb{P}^1$ by the transition map $\xi = g_q(x)$ gives*

$$D_{\pm1}(q,x) = qD_{\pm1}\big(q^{-1},\xi\big), \qquad D_0(q,x) = \big(q^2-q+1\big)\xi\partial_\xi.$$

The behavior of D_{-1} and D_1 is reminiscent of Proposition 2.6.

Proof. Using $\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{q^2-q+1}{(1+(q-1)x)^2}$ and $x = \frac{\xi+q-1}{q+(1-q)\xi}$, we get $1+(q-1)x = \frac{q^2-q+1}{q+(1-q)\xi}$. Hence

$$D_{-1}(x,q) = (1+(q-1)x)\frac{\mathrm{d}\xi}{\mathrm{d}x}\partial_\xi = \frac{q^2-q+1}{1+(q-1)x}\partial_\xi = (q+(1-q)\xi)\partial_\xi = qD_{-1}\big(q^{-1},\xi\big).$$

The computation for D_1 is similar. Finally,

$$D_0(q,x) = (1+(q-1)x)(1+(x-1)q)\partial_x = \big(q^2-q+1\big)g_q(x)\partial_\xi = \big(q^2-q+1\big)\xi\partial_\xi.$$ ∎

This proposition indicates that we can use the undeformed operator $x\partial$ together with $D_{\pm1}$ to get a deformation of $\mathfrak{sl}_2$ which is equivalent to our proposal. The importance of the q-rational transition map g_q, especially in the light of Proposition 2.6, justifies to use D_0 instead of $x\partial$.

2.3 Heisenberg algebra

The operators D_{-1} and g_q, seen as multiplication operator, give a deformation of the Heisenberg algebra. This strengthens the idea of considering g_q as a deformation of the identity.

Theorem 2.10. *The two operators D_{-1} and g_q satisfy*

$$[D_{-1}, g_q] = q + (1-q)g_q.$$

Hence together with the central element 1, they define a solvable 3-dimensional Lie algebra deforming the 3-dimensional Heisenberg algebra which we recover for $q = 1$.

The proof is a simple computation:

$$[D_{-1}, g_q] = D_{-1}(g_q) = \frac{q^2 - q + 1}{1 + (q-1)x} = q + (1-q)g_q.$$

The Lie algebra generated by $(1, g_q, D_{-1})$ is solvable since D_{-1} is not in the image of the Lie bracket. Hence the derived series becomes zero at the second step.

The previous theorem works since there is a nice expression for $D_{-1}(g_q)$. This holds true more generally:

Proposition 2.11. *The function g_q behaves well under the operators D_{-1}, D_0 and D_1:*

$$D_0(g_q) = \big(q^2 - q + 1\big) g_q, \qquad D_{-1}(g_q) = q + (1-q)g_q, \qquad D_1(g_q) = (q-1)g_q + g_q^2.$$

Proof. We only have to prove the last statement since we have already seen the first two. For that, we use the relation $g_q(S_q(x)) = -q/g_q(x)$, see Proposition 2.6. We get

$$D_1(g_q) = S_q D_{-1} S_q(g_q) = S_q D_{-1}(-q/g_q) = S_q\big(-q g_q^{-2}(q + (1-q)g_q)\big),$$

where we first used that S_q acts by precomposition, then Proposition 2.6 and finally the expression for $D_{-1}(g_q)$. Since S_q acts by precomposition applying 2.6 again concludes:

$$D_1(g_q) S_q\big(-q g_q^{-2}(q + (1-q)g_q)\big) = g_q^2 + (q-1)g_q. \qquad \blacksquare$$

We can use this proposition to express one operator in terms of another via the relation $D_i = \frac{D_i(g_q)}{D_j(g_q)} D_j$ for all $i, j \in \{-1, 0, 1\}$. This holds true since these differential operators are of order 1.

3 Deformed Witt algebra

Now that we have deformed the differential operators ∂, $x\partial$ and $x^2\partial$, we can do the same for all $x^n\partial$ for $n \in \mathbb{Z}$. These are a realization of the *Witt algebra*, the Lie algebra of complex polynomial vector fields on the circle (the centerless Virasoro algebra). Putting $\ell_n = x^{n+1}\partial$, the Lie algebra structure is given by

$$[\ell_n, \ell_m] = (m-n)\ell_{n+m}.$$

To get a deformation of the Witt algebra, we define for $n > 1$:

$$D_n = g_q^{n-1} D_1, \qquad D_{-n} = \big(q g_q^{-1}\big)^{n-1} D_{-1},$$

where $g_q^{-1} = 1/g_q$ denotes the inverse for multiplication (not composition).

