Non-Stationary Difference Equation
and Affine Laumon Space II:
Quantum Knizhnik–Zamolodchikov Equation

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Abstract. We show that Shakirov's non-stationary difference equation, when it is truncated, implies the quantum Knizhnik–Zamolodchikov (q-KZ) equation for $U_{\mathsf{v}}\big(A_1^{(1)}\big)$ with generic spins. Namely, we can tune mass parameters so that the Hamiltonian acts on the space of finite Laurent polynomials. Then the representation matrix of the Hamiltonian agrees with the R-matrix, or the quantum $6j$ symbols. On the other hand, we prove that the K theoretic Nekrasov partition function from the affine Laumon space is identified with the well-studied Jackson integral solution to the q-KZ equation. Combining these results, we establish that the affine Laumon partition function gives a solution to Shakirov's equation, which was a conjecture in our previous paper. We also work out the base-fiber duality and four-dimensional limit in relation with the q-KZ equation.

Key words: affine Laumon space; quantum affine algebra; non-stationary difference equation; quantum Knizhnik–Zamolodchikov equation

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1   Introduction

The quantum (or q-deformed) Knizhnik–Zamolodchikov (q-KZ) equation typically arises as a linear difference equation satisfied by matrix elements of the products of intertwining operators of highest weight representations of quantum affine algebra [23, 27]. It roughly takes the following form:

$$\Psi(u_1,\ldots,pu_j,\ldots,u_n) = R_{V_jV_{j-1}}\left(\frac{pu_j}{u_{j-1}}\right)\cdots R_{V_jV_1}\left(\frac{pu_j}{u_1}\right)D_j$$
$$\times R_{V_nV_j}^{-1}\left(\frac{u_n}{u_j}\right)\cdots R_{V_{j+1}V_j}^{-1}\left(\frac{u_{j+1}}{u_j}\right)\Psi(u_1,\ldots,u_n), \qquad (1.1)$$

where $\Psi(u_1,\ldots,u_n)$ is an appropriate matrix element of the intertwiners with the spectral parameters u_i and $R_{V_kV_\ell}$ denotes the R matrix on the tensor product $V_k \otimes V_\ell$ of highest weight representations. D_j is a diagonal operator acting on V_j. In the case of $U_{\mathsf{v}}(A^{(1)})$ with level k, the shift parameter is fixed to be $p = \mathsf{v}^{2(k+2)} = q^{k+2}$.[1] The case $n = 2$ is of our interest in this paper. By the scaling of the spectral parameters, which fixes the origin and the infinity, we can consider the two point function $\Phi(u) := \Psi(1,u)$, for which the q-KZ equation is

$$\Phi(pu) = R_{V_2V_1}(pu)D_2\Phi(u). \qquad (1.2)$$

For the q-difference equations including (1.2), various important results using the Jackson integrals have been developed based on the fundamental theory by K. Aomoto. In particular, the Jackson integral of symmetric Selberg type relevant for the equation (1.2) was thoroughly studied by Aomoto, Kato, Matsuo, Mimachi and Ito around early 1990's (see, e.g., [8, 29] and references therein). Around the same time, the relation to Bethe vectors was found by Reshetikhin [48] and the method was further developed by Tarasov, Varchenko, etc. These works play a crucial role in our study.

In [10], we show that the non-stationary difference equation proposed in [51] is transformed into a quantization of the discrete Painlevé VI equation. The original form of the non-stationary difference equation is

$$\mathsf{U}(t\Lambda,x) = \mathcal{A}_1(\Lambda,x)\cdot\mathcal{B}\cdot\mathcal{A}_2(\Lambda,x)\cdot\mathcal{B}\cdot\mathcal{A}_3(\Lambda,x)\mathsf{U}\left(\Lambda,\frac{x}{tqQ}\right), \qquad (1.3)$$

where $\mathcal{B}$ is the q-Borel transformation on a formal Laurent series in x:

$$\mathcal{B}\left(\sum_n c_n x^n\right) = \sum_n q^{n(n+1)/2}c_n x^n, \qquad (1.4)$$

and $\mathcal{A}_i(\Lambda,x)$ are products of infinite products $(x;q)_\infty$ and $(x;q,t)_\infty$ defined by (1.11) and (1.12), respectively. (See [51] and [10] for explicit forms of $\mathcal{A}_i(\Lambda,x)$). In Appendix A, we explain a relation of the operator $\mathcal{B}$ and the refined Chern–Simons theory [2], which gave the original motivation for introducing $\mathcal{B}$ in [51]. The non-stationary difference equation (1.3) was found by looking at the instanton partition function of five-dimensional SU(2) gauge theory with a surface defect. The parameters (q,t) are equivariant parameters of the torus action $(z_1,z_2) \longrightarrow (qz_1,t^{-1}z_2)$ on $\mathbb{C}^2$ and the defect is at $z_2 = 0$, which is the set of fixed points of t^{-1}-action.[2] The function $\mathsf{U}(\Lambda,x)$ is supposed to be a formal double power series in x and Λ/x:

$$\mathsf{U}(\Lambda,x) = \sum_{k,\ell\geq 0} c_{k,\ell}x^k(\Lambda/x)^\ell, \qquad (1.5)$$

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[1]The convention used in our papers [10, 51] is different from the standard convention for the deformation parameter of the quantum affine algebra. This is the reason why we denote $U_{\mathsf{v}}(A_1^{(1)})$ with $\mathsf{v}^2 = q$. According to the AGT dictionary for the current algebra [6], $k + 2 = -\epsilon_2/\epsilon_1$, where (ϵ_1,ϵ_2) are the Ω-background parameters on the gauge theory side. Hence, we have $p = e^{(k+2)\epsilon_1} = e^{-\epsilon_2} = t$.

[2]We can regard q and t as the Ω-background along the defect and transverse to the defect, respectively.

where the coefficients $c_{m,n}$ are functions of the SU(2) Coulomb parameter Q, mass parameters T_i and the equivariant parameters (q,t). Physically, Λ is the instanton expansion parameter and x is identified with the position of the degenerate field $\phi_{2,1}(x)$ that corresponds to the defect according to the AGT correspondence [4, 5].

In the case of the Virasoro algebra which corresponds to four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, the degenerate field $\phi_{2,1}(x)$ has a null state at level two, which implies a differential equation of the second order in x as the BPZ equation. The non-stationary difference equation (1.3) may be regarded as a q-deformed version of the BPZ equation. However, if we try to derive it, we encounter the problem, since the natural coproduct is not known for the deformed Virasoro algebra [13, 53]. Hence, we cannot define intertwiners and consequently cannot use the methods of representation theory. We can overcome such a difficulty by noticing the following fact. Namely, the instanton partition function with a surface defect allows another description in terms of the affine Laumon space [6, 35, 36]. In this case the instanton partition function gives a conformal block of the affine Kac–Moody algebra and its q deformation, the quantum affine algebra is a Hopf algebra! In [10] (and implicitly in [51]), it was conjectured that the instanton partition function coming from the equivariant character of the affine Laumon space solves the gauge transformed version of (1.3); See (1.6) below. More concretely, in [10] we have shown that the non-stationary difference equation (1.3) can be factorized into a coupled system, which is in accord with the formula of Hamiltonian of the discrete Painlevé VI equation in terms of the extended affine Weyl group of $D_5^{(1)}$. Then the conjecture is that a pair of the affine Laumon partition functions whose parameters are related by a simple transformation provides a solution to the coupled system, see [10, Conjecture 6.4]. In this paper, we prove this conjecture by using a connection through the q-KZ equation.

In order to see the relation to the q-KZ equation, we first make a gauge transformation of the "wave function" $\mathsf{U}(\Lambda, x) \to \Psi(\Lambda, x)$ to recast (1.3) into the following form [10]:

$$\Psi(\Lambda, x) = \mathcal{H}_{\mathrm{S}} T_{qtQ,x}^{-1} T_{t,\Lambda}^{-1} \cdot \Psi(\Lambda, x). \tag{1.6}$$

An explicit form of the Hamiltonian $\mathcal{H}_{\mathrm{S}}$ is given by (2.1). $\mathcal{H}_{\mathrm{S}}$ involves "renormalized" mass parameters d_i, $1 \le i \le 4$ which are monomials in the original mass parameters T_i, Q and (q,t). It is amusing that we can eliminate the double infinite product $(x; q, t)_\infty$ appeared in $\mathcal{A}_i(\Lambda, x)$ from $\mathcal{H}_{\mathrm{S}}$. It is also remarkable that the parameters Q and t appear explicitly only through the shift parameters for x and Λ. By tuning two of mass parameters, say $d_2 = q^{-m}$ and $d_3 = q^{-n}$, the Hamiltonian acts on the $(n+m+1)$-dimensional space of Laurent polynomials with a basis x^j, $-n \le j \le m$. Then, the representation matrix of $\mathcal{H}_{\mathrm{S}}$ agrees with the R-matrix of $U_{\mathsf{v}}(A_1^{(1)})$, which allows us to identify (1.6) as a q-KZ equation. Namely, for the matrix $r_{i,j}$ defined by

$$\mathcal{H}_{\mathrm{S}} x^i = \sum_{j=-n}^{m} r_{i,j}(\Lambda) x^j, \qquad -n \le i \le m, \tag{1.7}$$

we have the following statement.

Theorem 1.1 (Theorem 2.3). *Set $N = m + n$. The matrix $[r_{i,j}(\Lambda)]_{-n \le i,j \le m}$ coincides with the $(N+1) \times (N+1)$ block in the weight decomposition of the infinite-dimensional R-matrix for $U_{\mathsf{v}}(A_1^{(1)})$ for evaluation representations of generic weights.*

This is one of the fundamental results in this paper.

As we mentioned above, the Jackson integral of symmetric Selberg type solves the q-KZ equation. Recall that the Jackson integral is a pairing of the integration cycle ξ and the cocycle function $\phi(z)$

$$\langle \phi(z), \xi \rangle = (1-t)^N \sum_{\nu \in \mathbb{Z}^N} \left[\phi(z) \Phi(z) \prod_{1 \le i < j \le N} (z_i - z_j) \right]_{z_i = \xi_i t^{\nu_i}}. \tag{1.8}$$
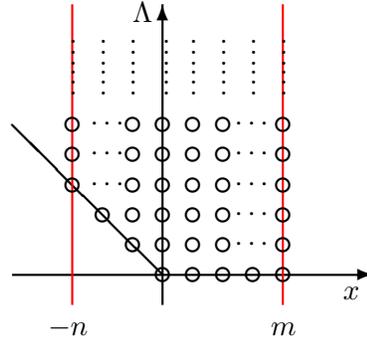
Figure 1. After the mass truncation $d_2 = q^{-m}$, $d_3 = q^{-n}$, the double series $\Psi(\Lambda, x)$ becomes a Laurent polynomial in x, while it is still a formal power series in Λ. The circles represent the positions of allowed terms in the (x, Λ)-lattice.

In this paper, we take the weight function $\Phi(z)$ which depends on four parameters (a_1, a_2, b_1, b_2); See Section 3 for details. On the other hand, the instanton partition function $\mathcal{Z}_{\mathrm{AL}}$ from the equivariant character of the affine Laumon space is defined by (4.5). Hence, if we can show the relation of the partition function $\mathcal{Z}_{\mathrm{AL}}$ to the Jackson integral (1.8), it gives a proof of the conjecture in our previous paper. Under the same tuning conditions for mass parameters as above, the partition function $\mathcal{Z}_{\mathrm{AL}}$ becomes a Laurent polynomial in x (see Figure 1 and Appendix E). By carefully examining combinatorial structure of the orbifolded Nekrasov factor in (4.5) (see Appendix F), we can identify the coefficients of $\mathcal{Z}_{\mathrm{AL}}$ in the expansion parameter x as the Jackson integral (1.8) with an appropriate choice of the cycle ξ. In this way, the affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$ agrees with the Jackson integral for the q-KZ equation term by term, if we choose suitable basis of (co-)cycles. More concretely, we have the following.

Theorem 1.2 (Theorem 4.1)**.** *Under the identification of the parameters*

$$d_1 = \frac{1}{q^{m-1}a_1 b_1}, \qquad d_2 = q^{-m}, \qquad d_3 = q^{-n}, \qquad d_4 = \frac{1}{q^{n-1}a_2 b_2}, \qquad Q = \frac{q^{m-n}a_1}{t a_2}, \ (1.9)$$

the affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$ coincides with the Jackson integral (1.8) with an appropriate choice of ξ and a suitable basis of $\phi(z)$.

A definition of the basis and the choice of ξ are the subjects of Section 3. In particular, we show that the combinatorial structure of $\mathcal{Z}_{\mathrm{AL}}$ as a sum over the pair of Young diagrams is in a nice correspondence with the cohomology basis called Matsuo basis in the literature.

By combining above two theorems under the specialization $d_2 = q^{-m}$, $d_3 = q^{-n}$, we can show that the affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$ is a solution to the gauge transformed version (1.6) of the Shakirov's equation. The result can be extended to generic mass parameters d_i also by the following reasons. We invoke the fact that the equation (1.3) inductively determines the expansion coefficients $c_{i,j}$ in (1.5) uniquely up to an overall normalization, which we fix by $c_{0,0} = 1$ (see [10, Proposition 2.7]). Moreover, from the recursion relations among $c_{i,j}$ we can see that for generic mass parameters d_i they are polynomials in d_i with coefficients in the field of rational functions of q, t and Q.[3] Hence, if $\mathcal{Z}_{\mathrm{AL}}$ is a solution under the specialization $d_2 = q^{-m}$, $d_3 = q^{-n}$ for any $m, n \in \mathbb{Z}_{\geq 0}$, it is valid for generic values of d_2 and d_3.

Theorem 1.3. *The affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$ is a solution to the non-stationary difference equation (1.6).*

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[3]If we expand the Hamiltonian $\mathcal{H}_{\mathrm{S}}$ in x and Λ/x, the leading term does not involve d_i.

In Appendix H, we recast (1.6) to a coupled system, which is essentially the same as the coupled system derived in [10] based on the structure of the Hamiltonian of the qq-Painlevé VI equation. Hence, our main Theorem 1.3 also proves [10, Conjecture 6.4]. Recall that the original conjecture in [51] was for the degenerate five point q-Virasoro conformal block, which is realized as the Nekrasov partition function of $\mathrm{U}(2) \times \mathrm{U}(2)$ linear quiver gauge theory with the Higgsing condition on mass parameters. One can express the partition function in question in terms of the topological vertex (intertwiners of the quantum toroidal $\mathfrak{gl}_1$-algebra) and confirm that it is similarly related to the Jackson integral of the same type as the present paper. This is regarded as a q-deformed analogue of the correspondence between the current block of $\mathfrak{sl}_2$ and the degenerate conformal block. Details will be reported elsewhere.

It is known that the Jackson integral relevant for q-KZ equation actually satisfies two kinds of difference equations (see, e.g., [29]). For the case in our hand, the second difference equation involves the shift of a "renormalized" Coulomb modulus $(qtQ)^{-1}$ of SU(2) gauge theory.[4] Two difference equations are exactly the same under the exchange

$$\Lambda \longleftrightarrow \frac{q^2}{d_1 d_2}(qtQ)^{-1}. \tag{1.10}$$

In the geometric engineering of SU(2) gauge theory by local $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, Λ and Q are identified with Kähler parameters of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, this is a manifestation of the base-fiber duality in topological strings.

We will also work out the four-dimensional limit of the gauge transformed equation (1.6). Then we can check that it agrees with the KZ equation, which has been derived from the regularity of the fractional fundamental qq-character of type A_1 [47].

Since the KZ equation can be regarded as a quantization of the Schlesinger system [49], the relation of the conformal block of the current algebra to the quantum monodromy preserving deformation has been believed as a folklore. The equation (1.6) is quite interesting object as an explicit example realizing such correspondence. In recent papers [20, 54], the quantum cohomology and (quantum) monodromy preserving deformation has also been discussed.

It is very likely that the q-Borel transformation $\mathcal{B}$ appears in some representation of double affine Hecke algebras (DAHA) on polynomials in $x_1, x_2, \ldots$ [51]. From such a speculation, it is interesting that in connection with the quantum Q system [21, 22], an operator very close to $\mathcal{B}$ is employed as Cherednik's Gaussian operator representing the Dehn twist on polynomial representations of spherical DAHA.

The present paper is organized as follows. In Section 2, we introduce a truncation of the non-stationary difference equation (1.3) by tuning two of mass parameters among four. We will call it mass truncation. After the mass truncation the Hamiltonian of (1.3) acts on the space of Laurent polynomials in x. We find that the matrix elements $r_{i,j}(\Lambda)$ satisfy the same relation as the quantum $6j$-symbol derived by Rosengren [50]. In Section 3, following the work by Ito [29], we first review the q-KZ equations satisfied by the Jackson integral (1.8), which is a paring of an integration cycle ξ and a cocycle function $\varphi(z)$. We also study a special integration cycle ξ which reduces the bilateral sum over the lattice in the definition of the Jackson integral to a sum over a positive cone of the lattice. This truncation (which we call lattice truncation) has been considered in literature (see, e.g., [31, 32] and references therein). The relevant cycles are called the characteristic cycles or the α-stable or α-unstable cycles by Aomoto, and play crucial roles in the study of asymptotic behavior. The lattice truncation is crucial, when we make an identification of the Jackson integral with the Nekrasov partition function which is expressed as a sum over a pair of Young diagrams. In Section 4, we consider the Nekrasov partition function with a surface defect, which can be obtained from the equivariant character at fixed

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[4]Recall that $(qtQ)^{-1}$ is the shift parameter of x in (1.3).

points of the toric action on the affine Laumon space. The partition function is expanded in x and Λ/x and we show that the expansion in x is in accord with a basis, which is called Matsuo basis, of the cocycle functions in the Jackson integral representation of solutions to the q-KZ equation. In Section 5, we compute the four-dimensional limit of the R-matrix that appears in the truncation of the non-stationary difference equation and show that the limit correctly reproduces the KZ equation for the conformal block of the $\mathfrak{sl}_2$ current algebra. There are many appendices in this paper, where some helpful clarifications and technical details are presented.