Proposition 3.1. *The operators D_n behave nicely when multiplied by g_q. By definition we have $g_q D_n = D_{n+1}$ for $n \geq 1$ and $g_q D_{-n} = q D_{-n+1}$ for $n \geq 2$. In addition,*

$$g_q D_0 = (1-q)D_0 + \big(q^2 - q + 1\big) D_1, \qquad g_q D_{-1} = D_0 + (1-q)D_1.$$

Similarly, there is a nice behavior when multiplied by $q g_q^{-1}$. By definition $q g_q^{-1} D_{-n} = D_{-n-1}$ for $n \geq 1$ and $q g_q^{-1} D_n = q D_{n-1}$ for $n \geq 2$. In addition,

$$q g_q^{-1} D_0 = (q-1)D_0 + \big(q^2 - q + 1\big) D_{-1}, \qquad q g_q^{-1} D_1 = D_0 + (q-1)D_{-1}.$$

Proof. From the definitions, we get $g_qD_0 = (1 + (x-1)q)^2\partial$. From Proposition 2.3 and its proof, we see that this is $[D_0, D_1]$. Therefore, $g_qD_0 = \big(q^2 - q + 1\big)D_1 + (1-q)D_0$. A direct computation also gives $g_qD_{-1} = (1 + (x-1)q)\partial = D_0 + (1-q)D_1$.

For the second half, note that

$$g_qD_0 + (q-1)g_qD_{-1} = \big(q^2 - q + 1 - (q-1)^2\big)D_1 = qD_1.$$

Dividing by g_q gives $qg_q^{-1}D_1 = D_0 + (q-1)D_{-1}$. Similarly,

$$(q-1)g_qD_0 + \big(q^2 - q + 1\big)D_{-1} = qD_0,$$

so dividing by g_q gives $qg_q^{-1}D_0 = (q-1)D_0 + \big(q^2 - q + 1\big)D_{-1}$. ■

Using Propositions 2.11 and 3.1, we get the bracket relations of all D_n:

Theorem 3.2. *The $(D_n)_{n\in\mathbb{Z}}$ form a Lie algebra with bracket given by (with $n, r > 0$):*

$$[D_0, D_n] = n\big(q^2 - q + 1\big)D_n + \big(q^2 - q + 1\big)\sum_{k=1}^{n-1}(1-q)^kD_{n-k} + (1-q)^nD_0,$$
$$[D_0, D_{-n}] = -n\big(q^2 - q + 1\big)D_{-n} - \big(q^2 - q + 1\big)\sum_{k=1}^{n-1}(q-1)^kD_{-n+k} - (q-1)^nD_0,$$
$$[D_n, D_{n+r}] = rD_{2n+r} + (q-1)rD_{2n+r-1},$$
$$[D_{-n}, D_{-n-r}] = -rD_{-2n-r} + (q-1)rD_{-2n-r+1},$$
$$[D_{-n}, D_n] = 2nq^{n-1}D_0 + (2n-1)q^{n-1}(q-1)(D_{-1} - D_1),$$
$$\begin{aligned}[D_{n+r}, D_{-n}] = {}& (q-1)q^{n-1}(2n+r-1)D_{r+1} - \big(q^2 + (2n+r-2)q + 1\big)q^{n-1}D_r,\\ &- q^{n-1}\big(q^2 - q + 1\big)\sum_{k=1}^{r-1}(1-q)^kD_{r-k} - (1-q)^rq^{n-1}D_0,\end{aligned}$$
$$\begin{aligned}[D_n, D_{-n-r}] = {}& -(q-1)q^{n-1}(2n+r-1)D_{-r-1} - \big(q^2 + (2n+r-2)q + 1\big)q^{n-1}D_{-r},\\ &- q^{n-1}\big(q^2 - q + 1\big)\sum_{k=1}^{r-1}(q-1)^kD_{-r+k} - (q-1)^rq^{n-1}D_0.\end{aligned}$$

For $q = 1$, one recovers the Witt algebra.

It is clear that the bracket of the operators D_n satisfies the Jacobi identity since these operators come from a realization as differential operators. We only have to check the bracket relations, which uses induction and all properties between g_q and D_{-1}, D_0, D_1.