We will use the following notations throughout the paper [28]:

$$\varphi(x) := (x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1-q^n}x^n\right), \qquad |x| < 1, \quad |q| < 1, \quad (1.11)$$

$$(x;q,t)_\infty = \prod_{n,m=0}^{\infty}(1-xq^nt^m) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{(1-q^n)(1-t^n)}x^n\right),$$

$$|x| < 1, \quad |q|, |t| < 1. \tag{1.12}$$

The q-shifted factorial is defined by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{i=0}^{n-1}(1-aq^i). \tag{1.13}$$

It is convenient to introduce the notation

$$(a_1, a_2, \ldots, a_k; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n. \tag{1.14}$$

We frequently use the short-hand notation $(a)_n := (a;q)_n$, when there is no risk of confusion. The following formulas are useful:

$$(a)_n = (-a)^n q^{n(n-1)/2}\left(\frac{1}{aq^{n-1}}\right)_n, \qquad (a)_{k+\ell} = (a)_k\left(aq^k\right)_\ell. \tag{1.15}$$

The q-exponential function is

$$e_q(x) := \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}. \tag{1.16}$$

Finally, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.17}$$

2 Shakirov's equation as a q-KZ equation with generic spins

In [10], we show that by an appropriate gauge transformation we can recast the original equation (1.3) to the form $\mathcal{H}_S T_{qtQ,x}^{-1} T_{t,\Lambda}^{-1} \cdot \Psi(\Lambda, x) = \Psi(\Lambda, x)$ with the Hamiltonian

$$\mathcal{H}_S = \frac{1}{\varphi(qx)\varphi(\Lambda/x)} \cdot \mathcal{B} \cdot \frac{\varphi(\Lambda)\varphi\left(q^{-1}d_1 d_2 d_3 d_4 \Lambda\right)}{\varphi(-d_1 x)\varphi(-d_2 x)\varphi(-d_3 \Lambda/x)\varphi(-d_4 \Lambda/x)}$$

$$\times \mathcal{B} \cdot \frac{1}{\varphi\left(q^{-1}d_1 d_2 x\right)\varphi(d_3 d_4 \Lambda/x)}, \tag{2.1}$$

where $\varphi(z) := (z; q)_\infty$. We have made the following change of variables:[5]

$$\Lambda' = tT_2T_3\Lambda, \qquad x' = T_2q^{-1/2}t^{1/2}x, \tag{2.2}$$

and mass parameters:[6]

$$d_1 = T_1q^{1/2}t^{-1/2}, \qquad d_2 = T_2^{-1}q^{1/2}t^{-1/2}Q^{-1},$$
$$d_3 = T_3^{-1}q^{1/2}t^{-1/2}, \qquad d_4 = T_4q^{1/2}t^{1/2}Q. \tag{2.3}$$

The Hamiltonian $\mathcal{H}_S$ is manifestly symmetric under the exchange $d_1 \leftrightarrow d_2$, $d_3 \leftrightarrow d_4$. Let $\vartheta_x := x\partial_x$ be the Euler operator in x. We have the following commutation relations among $x, p = q^{\vartheta_x}$ and $\mathcal{B} = q^{\vartheta_x(\vartheta_x+1)/2}$:

$$px = qxp, \qquad \mathcal{B}p = p\mathcal{B}, \qquad \mathcal{B}x = px\mathcal{B}. \tag{2.4}$$

Lemma 2.1. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{H}_S \cdot x^m\mathbb{C}\left[\left[\frac{1}{x}\right]\right] \subset x^m\mathbb{C}\left[\left[\frac{1}{x}\right]\right] \qquad \text{for } d_1 = q^{-m} \text{ or } d_2 = q^{-m},$$
$$\mathcal{H}_S \cdot x^{-m}\mathbb{C}[[x]] \subset x^{-m}\mathbb{C}[[x]] \qquad \text{for } d_3 = q^{-m} \text{ or } d_4 = q^{-m}. \tag{2.5}$$

Proof. Due to the relation

$$\mathcal{H}_S(d_1, d_2, d_3, d_4, \Lambda, x) = q^2x\,\mathcal{H}_S(qd_1, qd_2, d_3/q, d_4/q, \Lambda, x)\,p^2x^{-1},$$

the statements inductively reduce to the case $m = 0$. Hence, it is enough to show the lemma for $m = 0$. Note that one can omit the factors of the form $\varphi(*\Lambda/x)$ [$\varphi(*x)$ (resp.)] in the first [second (resp.)] relation in (2.5), since they do not affect the consequence. By omitting these factors, the relations (2.5) follow from the identities ($k \geq 0$)

$$\frac{1}{\varphi(-aqx)}\mathcal{B}\frac{1}{\varphi(ax)} \cdot x^{-k} = x^{-k}\prod_{j=0}^{k-1}(q^j + ax),$$

$$\frac{1}{\varphi(-a/x)}\mathcal{B}\frac{1}{\varphi(a/x)} \cdot x^k = x^k\prod_{j=1}^{k}\left(q^j + \frac{a}{x}\right),$$

which are a special case of the formulas (B.1) in Appendix B. ∎

Due to Lemma 2.1, we have the following action of $\mathcal{H}_S$ with finite support:

$$\mathcal{H}_S x^i = \sum_{j=-n}^{m} r_{i,j}(\Lambda)x^j, \qquad -n \leq i \leq m, \tag{2.6}$$

when the parameters are specialized as $d_2 = q^{-m}$, $d_3 = q^{-n}$, $m, n \in \mathbb{Z}_{\geq 0}$.[7] The coefficient $r_{i,j}(\Lambda)$ is a rational function in the spectral parameter Λ. It also depends on the remaining mass parameters d_1 and d_4, which are to be identified with the highest weights of the in-state and the out-state, when we consider a matrix element (two point function) of the intertwiners.

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[5]In (2.1), $'$ on x and Λ are deleted for simplicity.
[6]We made an exchange $d_1 \leftrightarrow d_2$ in [10, equation (5.5)]. We also exchanged T_1 and T_2, which corresponds to the action of a Weyl reflection on the Painlevé side, namely (2.1) is derived from [10, equation (5.2)] by the change of variables (2.2) and (2.3).
[7]Due to the symmetry $d_1 \leftrightarrow d_2$, $d_3 \leftrightarrow d_4$, the cases $d_1 = q^{-m}$, $d_4 = q^{-n}$ etc. have the similar truncation.

Example 2.2. For $(m, n) = (1, 0), (2, 0)$, we have

$$
\begin{bmatrix} r_{0,0} & r_{0,1} \\ r_{1,0} & r_{1,1} \end{bmatrix} = \begin{bmatrix} \dfrac{(\frac{d_1 \Lambda}{q})_1}{(\frac{\Lambda}{q})_1} & -\dfrac{(d_1)_1}{(\frac{\Lambda}{q})_1} \\[12pt] -\dfrac{\Lambda q (\frac{d_4}{q})_1}{(\frac{\Lambda}{q})_1} & \dfrac{q^2 (\frac{d_4 \Lambda}{q^2})_1}{(\frac{\Lambda}{q})_1} \end{bmatrix}
\tag{2.7}
$$

and

$$
\begin{bmatrix} r_{0,0} & r_{0,1} & r_{0,2} \\ r_{1,0} & r_{1,1} & r_{1,2} \\ r_{2,0} & r_{2,1} & r_{2,2} \end{bmatrix} = \begin{bmatrix} \dfrac{(\frac{d_1\Lambda}{q^2})_1(\frac{d_1\Lambda}{q})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & -\dfrac{(1+q)(d_1)_1(\frac{d_1\Lambda}{q})_1}{q(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & \dfrac{(d_1)_1(d_1q)_1}{q(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} \\[12pt] -\dfrac{\Lambda q(\frac{d_4}{q})_1(\frac{d_1\Lambda}{q})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & \dfrac{q^2(\frac{d_1\Lambda}{q})_1(\frac{d_4}{q^2})_1+\Lambda q(d_1q)_1(\frac{d_4}{q^2})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & -\dfrac{q^2(d_1q)_1(\frac{d_4\Lambda}{q^3})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} \\[12pt] \dfrac{\Lambda^2 q^3(\frac{d_4}{q^2})_1(\frac{d_4}{q})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & -\dfrac{\Lambda q^4(1+q)(\frac{d_4}{q^2})_1(\frac{d_4\Lambda}{q^3})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} & \dfrac{q^6(\frac{d_4\Lambda}{q^4})_1(\frac{d_4\Lambda}{q^3})_1}{(\frac{\Lambda}{q^2})_1(\frac{\Lambda}{q})_1} \end{bmatrix}, \tag{2.8}
$$

where $r_{i,j} = r_{i,j}(\Lambda)$, $(x)_1 = (1 - x)$.

Theorem 2.3. *Set $N = m + n$. The matrix $[r_{i,j}(\Lambda)]_{-n \le i,j \le m}$ coincides with the $(N + 1) \times (N + 1)$ block in the weight decomposition of the infinite-dimensional R-matrix for $U_{\mathsf{v}}\big(A_1^{(1)}\big)$ for evaluation representations of generic weights* [14, 38].[8]

Proof. The defining relation (2.6) of $r_{i,j}(\Lambda)$ can be written as

$$
\frac{\varphi(\Lambda)\varphi\big(q^{-1}d_1 d_2 d_3 d_4 \Lambda\big)}{\varphi(-d_1 x)\varphi(-d_2 x)\varphi(-d_3 \Lambda/x)\varphi(-d_4 \Lambda/x)} \mathcal{B} \frac{1}{\varphi(q^{-1}d_1 d_2 x)\varphi(d_3 d_4 \Lambda/x)} x^i
$$
$$
= \mathcal{B}^{-1}\varphi(qx)\varphi\left(\frac{\Lambda}{x}\right) \sum_{j=-n}^{m} r_{i,j}(\Lambda) x^j. \tag{2.9}
$$

Using the relations (see Appendix B)

$$
\mathcal{B} \frac{1}{\varphi\big(q^{-1}d_1 d_2 x\big)\varphi\big(d_3 d_4 \frac{\Lambda}{x}\big)} x^i = \frac{\varphi\big(-q^i d_1 d_2 x\big)\varphi\big(-q^{-i}d_3 d_4 \frac{\Lambda}{x}\big)}{\varphi\big(q^{-1}d_1 d_2 d_3 d_4 \Lambda\big)} q^{i(i+1)/2} x^i,
$$
$$
\mathcal{B}^{-1}\varphi(qx)\varphi\left(\frac{\Lambda}{x}\right) x^j = q^{-j(j+1)/2} x^j \frac{\varphi(\Lambda)}{\varphi\big(-q^{-j}x\big)\varphi\big(-q^j \frac{\Lambda}{x}\big)}, \tag{2.10}
$$

we can rewrite (2.9) as

$$
\frac{\varphi\big(-q^i d_1 d_2 x\big)\varphi\big(-q^{-i}d_3 d_4 \frac{\Lambda}{x}\big)}{\varphi(-d_3 \Lambda/x)\varphi(-d_4 \Lambda/x)} q^{i(i+1)/2} x^i
$$
$$
= \sum_{j=-n}^{m} r_{i,j}(\Lambda) q^{-j(j+1)/2} x^j \frac{\varphi(-d_1 x)\varphi(-d_2 x)}{\varphi\big(-q^{-j}x\big)\varphi\big(-q^j \frac{\Lambda}{x}\big)}. \tag{2.11}
$$

Since $d_2 = q^{-m}$, $d_3 = q^{-n}$, $m, n \in \mathbb{Z}_{\ge 0}$, this equation can be simplified to

$$
q^{\frac{1}{2}i(i+1)} x^i \big(-d_1 q^{i-m}x\big)_{m-i}\left(-d_4 q^{-i-n}\frac{\Lambda}{x}\right)_{i+n}
$$
$$
= \sum_{j=-n}^{m} r_{i,j}(\Lambda) q^{-\frac{1}{2}j(j+1)} x^j (-q^{-m}x)_{m-j}\left(-q^{-n}\frac{\Lambda}{x}\right)_{j+n}, \tag{2.12}
$$

where $(a)_n = (a; q)_n$. This relation is essentially the same as the equation for the q-$6j$ symbols [50]. Hence, the coefficients $r_{i,j}(\Lambda)$ is given by the q-$6j$ (i.e., R-matrix) for $U_{\mathsf{v}}\big(A_1^{(1)}\big)$. ∎

---

[8]See Remark 2.4 below and computations in Section 2.1 for more precise identification.

Remark 2.4. Some explicit formulas for $U_{\mathsf{v}}\big(A_1^{(1)}\big)$ R-matrix are known [14, 38]. One of such expressions is related to the hypergeometric series ${}_4\phi_3$ with the base q,[9]

$$R_{i,j}^{\mathrm{HG}} = \beta^{-j} \frac{(q)_N \big(\frac{\alpha}{z}\big)_{N-i} \big(\frac{1}{\beta}\big)_{N-j} \big(\frac{\beta}{z}\big)_j}{(q)_j (q)_{N-j} \big(\frac{1}{z}\big)_N \big(\frac{1}{\beta}\big)_{N-i}} \sum_{k=0}^{j} \frac{(q^{-j})_k (q^{i-N})_k (q^{1-N}z)_k \big(\frac{z}{\alpha\beta}\big)_k}{(q)_k (q^{-N})_k \big(\frac{q^{1+i-N}z}{\alpha}\big)_k \big(\frac{q^{1-j}z}{\beta}\big)_k} q^k. \tag{2.13}$$

For $d_2 = q^{-m}$, $d_3 = q^{-n}$ and $i, j \in [-n, m]$, one can show (see Section 2.1)

$$r_{i,j}(\Lambda) = d_1^{m-i} q^{(m+1)i} R_{i+n,j+n}^{\mathrm{HG}}\Big|_{\{N=m+n, z=\frac{\Lambda}{q}, \alpha=\frac{q^n}{d_1}, \beta=\frac{q^m}{d_4}\}}, \tag{2.14}$$

hence, the matrices $\{r_{i,j}(\Lambda)\}_{i,j\in[-n,m]}$ with fixed $N = m + n$ give essentially the same $(N+1) \times (N+1)$ R-matrix.

The above result implies that the equation

$$\Psi(\Lambda, x) = \mathcal{H}_{\mathrm{S}} \Psi\Big(\frac{\Lambda}{t}, \frac{x}{qtQ}\Big), \tag{2.15}$$

for $d_2 = q^{-m}$, $d_3 = q^{-n}$, $N = m + n$, is written as the t-difference equation with $(N+1) \times (N+1)$ matrix coefficient:

$$\psi_j(\Lambda) = \sum_{i=-n}^{m} \psi_i\Big(\frac{\Lambda}{t}\Big) r_{i,j}(\Lambda)(qtQ)^{-i}, \tag{2.16}$$

where $\Psi(\Lambda, x) = \sum_{i=-n}^{m} x^i \psi_i(\Lambda)$. This is exactly the q-KZ equation for the four point function of $U_{\mathsf{v}}\big(A_1^{(1)}\big)$ in N-down spin sector.[10] The cases (m, n) with $m + n = N$ correspond to an equivalent equation of size $N + 1$. In Section 4, we will show that the specializations of the affine Laumon partition function corresponding to the condition $d_2 = q^{-m}$, $d_3 = q^{-n}$ give a set of $N + 1$ solutions which form the fundamental solutions. In this sense, the Laumon partition function can be viewed as the "universal fundamental solution" for q-KZ equation.

2.1   Computation of $r_{i,j}$

We will explain how to compute $(r_{i,j})_{i,j=-n}^{m}$ from (2.12). By using the formula (1.15), the relation (2.12) can be written as

$$q^{-in}(\Lambda d_4)^{i+n} U_i(x) = \sum_{j=-n}^{m} r_{i,j} q^{-j} \Lambda^{j+n} V_j(x), \tag{2.17}$$

where

$$U_i(x) = \big(-d_1 q^{i-m} x\big)_{m-i} \Big(-\frac{qx}{\Lambda d_4}\Big)_{i+n}, \qquad V_j(x) = (-q^{-m}x)_{m-j} \Big(-\frac{q^{1-j}x}{\Lambda}\Big)_{j+n}. \tag{2.18}$$

Since both $\{U_i(x)\}_{i=-n}^{m}$ and $\{V_j(x)\}_{j=-n}^{m}$ are bases of polynomials of degree $m + n$, our task is to compute the transition matrix between these two bases. For this purpose it is convenient to introduce an intermediate basis $\{W_k(x)\}_{k=-n}^{m}$ defined by[11]

$$W_k(x) = U_k(x)\Big|_{d_1 \mapsto \frac{q^{n+1}}{\Lambda}} = \Big(-\frac{q^{n+1+k-m}}{\Lambda}x\Big)_{m-k} \Big(-\frac{qx}{\Lambda d_4}\Big)_{k+n}. \tag{2.19}$$

---

[9]The parameter q in [14, 38] is denoted by $\mathsf{v} = q^{\frac{1}{2}}$ in this paper.

[10]The parameter Q specifies the intermediate state of the correlation function $\langle j_4|\Phi_{j_3}(1)\Phi_{j_2}(\Lambda)|j_1\rangle$. The variable x is merely a book-keeping parameter for the spin direction.

[11]Another choice $\tilde{W}_k(x) = U_k(x)\big|_{d_4 \mapsto \frac{q^{m+1}}{\Lambda}}$ also works as well, which gives different decomposition of the same matrix $(r_{i,j})$.

The polynomial $W_k(x)$ is defined so that the zeros of $U_i(x)$, $W_k(x)$, $V_j(x)$ overlap with each other as follows:

$$
\begin{aligned}
U_i(x)\colon\quad -x &= \underbrace{\frac{q}{d_1},\ldots,\frac{q}{d_1}q^{m-i-1}}_{m-i},\ \underbrace{\frac{d_4\Lambda}{q^{n+i}},\ldots,\frac{d_4\Lambda}{q}}_{n+i}, \\
W_k(x)\colon\quad -x &= \underbrace{\frac{\Lambda}{q^n},\ldots,\frac{\Lambda}{q^n}q^{m-k-1}}_{m-k},\ \underbrace{\frac{d_4\Lambda}{q^{n+k}},\ldots,\frac{d_4\Lambda}{q}}_{n+k}, \\
V_j(x)\colon\quad -x &= \underbrace{\frac{\Lambda}{q^n},\ldots,\frac{\Lambda}{q^n}q^{n+j-1}}_{n+j},\ \underbrace{q^{j+1},\ldots,q^m}_{m-j}.
\end{aligned}
\tag{2.20}
$$

Let us compute the transition coefficients $r_{i,k}^{UW}$ and $r_{k,j}^{WV}$ defined by

$$
U_i(x) = \sum_{k=-n}^{m} r_{i,k}^{UW} W_k(x), \qquad W_k(x) = \sum_{j=-n}^{m} r_{k,j}^{WV} V_j(x).
\tag{2.21}
$$

Due to the cancellation of common factors arising from (2.20), the first relation of (2.21) can be written as

$$
\left(-d_1 q^{i-m}x\right)_{m-i} = \sum_{k=i}^{m} r_{i,k}^{UW}\left(-\frac{q^{n+1+k-m}}{\Lambda}x\right)_{m-k}\left(-\frac{q^{i+n+1}x}{\Lambda d_4}\right)_{k-i}.
\tag{2.22}
$$