Proof. We prove the first relation by induction on n. The case $n = 1$ is true by Proposition 2.3. Then for $n > 1$,

$$\begin{aligned}[D_0, D_n] &= [D_0, g_qD_{n-1}] = D_0(g_q)D_{n-1} + g_q[D_0, D_{n-1}]\\ &= \big(q^2 - q + 1\big)g_qD_{n-1} + g_q(n-1)\big(q^2 - q + 1\big)D_{n-1}\\ &\quad + g_q\left(\big(q^2 - q + 1\big)\sum_{k=1}^{n-2}(1-q)^kD_{n-1-k} + (1-q)^{n-1}D_0\right)\\ &= n\big(q^2 - q + 1\big)D_n + \big(q^2 - q + 1\big)\sum_{k=1}^{n-1}(1-q)^kD_{n-k} + (1-q)^nD_0,\end{aligned}$$

where we used that g_q is an eigenfunction of D_0, and Proposition 3.1. The second statement is a similar computation. The third relation comes as follows:

$$\begin{aligned}[D_n, D_{n+r}] &= [D_n, g_q^r D_n] = D_n(g_q^r)D_n = rg_q^{r-1}g_q^{n-1}D_1(g_q)D_n \\ &= rg_q^{r+n-2}\big(g_q^2 + (q-1)g_q\big)D_n = rD_{2n+r} + r(q-1)D_{2n+r-1},\end{aligned}$$

where we used Proposition 2.11 for $D_1(g_q)$. The fourth bracket is a similar computation. To prove the fifth relation, we use induction on n again. The initial $n = 1$ is done by Proposition 2.3. Then for $n \geq 1$,

$$\begin{aligned}[D_{-n-1}, D_{n+1}] &= \big[qg_q^{-1}D_{-n}, g_qD_n\big] = qg_q^{-1}D_{-n}(g_q)D_n + q[D_{-n}, D_n] - qg_qD_n\big(g_q^{-1}\big)D_{-n} \\ &= q^n g_q^{-n}(q + (1-q)g_q)D_n + q\big(2nq^{n-1}D_0 \\ &\quad + (2n-1)q^{n-1}(q-1)(D_{-1} - D_1)\big) + qg_q^{n-2}\big(g_q^2 + (q-1)g_q\big)D_{-n} \\ &= 2nq^nD_0 + q^n(2n(q-1) + q_q)D_{-1} - q^n\big(2n(q-1) - qg_q^{-1}\big)D_1 \\ &= (2n+2)q^nD_0 + (2n+1)q^n(q-1)(D_{-1} - D_1),\end{aligned}$$

where we used several times Propositions 2.11 and 3.1. Finally, for the last two brackets, we start from (where $a, b > 0$)

$$\begin{aligned}[D_a, D_{-b}] &= \big[g_q^{a-1}D_1, \big(qg_q^{-1}\big)^{b-1}D_{-1}\big] \\ &= q^{b-1}\big(g_q^{a-1}D_1\big(g_q^{1-b}\big)D_{-1} - g_q^{1-b}D_{-1}\big(g_q^{a-1}\big)D_1 + g_q^{a-b}[D_1, D_{-1}]\big) \\ &= q^{b-1}g_q^{a-b}(-(a+b)D_0 + (a+b-1)(q-1)(D_1 - D_{-1})).\end{aligned} \tag{3.1}$$

An easy induction gives for $r > 0$,

$$g_q^rD_0 = \big(q^2 - q + 1\big)\sum_{k=0}^{r-1}(1-q)^kD_{r-k} + (1-q)^rD_0. \tag{3.2}$$

Also we get $g_q^rD_{-1} = g_q^{r-1}D_0 + (1-q)D_r$, where we can use equation (3.2) to express the first term. Similar results hold for $\big(qg_q^{-1}\big)^rD_0$ and $\big(qg_q^{-1}\big)^rD_1$. Putting $a = n + r$ and $b = n$ in equation (3.1) and using (3.2) gives the bracket $[D_{n+r}, D_{-n}]$. Putting $a = n$ and $b = n + r$ gives in a similar way the last bracket $[D_n, D_{-n-r}]$. ■

Remark 3.3. Regarding Remark 2.5, we could try to simplify the defining relations of the q-deformed Witt algebra by considering a formal parameter q satisfying $(q-1)^2 = 0$ (and then forget about this relation again). In contrast to the q-deformed $\mathfrak{sl}_2$, the result here is not a Lie algebra anymore. The Jacobi identity does not hold exactly, but only modulo $(q-1)^2 = 0$.