Note that we may delete the coefficient of $W_k(x)$ which does not have the common zeros with $U_i(x)$. Similarly, from the second equation of (2.21), we have

$$
\left(-\frac{qx}{\Lambda d_4}\right)_{k+n} = \sum_{j=m-k-n}^{m} r_{k,j}^{WV}(-q^{-m}x)_{m-j}\left(-\frac{q^{1-j}x}{\Lambda}\right)_{j+n-m+k}.
\tag{2.23}
$$

Proposition 2.5. *The nonzero coefficients of $r_{i,k}^{UW}$, $i \le k$, can be determined as*

$$
\begin{aligned}
r_{i,k}^{UW} &= q^{\frac{1}{2}(i-k)(2m+1+i-k)}(-d_4)^{k-i}\frac{(q)_{m-i}}{(q)_{k-i}(q)_{m-k}} \\
&\quad \times \frac{\left(d_1\Lambda q^{i-k-n}\right)_{k-i}\left(d_1 d_4\Lambda q^{-m-n-1}\right)_{m-k}}{(d_4 q^{-m})_{m-i}}.
\end{aligned}
\tag{2.24}
$$

Hence, $\left(r_{i,k}^{UW}\right)$ is upper triangular matrix with factorized elements. Similarly, the nonvanishing coefficients of $r_{k,j}^{WU}$, $N-(k+n) \le j+n$, are

$$
\begin{aligned}
r_{k,j}^{WV} &= q^{\frac{1}{2}(j-k-m-n-1)(j+k-m+n)}(-\Lambda)^{j+k-m+n}\frac{(q)_{k+n}}{(q)_{k+j-m+n}(q)_{m-j}} \\
&\quad \times \frac{\left(\frac{q^{m+1}}{d_4\Lambda}\right)_{k+j-m+n}\left(\frac{q^{j+1}}{d_4}\right)_{m-j}}{\left(q^{-k-n}\Lambda\right)_{k+n}}.
\end{aligned}
\tag{2.25}
$$

The matrix $\left(r_{k,j}^{WV}\right)$ is "lower triangular along anti-diagonal" with factorized elements.

Proof. We follow the method in [50, Section 3]. By the change of the parameters

$$
N = m-i, \qquad r = m-k, \qquad (a,b,c) = \left(-d_1 q^{i-m}, -\frac{q^{i+n+1}}{d_4\Lambda}, -\frac{q^{n+1}}{\Lambda}\right)
\tag{2.26}
$$

for (2.22), and

$$N = k + n, \qquad r = j + k - m + n, \qquad (a, b, c) = \left(-\frac{q}{d_4 \Lambda}, -q^{-m}, -\frac{q^{k-m+n+1}}{\Lambda} \right) \qquad (2.27)$$

for (2.23), respectively, both equations are written as the binomial expansion

$$(ax)_N = \sum_{r=0}^{N} C_{N,r} (bx)_{N-r} (cq^{-r} x)_r. \qquad (2.28)$$

This implies that the coefficients $C_{N,r}$ are given by

$$C_{N,r} = q^{r(r+1)/2} \left(-\frac{b}{c} \right)^r \frac{(q)_N}{(q)_r (q)_{N-r}} \frac{\left(\frac{a}{b} \right)_r \left(q^{r+1} \frac{a}{c} \right)_{N-r}}{\left(\frac{bq}{c} \right)_N}. \qquad (2.29)$$

To see this, note that

$$1 - q^N ax = A_r \left(1 - q^{N-r} bx \right) + B_r \left(1 - q^{-r-1} cx \right),$$
$$A_r = \frac{c - aq^{N+1+r}}{c - bq^{N+1}}, \qquad B_r = \frac{b - aq^r}{b - cq^{-N-1}}. \qquad (2.30)$$

Hence,

$$\frac{(ax)_{N+1}}{(ax)_N} = A_r \frac{(bx)_{N+1-r}}{(bx)_{N-r}} + B_r \frac{\left(cq^{-r-1} x \right)_{r+1}}{\left(cq^{-r} x \right)_r}. \qquad (2.31)$$

From this, the coefficients $C_{N,r}$ are uniquely determined by "Pascal's triangle"

$$C_{N+1,r} = A_r C_{N,r} + B_{r-1} C_{N,r-1}, \qquad (2.32)$$

with the boundary conditions $C_{0,0} = 1$, $C_{N,-1} = C_{N,N+1} = 0$. For $C_{N,r}$ given by (2.29), the boundary conditions are obvious. The relation (2.32) follows from

$$\frac{C_{N+1,r}}{C_{N,r}} = \frac{\left(1 - q^{N+1} \right) \left(1 - \frac{aq^{N+1}}{c} \right)}{\left(1 - q^{N+1-r} \right) \left(1 - \frac{bq^{N+1}}{c} \right)}, \qquad \frac{C_{N,r-1}}{C_{N,r}} = -\frac{cq^{-r} (1 - q^r) \left(1 - \frac{aq^r}{c} \right)}{b \left(1 - q^{N+1-r} \right) \left(1 - \frac{aq^{r-1}}{b} \right)} \qquad (2.33)$$

and the identity

$$\left(1 - q^{N+1} \right) \left(1 - \frac{a}{c} q^{N+1} \right) - q^{N+1-r} (1 - q^r) \left(1 - \frac{a}{c} q^r \right)$$
$$- \left(1 - q^{N+1-r} \right) \left(1 - \frac{a}{c} q^{N+1+r} \right) = 0. \qquad (2.34)$$

Hence, (2.29) is proved. Back to the original parameters, we obtain the desired results. ∎

Combining (2.24) and (2.25), we have

$$r_{i,j} = q^{-in} (\Lambda d_4)^{i+n} q^j \Lambda^{-j-n} r_{i,j}^{UV} = q^{-in} (\Lambda d_4)^{i+n} q^j \Lambda^{-j-n} \sum_{k=-n}^{m} r_{i,k}^{UW} r_{k,j}^{WV}$$

$$= \sum_{k=m-n-j}^{m} q^{-in} q^j q^{\frac{1}{2}(i-k)(2m+1+i-k)} q^{\frac{1}{2}(j-k-m-n-1)(j+k-m+n)} (-1)^{-m+n-i+j}$$

$$\times d_4^{k+n} \Lambda^{i+n+k-m} \frac{(q)_{m-i}}{(q)_{k-i}(q)_{m-k}} \frac{\left(d_1 \Lambda q^{i-k-n}\right)_{k-i}\left(d_1 d_4 \Lambda q^{-m-n-1}\right)_{m-k}}{(d_4 q^{-m})_{m-i}}$$

$$\times \frac{(q)_{k+n}}{(q)_{k+j-m+n}(q)_{m-j}} \frac{\left(\frac{q^{m+1}}{d_4\Lambda}\right)_{k+j-m+n}\left(\frac{q^{j+1}}{d_4}\right)_{m-j}}{(q^{-k-n}\Lambda)_{k+n}}. \tag{2.35}$$

To show (2.14), we make the change of variables

$$z = \frac{\Lambda}{q}, \qquad \alpha = \frac{q^n}{d_1}, \qquad \beta = \frac{q^m}{d_4}. \tag{2.36}$$

Using $(a)_k = (-a)^k q^{k(k-1)/2}\left(\frac{1}{aq^{k-1}}\right)_k$ and $(a)_{k+l} = (a)_k\left(aq^k\right)_l$, we compute

$$r_{i,j} = d_1^{m-i}\beta^{-n-j}q^{(m+1)i} \frac{\left(\frac{\alpha}{z}\right)_{m-i}\left(\frac{1}{\beta}\right)_{m-j}\left(\frac{\beta}{z}\right)_{j+n}}{\left(\frac{1}{z}\right)_{m+n}\left(\frac{1}{\beta}\right)_{m-i}} \sum_{k=m-n-j}^{m} q^{(m-k)(i+n)}$$

$$\times \left(\frac{\beta}{z}\right)^{m-k} \frac{(q)_{m-i}}{(q)_{k-i}(q)_{m-k}} \frac{(q)_{k+n}}{(q)_{k+j-m+n}(q)_{m-j}} \frac{\left(\frac{z}{\alpha\beta}\right)_{m-k}}{\left(\frac{z}{\alpha}q^{1+i-m}\right)_{m-k}} \frac{\left(zq^{1-N}\right)_{m-k}}{\left(\frac{\beta}{z}q^{j+n+k-m}\right)_{m-k}}. \tag{2.37}$$

One can check these coefficients $r_{i,j}$ coincide with the results in Remark 2.4.

3 Jackson integral representation of solutions to the q-KZ equation

3.1 Jackson integral of symmetric Selberg type

Solutions to the q-KZ equation allow a representation by the Jackson integral. In this subsection we review the results of [29] which are needed later.

In the construction of solutions to the q-KZ equation for $U_{\mathsf{v}}\big(A_1^{(1)}\big)$ by M. Ito [29], the N-tuple Jackson integral with the integration variables $z = (z_1, \ldots, z_N)$

$$\langle \phi(z), \xi \rangle = (1-t)^N \sum_{\nu \in \mathbb{Z}^N} [\phi(z)\Phi(z)\Delta(1,z)]_{z_i=\xi_i t^{\nu_i}} \tag{3.1}$$

is considered.[12] Here,

$$\Delta(q,z) := \prod_{1 \le i < j \le N} \left(z_i - q^{-1}z_j\right), \tag{3.2}$$

and

$$\Phi(z) = \prod_{i=1}^{N} z_i^{\alpha} \frac{(ta_1^{-1}z_i, ta_2^{-1}z_i; t)_\infty}{(b_1 z_i, b_2 z_i; t)_\infty} \cdot \prod_{1 \le i < j \le N} z_i^{2(\log_t q)-1} \frac{(tq^{-1}z_j/z_i; t)_\infty}{(qz_j/z_i; t)_\infty} \tag{3.3}$$

is a common weight factor with four parameters a_1, a_2, b_1 and b_2. Note that $\Delta(1,z)$ is nothing but the Vandermonde determinant. In the pairing (3.1), $\xi = (\xi_1, \ldots, \xi_N)$ has the meaning of parameters for an integration cycle. On the other hand, $\phi(z)$ defines a cocycle function. In order to give a basis of the space of cohomology classes, in [29] the symmetric polynomial

$$\tilde{E}_{k,i}(a,b;z) = \frac{1}{\Delta(1,z)}\mathcal{A}(E_{k,i}(a,b;z)) = \mathcal{S}\frac{E_{k,i}(a,b;z)}{\Delta(1,z)}, \qquad a,b \in \mathbb{C}^\times, \tag{3.4}$$

---

[12]In view of the application in the next section, we exchange q and t in the original paper [29].

was employed by introducing

$$E_{k,i}(a,b;z) = z_1 \cdots z_k \ \Delta(q,z) \prod_{j=1}^{N-i}(1-bz_j) \prod_{j=N-i+1}^{N}(1-a^{-1}z_j).$$ (3.5)

Here $\mathcal{S}/\mathcal{A}$ denotes the symmetrization/anti-symmetrization of the variables $(z_1, \ldots, z_N)$, respectively. In particular, $e_i(a,b;z) = \tilde{E}_{0,i}(a,b;z)$, $0 \le i \le N$, gives a basis which is called Matsuo basis [39, 40] (see also [41, 55]).

The function $c(x) = x^{\log_t(\frac{a}{b})} \frac{\vartheta(ax;t)}{\vartheta(bx;t)}$ with $\vartheta(x;t) = (x;t)_\infty (t/x;t)_\infty$ satisfies $c(tx) = c(x)$. Such a function is called pseudo constant. The following lemma shows that the function $\Phi(z)\Delta(1,z)$ in (3.1) can be written in a symmetric form, up to some pseudo constant factor. Due to this property, the integral is sometimes called Jackson integral of symmetric Selberg type.

Lemma 3.1. *Let* $\tau = \log_t q$ *and* $i,j \in [N] := \{1,2,\ldots,N\}$. *We have*

$$\prod_{i \ne j} \frac{(tq^{-1}z_j/z_i;t)_\infty}{(tz_j/z_i;t)_\infty} = C(z)\Delta(1,z) \prod_{i=1}^{N} z_i^{-\tau(N-1)} \prod_{i<j} z_i^{2\tau-1} \frac{(tq^{-1}z_j/z_i;t)_\infty}{(qz_j/z_i;t)_\infty},$$ (3.6)

where $C(z) = \prod_{i<j}\left(\frac{z_j}{z_i}\right)^{\tau} \frac{\vartheta(tq^{-1}z_i/z_j;t)}{\vartheta(tz_i/z_j;t)}$ *is a pseudo constant for each variable* z_i.

Proof. The left-hand side is computed as follows:

$$\prod_{i \ne j} \frac{(tq^{-1}z_j/z_i;t)_\infty}{(tz_j/z_i;t)_\infty} = \prod_{i<j} \frac{(tq^{-1}z_i/z_j;t)_\infty (tq^{-1}z_j/z_i;t)_\infty}{(tz_i/z_j;t)_\infty (tz_j/z_i;t)_\infty}$$

$$= \prod_{i<j} \frac{\vartheta(tq^{-1}z_i/z_j;t)(z_j/z_i;t)_\infty (tq^{-1}z_j/z_i;t)_\infty}{\vartheta(tz_i/z_j;t)(qz_j/z_i)_\infty (tz_j/z_i;t)_\infty}$$

$$= C(z) \prod_{i<j} \left(\frac{z_j}{z_i}\right)^{-\tau}(1-z_j/z_i) \frac{(tq^{-1}z_j/z_i;t)_\infty}{(qz_j/z_i;t)_\infty}$$

$$= C(z)\Delta(1,z) \prod_{i=1}^{N} z_i^{-\tau(N-1)} \prod_{i<j} z_i^{2\tau-1} \frac{(tq^{-1}z_j/z_i;t)_\infty}{(qz_j/z_i;t)_\infty}.$$

The last line follows from

$$\prod_{i<j}\left(\frac{z_j}{z_i}\right)^{-\tau} z_i^{-1} = \prod_{i<j}(z_j z_i)^{-\tau} z_i^{2\tau-1} = \prod_{i=1}^{N} z_i^{-\tau(N-1)} \prod_{i<j} z_i^{2\tau-1}. \qquad \blacksquare$$

For the cocycle function $\phi(z)$ in the pairing (3.1), we will take the Matsuo basis $e_0(a_2,b_1), \ldots, e_N(a_2,b_1)$.[13] Namely, we consider the following $\mathbb{C}^{N+1}$-valued function

$$\Psi = [\Psi_{-n}, \ldots, \Psi_m] = [\langle e_N(a_2,b_1), \xi \rangle, \ldots, \langle e_0(a_2,b_1), \xi \rangle].$$ (3.7)

In order to identify the instanton partition function to be introduced in the next section with the Jackson integral, it is important to keep the factorized structure of the integrand as far as possible, hence we use the following expression:

$$\hat{e}_k(a,b;z) = e_{N-k}(a,b;z)$$ (3.8)

---

[13]Since the common factor (3.3) is symmetric under the exchanges $a_1 \leftrightarrow a_2$ and $b_1 \leftrightarrow b_2$, there are four choices of the Matsuo basis. Here we follow the choice in [29].

$$= [k]_{q^{-1}}! [N-k]_{q^{-1}}! \sum_{I \sqcup J = [N], |J| = k} \prod_{i \in I} \left(1 - \frac{z_i}{a}\right) \prod_{j \in J} (1 - b z_j) \prod_{i \in I, j \in J} \frac{z_j - q^{-1} z_i}{z_j - z_i},$$

where the sum is taken over the $\binom{N}{k}$ terms corresponding to disjoint union $I \sqcup J = [N] = \{1, 2, \ldots, N\}$ with $|I| = N - k$, $|J| = k$. The formula (3.8) follows from Proposition G.1 in Appendix G for the case $s = 2$ with $f_1(x) = 1 - b_1 x$ and $f_2(x) = 1 - \frac{x}{a_2}$.[14] Thus the summation in Jackson integral is written as the sum over the cone (3.22) and additional $2^N = \sum_{k=0}^{N} \binom{N}{k}$ sums. As we will see in the next section, in the relevant instanton partition function, the first sum corresponds to the sum over the two Young diagrams with even lengths for all the columns and the second sum comes from their even/odd variants depending on the number of columns with odd length.

The expression (3.8) looks like rational, but it is a polynomial in $z = (z_1, \ldots, z_N)$. In fact, the base $e_k(a, b; z)$ can be characterized as the linear combination of the elementary symmetric functions in z having the following specialization [29, 32]:

$$e_k\left(a, b; (x, xq \ldots, xq^{N-1})\right) = [N]_{q^{-1}}! \prod_{i=1}^{N-k} \left(1 - q^i b x\right) \prod_{i=1}^{k-1} \left(1 - q^i \frac{x}{a}\right). \tag{3.9}$$

In [29], the following fact is proved.

Theorem 3.2 ([29]). *With respect to the shifts $T_\alpha \colon \alpha \to \alpha + 1$ and $T_i = T_{t,b_i}^{-1} T_{t,a_i}$, $i = 1, 2$, the Jackson integral Ψ satisfies the following system of difference equations:*

$$T_\alpha \Psi = \Psi K_0, \qquad T_1 \Psi = \Psi K_1, \qquad T_2 \Psi = \Psi K_2, \tag{3.10}$$

where

$$K_0 = R^{-1} A R = D_2 A D_2^{-1}, \qquad K_1 = R^{-1} D_1, \qquad K_2 = D_2(T_2 R),$$
$$D_1 = \left(\left(t^\alpha q^{N-1}\right)^{N-i} \delta_{ij}\right)_{0 \le i, j \le N}, \qquad D_2 = \left(\left(t^\alpha q^{N-1}\right)^i \delta_{ij}\right)_{0 \le i, j \le N}, \tag{3.11}$$

and δ_{ij} is the Kronecker symbol. An explicit form of the matrix R is given by[15]

$$R = L_R D_R U_R,$$
$$L_R = \left(l_{ij}^R\right)_{0 \le i, j \le N}, \qquad D_R = \left(\delta_{ij} d_j^R\right)_{0 \le i, j \le N}, \qquad U_R = \left(u_{ij}^R\right)_{0 \le i, j \le N}, \tag{3.12}$$

where

$$l_{ij}^R = \begin{bmatrix} N-j \\ N-i \end{bmatrix}_{q^{-1}} \frac{(-1)^{i-j} q^{-\binom{i-j}{2}} (a_2 b_2 q^j; q)_{i-j}}{\left(a_1^{-1} a_2 q^{-(N-2j-1)}; q\right)_{i-j}}, \qquad N \ge i \ge j \ge 0,$$

$$d_j^R = \frac{\left(a_1 a_2^{-1} q^{-j}; q\right)_{N-j} (a_2 b_1; q)_j}{(a_1 b_2; q)_{N-j} \left(a_1^{-1} a_2 q^{-(N-j)}; q\right)_j}, \qquad 0 \le j \le N,$$

$$u_{ij}^R = \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-1}} \frac{\left(a_1 b_1 q^{N-j}; q\right)_{j-i}}{\left(a_1 a_2^{-1} q^{N-i-j}; q\right)_{j-i}}, \qquad 0 \le i \le j \le N, \tag{3.13}$$

and other elements are zero. $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}}$ denotes q^{-1}-binomial coefficient (see (1.17)). The matrix A is also given in a Gauss decomposed form as

$$A = L_A D_A U_A,$$

---

[14]As noted in [48], these cocycle functions naturally arise as the Bethe vector. See [1] for the developments based on the geometric method.