4 Möbius transformations

The differential operators ∂, $x\partial$, $x^2\partial$ can be interpreted in at least three different ways: first as differential operators on $\mathbb{P}^1$ written in one chart (this was our approach). Second they can be seen as complex vector fields on the circle $\mathbb{S}^1 \subset \mathbb{C}$ (this approach was used for the Witt algebra). Third, a Lie algebra can be realised as Killing vector fields on the associated symmetric space of non-compact type. For $\mathfrak{sl}_2$ this symmetric space is the hyperbolic plane $\mathbb{H}^2$.

In this section, we integrate the operators D_{-1}, D_0 and D_1 seen as vector fields of $\mathbb{H}^2$. The result gives interesting Möbius transformations with q-parameter. In the Taylor expansion around $q \to 1$ (the "semi-classical limit"), we recover the deformed translation T_q. Conjecturally there should be a q-deformation of $\mathbb{H}^2$ on which these transformations act, such that the boundary can be identified with the q-deformed real numbers of [13].

4.1 Classical setting

Consider first the classical setup with the operators ∂, $x\partial$ and $x^2\partial$. These are Killing vector fields on the hyperbolic plane $\mathbb{H}^2$, whose integration determines isometries of $\mathbb{H}^2$. Here, we consider $\mathbb{H}^2 \subset \mathbb{C}$ in the upper half-plane model and use the coordinate $x \in \mathbb{C}$.

To start, consider the case of the vector field $V = \partial = \frac{\partial}{\partial x}$. A curve γ integrates this vector field iff $\gamma'(t) = V(\gamma(t)) = 1$ for all $t \in \mathbb{R}$ (where we identified 1 with the constant vector field ∂). With initial condition $\gamma(0) = x$ we get $\gamma(t) = x + t$. We should think of this as a function $\gamma_x(t)$ of the initial condition. For time 1, we get the translation $\gamma_x(1) = x + 1 = T(x)$.

Another important case is $x\partial$, for which we have to solve $\gamma'(t) = \gamma(t)$. With initial condition $\gamma(0) = x$, we obviously get $\gamma(t) = \mathrm{e}^t x$. The function $x \mapsto \mathrm{e}^t x$ is the hyperbolic isometry of $\mathbb{H}^2$ associated to the geodesic joining 0 to ∞. Its matrix is given by

$$\begin{pmatrix} \mathrm{e}^{t/2} & 0 \\ 0 & \mathrm{e}^{-t/2} \end{pmatrix}. \tag{4.1}$$

We can immediately generalize to the generators of the Witt algebra. Consider the operator $x^n\partial$ with $n \in \mathbb{Z}$, $n \neq 1$. A curve γ integrates the associated vector field if $\gamma'(t) = \gamma(t)^n$. The solution with initial condition $\gamma(0) = x$ is given by

$$\gamma_x(t) = \frac{x}{\left(1 - (n-1)tx^{n-1}\right)^{1/(n-1)}}.$$

Apart from $n = 0$ and $n = 2$, the associated transformations in x are not Möbius transformations. We get Möbius transformations though when passing to a ramified covering. Putting $y = x^{n-1}$, we get

$$\gamma_x(t)^{n-1} = \frac{y}{1 - (n-1)ty}.$$

4.2 Deformed transformations

We repeat the method of the previous subsection to deduce the transformations associated to D_{-1}, D_0 and D_1. Since these operators are still of the form $p(x)\partial$ with p a polynomial in x of degree at most 2, the vector fields D_i are still Killing vector fields, so their integration gives Möbius transformations.

Start with $D_{-1} = (1 + (q-1)x)\partial$. The associated differential equation is $\gamma'(t) = 1 + (q-1)\gamma(t)$ with initial condition $\gamma(0) = x$. Solving this equation is standard: first one solves the homogeneous equation, then one uses the variation of the constant to finally get

$$\gamma(t) = -\frac{1}{q-1} + \left(x + \frac{1}{q-1}\right)\mathrm{e}^{(q-1)t}.$$

For $t = 1$, we get the associated map $x \mapsto -\frac{1}{q-1} + \left(x + \frac{1}{q-1}\right)\mathrm{e}^{q-1}$. The Taylor expansion around $q - 1$ at order 1 gives

$$x \mapsto -\frac{1}{q-1} + \left(x + \frac{1}{q-1}\right)q = qx + 1,$$

which is nothing but $T_q(x)$. For a general time t, the same procedure gives $x \mapsto (1 - t + qt)x + t$. To sum up:

Proposition 4.1. *The time* 1 *flow of the operator* D_{-1} *seen as vector field on* $\mathbb{H}^2$ *is the affine map* $x \mapsto -\frac{1}{q-1} + \left(x + \frac{1}{q-1}\right)\mathrm{e}^{q-1}$ *whose Taylor expansion at order* 1 *in* $q - 1$ *is* T_q.