[15]The matrix R in this section is slightly different from the R matrix in the previous section. See Appendix C for a summary of various R-like matrices used in this paper.

$$L_A = \left(l_{ij}^A\right)_{0 \leq i,j \leq N}, \qquad D_A = \left(\delta_{ij} d_j^A\right)_{0 \leq i,j \leq N}, \qquad U_A = \left(u_{ij}^A\right)_{0 \leq i,j \leq N}, \tag{3.14}$$

where the nonzero elements are

$$l_{ij}^A = (-1)^{i-j} q^{\binom{N-i}{2}-\binom{N-j}{2}} \begin{bmatrix} N-j \\ N-i \end{bmatrix}_q \frac{(a_2 b_2 q^j; q)_{i-j}}{(t^\alpha a_2 b_2 q^{2j}; q)_{i-j}}, \qquad N \geq i \geq j \geq 0,$$

$$d_j^A = a_1^{N-j} a_2^j q^{\binom{j}{2}+\binom{N-j}{2}} \frac{(t^\alpha; q)_j (t^\alpha a_2 b_2 q^{2j}; q)_{N-j}}{(t^\alpha a_2 b_2 q^{j-1}; q)_j (t^\alpha a_1 a_2 b_1 b_2 q^{n+j-1}; q)_{N-j}}, \qquad 0 \leq j \leq N,$$

$$u_{ij}^A = \left(-t^\alpha a_1^{-1} a_2\right)^{j-i} q^{\binom{j}{2}-\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{(a_1 b_1 q^{N-j}; q)_{j-i}}{(t^\alpha a_2 b_2 q^{2i}; q)_{j-i}}, \qquad 0 \leq i \leq j \leq N. \tag{3.15}$$

Remark 3.3. The difference equations for the shift $T_i = T_{t,b_i}^{-1} T_{t,a_i}$ are nothing but the traditional (original) q-KZ equation. The components of the R matrix appearing in this type of q-KZ equation are nothing but the connection coefficients between two Matsuo bases $\{e_i(a,b)\}$ with different parameters a_i and b_i. Due to the specialization given in equation (3.9), the computation can be reduced to the connection problem of single variable polynomials as in [50]. On the other hand, the matrix A for the shift of $T_\alpha \colon \alpha \to \alpha + 1$ is obtained in a different manner and the basis $E_{k,i}$ was introduced for this purpose in [29]. It is the q-KZ equation for T_α that is related to the Shakirov's equation.

The following relation between the matrices R and A seems to be noticed long ago at least among specialists (e.g., [30]). We give a proof since the relation plays an important role in Section 3.3.

Proposition 3.4. *We have*

$$A = sT(R)D, \tag{3.16}$$

where s is a scalar and D is a diagonal matrix given by

$$s = q^{N(N-1)/2} (a_1 a_2 b_2)^N \frac{(t^\alpha; q)_N}{(t^\alpha q^{N-1} a_1 a_2 b_1 b_2; q)_N}, \qquad D = \left((a_1 b_2)^{-i} \delta_{ij}\right)_{0 \leq i,j \leq N}. \tag{3.17}$$

T is a shift operator acting only on the parameters a_2, b_2 as

$$T = \left\{a_2 \to a_2\left(t^\alpha q^{N-1} a_1 b_2\right), \ b_2 \to b_2\left(t^\alpha q^{N-1} a_1 b_2\right)^{-1}\right\}. \tag{3.18}$$

Proof. Using the relation $\left(x^{-1}; q\right)_k = (-1)^k q^{k(k-1)/2} x^{-k} \left(xq^{1-k}; q\right)_k$ and the explicit forms (3.12) and (3.14), one can check

$$L_A = T(L_R), \qquad D_A = sT(D_R)D, \qquad U_A = D^{-1} T(U_R)D. \tag{3.19}$$

Then we obtain

$$A = L_A D_A U_A = T(L_R) sT(D_R) D D^{-1} T(U_R) D = sT(L_R D_R U_R) D = sT(R)D, \tag{3.20}$$

as desired. ∎

The compatibility of the dual pair of difference equations in Theorem 3.2 implies $R D_2 A = A R D_2$. In Appendix D, we give a direct check of the compatibility based on the matrix inversion formula of Andrews and Bressoud. It is amusing that the compatibility condition follows from Bailey's transformation formula for terminating very-well-poised balanced series $_{10}W_9$.

3.2   Lattice truncation by a choice of the cycle

The sum in the Jackson integral (3.1) considered in [29] is bilateral, namely it is taken over $\{\nu_i\} \in \mathbb{Z}^N$. On the other hand, the Nekrasov partition function involves a sum over Young diagrams. For the function Ψ, the discrepancy is remedied by an appropriate choice of the cycle $\xi = (\xi_1, \ldots, \xi_N)$. The suitable cycle is already known (see [32, equation (3.19)] and [31, equations (4.4) and (4.5)]). In fact, we have the following.

Proposition 3.5. *For ξ given by*

$$\xi = \xi_{m,n} = (\underbrace{a_2, a_2 q, \ldots, a_2 q^{n-1}}_{n}, \underbrace{a_1, a_1 q, \ldots, a_1 q^{m-1}}_{m}), \qquad m + n = N, \tag{3.21}$$

the lattice summation over $\{z_i = \xi_i t^{\nu_i}\}$ is truncated to a cone

$$\begin{aligned}
& 0 \le \nu_1 \le \nu_2 \le \cdots \le \nu_{n-1} \le \nu_n, \\
& 0 \le \nu_{n+1} \le \nu_{n+2} \le \cdots \le \nu_{N-1} \le \nu_N,
\end{aligned} \tag{3.22}$$

and we can normalize the $\mathbb{C}^{N+1}$-valued function (3.7) as

$$\Psi^T = \begin{bmatrix} \Psi_m \\ \vdots \\ \Psi_0 \\ \vdots \\ \Psi_{-n} \end{bmatrix} = \begin{bmatrix} * & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & * & * & * & \cdots \\ & \ddots & \ddots & \ddots & \cdots \\ O & & * & * & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ \Lambda \\ \Lambda^2 \\ \Lambda^3 \\ \vdots \end{bmatrix}, \tag{3.23}$$

where $\Lambda = t^\alpha$.

When we impose the mass truncation condition $d_2 = q^{-m}$, $d_3 = q^{-n}$ the partition function becomes a Laurent polynomial in x. With the normalization (3.23) the component Ψ_i, $-n \le i \le m$, is to be identified with the coefficient of x^i-term in the Laurent polynomial (see also Figure 1 for the structure of the partition function).

Proof. The condition (3.22) easily follows from the explicit form (3.3) of the function $\Phi(z)$. We will show (3.23). From the expression (3.8) for $e_k = e_k(a_2, b_1, z)$ with

$$z = (\underbrace{a_2 t^{\nu_1}, a_2 q t^{\nu_2}, \ldots, a_2 q^{n-1} t^{\nu_n}}_{n}, \underbrace{a_1 t^{\nu_{n+1}}, a_1 q t^{\nu_{n+2}}, \ldots, a_1 q^{m-1} t^{\nu_{n+m}}}_{m}),$$

we see that the leading term in Λ-expansion of each component of Ψ has a single contribution arising from a specific $\{\nu_i\}$ and $J \subset [N]$, $|J| = k$. Explicitly, for $0 \le k \le m$, we have

$$e_k = g_k + O(\Lambda^1), \qquad g_k = \frac{(q^{-1}; q^{-1})_m (b_1 a_2; q)_n (q^{-N+k}; q)_n (q^k b_1 a_1; q)_{m-k} (q^{-n} \frac{a_1}{a_2}, q)_k}{(1 - q^{-1})^N},$$

arising from $J = \{n+1, n+2, \ldots, n+k\}$, $(\nu_i) = (0^N)$, and for $0 \le l \le n$, we have

$$e_{m+l} = h_l \Lambda^l + O(\Lambda^{l+1}),$$
$$h_l = \frac{(q^{-1}; q^{-1})_{n-l} (q^{-1}; q^{-1})_{m+l} (t; q)_l (b_1 a_2; q)_{n-l} (q^{l-n} \frac{a_1}{a_2}; q)_m}{(1 - q^{-1})^N},$$

arising from $J = \{n-l+1, n-l+2, \ldots, N\}$, $(\nu_i) = (0^l, 1^{n-l}, 0^m)$. Dividing by the scalar factor

$$h_0 = \frac{(q^{-1}; q^{-1})_n (q^{-1}; q^{-1})_m (b_1 a_2; q)_n (q^{-n} \frac{a_1}{a_2}; q)_m}{(1 - q^{-1})^N},$$