For the operator D_1, it is not necessary to do any computation since $D_1 = S_q D_{-1} S_q$. We can simply conjugate by S_q the previous computations. In particular, the associated transformation in the Taylor expansion is $S_q T_q S_q$.

Consider now the operator $D_0 = (1+(q-1)x)(1+(x-1)q)\partial$. The associated differential equation reads

$$\gamma'(t) = 1-q+\left(-1+3q-q^2\right)\gamma + q(q-1)\gamma^2,$$

which is a Ricatti equation.

To solve a Ricatti equation, put $a = 1-q$, $b = -1+3q-q^2$, $c = q(q-1)$ and introduce the new function u such that $c\gamma(t) = -u'(t)/u(t)$. Then u satisfies $u''(t) - bu'(t) + acu(t) = 0$. The discriminant has the nice expression $b^2 - 4ac = \left(q^2-q+1\right)^2$. The two roots of the characteristic equation are q and $-(q-1)^2$. Hence we get $u(t) = C_1 \mathrm{e}^{qt} + C_2 \mathrm{e}^{-(q-1)^2 t}$, where C_1, C_2 are two constants. Since $\gamma = -u'/(cu)$, we can scale C_1 and C_2 by the same number without changing γ. Putting $C_1 = 1-q$, we get

$$\gamma(t) = \frac{\mathrm{e}^{qt} + C_2 \frac{q-1}{q} \mathrm{e}^{-(q-1)^2 t}}{(1-q)\mathrm{e}^{qt} + C_2 \mathrm{e}^{-(q-1)^2 t}}.$$

The initial condition $\gamma(0) = x$ gives

$$C_2 = \frac{q(1-q)x - q}{q-1-qx}.$$

We already see that $\gamma_x(t)$ is a Möbius transformation in x since C_2 is. For time $t = 1$, we get the following.

Proposition 4.2. *Integrating to time $t = 1$ the operator D_0 seen as vector field in $\mathbb{H}^2$ gives the Möbius transformation*

$$\gamma_x(1) = \frac{\left(q\mathrm{e}^q + (q-1)^2 \mathrm{e}^{-(q-1)^2}\right)x + (1-q)\left(\mathrm{e}^q - \mathrm{e}^{-(q-1)^2}\right)}{q(1-q)\left(\mathrm{e}^q - \mathrm{e}^{-(q-1)^2}\right)x + (1-q)^2\mathrm{e}^q + q\mathrm{e}^{-(q-1)^2}}.$$

If we Taylor expand $\gamma_x(1)$ around $q-1$ to order 1, we get a quadratic polynomial in x. In order to keep a Möbius transformation, we Taylor expand all entries of the associated 2×2 matrix to order 1 in $q-1$. The result is

$$W_q := \begin{pmatrix} e(1-2q) & (e-1)(q-1) \\ (e-1)(q-1) & -q \end{pmatrix},$$

where we used $q^2 = 2q-1$ coming from the Taylor expansion. We see that $q = 1$ gives the transformation $x \mapsto ex$.

A similar computation with arbitrary time t gives

$$W_q^t = \begin{pmatrix} \mathrm{e}^t(t-qt-q) & (\mathrm{e}^t-1)(q-1) \\ (\mathrm{e}^t-1)(q-1) & -q \end{pmatrix},$$

which for $q = 1$ gives the transformation $x \mapsto \mathrm{e}^t x$ from equation (4.1).

4.3 Speculations about a q-deformed hyperbolic plane

The above computations seem to indicate the existence of a q-deformed version of the hyperbolic plane $\mathbb{H}_q^2$ on which the transformations T_q, S_q, g_q and W_q act. A similar idea is developed in [2] where a compactification of the space of stability conditions for type A_2 is constructed.

The transformation $S_q(x) = -1/(qx)$ has only one fixed point given by $iq^{-1/2}$. This equals $[i]_q$, the q-deformed version of i from [17, formula (9)]. The translation $T_q(x) = qx+1$ has two fixed points at the (usual) boundary at infinity, given by ∞ and $1/(1-q)$. However, we expect the boundary of $\mathbb{H}_q^2$ to be $\mathbb{R}_q \cup \{\infty\}$, where $\mathbb{R}_q$ denotes the q-reals. On $\mathbb{R}_q$, the transformation T_q has no fixed point since $T_q[x]_q = [x+1]_q$.

An important role should play the q-rational transition map $g_q(x) = \frac{1+(x-1)q}{1+(q-1)x}$. Since it deforms the identity, there are strictly more transformations in the deformed setting. For $q \neq 1$, g_q is an elliptic transformation with only fixed point on $\mathbb{H}^2$ given by $\frac{1+i\sqrt{3}}{2}$ which is independent of q. In [17, Part 2.3], it is shown that this complex number stays itself under q-deformation. Note that both transformations g_q and T_qS_q are rotations around the same center. Hence they commute. Similarly, the matrix of g_q^{-1} anti-commutes with the matrix of S_q.

These links between q-deformed numbers and the q-deformed $\mathfrak{sl}_2$-algebra are intriguing and might point towards a deeper relation.

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References

[1] Avan J., Frappat L., Ragoucy E., Deformed Virasoro algebras from elliptic quantum algebras, *Comm. Math. Phys.* **354** (2017), 753–773, arXiv:1607.05050.

[2] Bapat A., Becker L., Licata A.M., q-deformed rational numbers and the 2-Calabi–Yau category of type A_2, *Forum Math. Sigma* **11** (2023), e47, 41 pages, arXiv:2202.07613.

[3] Bharathram V., Birman J., On the Burau representation of B_4, *Involve* **14** (2021), 143–154, arXiv:2208.12378.

[4] Burau W., Über Zopfgruppen und gleichsinnig verdrillte Verkettungen, *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 179–186.

[5] Chaichian M., Isaev A.P., Lukierski J., Popowicz Z., Prešnajder P., q-deformations of Virasoro algebra and conformal dimensions, *Phys. Lett. B* **262** (1991), 32–38.

[6] Curtright T.L., Zachos C.K., Deforming maps for quantum algebras, *Phys. Lett. B* **243** (1990), 237–244.

[7] Frenkel E., Reshetikhin N., Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$-algebras, *Comm. Math. Phys.* **178** (1996), 237–264, arXiv:q-alg/9505025.

[8] Hartwig J.T., Larsson D., Silvestrov S.D., Deformations of Lie algebras using σ-derivations, *J. Algebra* **295** (2006), 314–361, arXiv:math.QA/0408064.

[9] Hu N., q-Witt algebras, q-Lie algebras, q-holomorph structure and representations, *Algebra Colloq.* **6** (1999), 51–70, arXiv:math.QA/0512526.

[10] Kassel C., Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, *Comm. Math. Phys.* **146** (1992), 343–356.

[11] Khesin B., Lyubashenko V., Roger C., Extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle, *J. Funct. Anal.* **143** (1997), 55–97, arXiv:hep-th/9403189.

[12] Morier-Genoud S., Ovsienko V., q-deformed rationals and q-continued fractions, *Forum Math. Sigma* **8** (2020), e13, 55 pages, arXiv:1812.00170.

[13] Morier-Genoud S., Ovsienko V., On q-deformed real numbers, *Exp. Math.* **31** (2022), 652–660, arXiv:1908.04365.

[14] Morier-Genoud S., Ovsienko V., Veselov A.P., Burau representation of braid groups and q-rationals, *Int. Math. Res. Not.* **2024** (2024), 8618–8627, arXiv:2309.04240.

[15] Nedelin A., Zabzine M., q-Virasoro constraints in matrix models, *J. High Energy Phys.* **2017** (2017), no. 3, 098, 17 pages, arXiv:1511.03471.

[16] Nigro A., A q-Virasoro algebra at roots of unity, free fermions, and Temperley–Lieb hamiltonians, *J. Math. Phys.* **57** (2016), 041702, 12 pages, arXiv:1211.1067.

[17] Ovsienko V., Towards quantized complex numbers: q-deformed Gaussian integers and the Picard group, *Open Commun. Nonlinear Math. Phys.* **1** (2021), 73–93, arXiv:2103.10800.

[18] Scherich N., Classification of the real discrete specialisations of the Burau representation of B_3, *Math. Proc. Cambridge Philos. Soc.* **168** (2020), 295–304, arXiv:1801.08203.

[19] Shiraishi J., Kubo H., Awata H., Odake S., A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions, *Lett. Math. Phys.* **38** (1996), 33–51, arXiv:q-alg/9507034.

[20] Tsallis C., Possible generalization of Boltzmann–Gibbs statistics, *J. Stat. Phys.* **52** (1988), 479–487.