we have the expression (3.23).   ∎

For the function Ψ normalized as above, the two types of q-KZ equations in Theorem 3.2 can be written as

$$\left(T_{t,a_2}^{-1}T_{t,b_2}\Psi\right)\tilde{D}_1 R = \Psi, \qquad T_{t,\Lambda}\Psi = \Psi\tilde{A}\tilde{D}_2, \tag{3.24}$$

where the matrices $\tilde{A}$ and R are given as the following connection matrices[16]

$$[e_0(a_2,b_1),\ldots,e_N(a_2,b_1)] = [e_N(a_1,b_2),\ldots,e_0(a_1,b_2)]R,$$
$$[e_N(a,b),\ldots,e_0(a,b)]\underset{\substack{a=a_1b_1\Lambda\\b=q^{N-1}a_2b_2}}{} = [e_0(c,d),\ldots,e_N(c,d)]\underset{\substack{c=q^{1-N}\\d=\Lambda^{-1}}}{}\tilde{A},$$

and $\tilde{D}_1$, $\tilde{D}_2$ are diagonal matrices given by

$$\tilde{D}_1 = (-1)^N q^{\frac{(N-1)(n-m)}{2}}\left(\frac{a_2}{a_1}\right)^N \prod_{i=0}^{n-1}\frac{1-q^{m-n+1+i}\frac{a_1}{a_2}}{1-q^i a_2 b_1}$$

$$\times \prod_{i=0}^{m-1}\frac{1-q^i a_1 b_2}{1-q^{n-m+1+i}\frac{a_2}{a_1}}\cdot \mathrm{diag}\left(\left\{\left(\Lambda q^{N-1}\right)^{m-i}\right\}_{i=0}^N\right),$$

$$\tilde{D}_2 = \left(\frac{q^m a_1}{a_2}\right)^n \prod_{i=0}^{N-1}\frac{1-q^i\Lambda}{1-q^{N-1+i}a_1a_2b_1b_2\Lambda}\cdot\mathrm{diag}\left(\left\{\left(a_2 b_1\right)^{N-i}\right\}_{i=0}^N\right).$$

3.3 Base-fibre duality of the q-KZ equation

Let us look at the two types of the q-KZ equation explicitly in the case at our hand. It turns out that these q-KZ equations are related by the duality which exchanges the instanton expansion parameter Λ and the Coulomb parameter Q. The q-KZ equation arising from Shakirov's equation was given by

$$\psi_j(\Lambda) = \sum_{i=-n}^m \psi_i\left(\frac{\Lambda}{t}\right)\left(Q^\vee\right)^i r_{i,j}(\Lambda), \qquad j = -n,\ldots,m, \tag{3.25}$$

where $Q^\vee := (qtQ)^{-1}$ is the shift parameter for x and $[r_{i,j}(\Lambda)]_{i,j=-n}^m$ is the R-matrix obtained by the specialization $d_2 = q^{-m}$, $d_3 = q^{-n}$. Since the leading term $r_{i,j}(0)$ of the expansion $r_{i,j}(\Lambda) = r_{i,j}(0) + O(\Lambda)$ is upper triangular with $r_{i,i}(0) = q^{i(i+1)}$, the equation (3.25) has a set of fundamental solutions of the form

$$\psi_j^{(i)}(\Lambda) = \Lambda^{\rho_i}Y_{i,j}(\Lambda), \qquad Y_{i,j}(\Lambda) = Y_{i,j}(0) + O(\Lambda), \qquad i,j = -n,\ldots,m, \tag{3.26}$$

where $t^{\rho_i} = \left(Q^\vee\right)^i r_{i,i}(0) = \left(Q^\vee\right)^i q^{i(i+1)}$ and the leading coefficients $Y_{i,j}(0)$ are upper triangular. One can normalize them as $Y_{i,i}(0) = 1$. Then the equation for $Y_{i,j}(\Lambda)$ is written as

$$Y_{k,j}(\Lambda) = \sum_{i=-n}^m \left(Q^\vee\right)^{-k}q^{-k(k+1)}Y_{k,i}\left(\frac{\Lambda}{t}\right)\left(Q^\vee\right)^i r_{i,j}(\Lambda), \qquad k,j = -n,\ldots,m. \tag{3.27}$$

Since $r_{i,j}(\Lambda)$ is independent of $Q^\vee$, this equation depends on the parameter $Q^\vee$ only through simple power factors. The following proposition[17] shows that the fundamental solution $Y(\Lambda) = Y\left(\Lambda, Q^\vee\right)$ satisfies another q-KZ equation with respect to the parameter $Q^\vee$.

---

[16]Contrary to the matrix R, the fact that the matrix $\tilde{A}$ is obtained as the connection matrix of Matsuo base is not obvious from the definition.

[17]Proposition 3.6 is a consequence of Theorems 3.2 and 4.1. The proof is omitted, since it is not used in other parts of the paper.
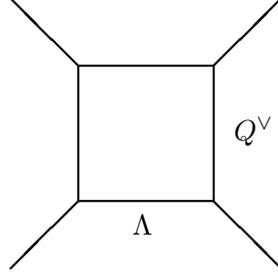
Figure 2. Geometric engineering of U(2) gauge theory by local $\mathbb{P}^1 \times \mathbb{P}^1$. Λ is the Kähler parameter of the base $\mathbb{P}^1$ and $Q^\vee$ is that of the fibre. We can introduce four matter hypermultiplets by blow-ups.

Proposition 3.6. *The fundamental solution* $Y(\Lambda, Q^\vee)$ *satisfies the following equation:*

$$\sum_{j=-n}^{m} Y_{i,j}\left(\Lambda, \frac{Q^\vee}{t}\right)\left(\frac{\Lambda d_1}{q^{m+2}}\right)^j \tilde{r}_{j,k}(Q^\vee) = v_i(\Lambda, Q^\vee) Y_{i,k}(\Lambda, Q^\vee), \qquad -n \leq i, k \leq m, \quad (3.28)$$

where

$$\tilde{r}_{i,j}(Q^\vee) = r_{i,j}(\Lambda)\big|_{\Lambda \mapsto \frac{q^{m+2}}{d_1}Q^\vee} \tag{3.29}$$

and

$$v_i(\Lambda, Q^\vee) = q^{i(i+1)}\left(\frac{\Lambda d_1}{q^{m+2}}\right)^i \frac{(Q^\vee q^{2+2i}; q)_{m-i}(d_4 Q^\vee q^{1-n}; q)_{n+i}}{(d_1^{-1} Q^\vee q^{2+i}; q)_{m-i}(Q^\vee q^{1-n+i}; q)_{n+i}}. \tag{3.30}$$

The equations (3.27) and (3.28) look very similar. To make this similarity more explicit we make the gauge transformation $Y_{i,j}(\Lambda, Q^\vee) = G_i(Q^\vee)\tilde{Y}_{i,j}(\Lambda, Q^\vee)$ with

$$G_i(Q^\vee) = \prod_{k=1}^{\infty} \frac{(Q^\vee q^{2+2i} t^k; q)_{m-i}(d_4 Q^\vee q^{1-n} t^k; q)_{n+i}}{(d_1^{-1} Q^\vee q^{2+i} t^k; q)_{m-i}(Q^\vee q^{1-n+i} t^k; q)_{n+i}}. \tag{3.31}$$

Since

$$G_i^{-1}\left(\frac{Q^\vee}{t}\right) G_i(Q^\vee) v_i(\Lambda, Q^\vee) = q^{i(i+1)}\left(\frac{\Lambda d_1}{q^{m+2}}\right)^i,$$

one can rewrite the equation (3.28) for $-n \leq i, k \leq m$ as

$$\sum_{j=-n}^{m} \tilde{Y}_{i,j}\left(\Lambda, \frac{Q^\vee}{t}\right)\left(\frac{\Lambda d_1}{q^{m+2}}\right)^j \tilde{r}_{j,k}(Q^\vee) = q^{i(i+1)}\left(\frac{\Lambda d_1}{q^{m+2}}\right)^i \tilde{Y}_{i,k}(\Lambda, Q^\vee). \tag{3.32}$$

This equation is exactly the same form as (3.27) under the replacement

$$\Lambda \mapsto \frac{q^{m+2}}{d_1}Q^\vee, \qquad Q^\vee \mapsto \frac{d_1}{q^{m+2}}\Lambda. \tag{3.33}$$

Remark 3.7. For the five-dimensional quantum Seiberg–Witten curve with coefficient matrix $\mathcal{D}$ or $\mathcal{D}_3$ in the proof of Proposition 5.1 in [10], the positions of the external lines (tentacles of the corresponding amoeba) are given by

$$(x,p) = (0,d_3), (0,d_4), \left(\infty, \frac{1}{d_1}\right), \left(\infty, \frac{1}{d_2}\right), (1,0), (d_3 d_4 \Lambda \mu, 0), \left(\frac{\mu q}{d_1 d_2}, \infty\right), \left(\frac{\Lambda}{q}, \infty\right)$$

or

$$(x,p) = (0,d_3), \left(0, \frac{1}{\mu d_4}\right), \left(\infty, \frac{1}{d_1}\right), \left(\infty, \frac{d_2}{q^2 \mu}\right), \left(\frac{q}{d_2}, 0\right), (d_3 \Lambda, 0), \left(\frac{1}{d_1}, \infty\right), \left(\frac{d_4 \Lambda}{q}, \infty\right),$$

where $\mu = Q^\vee$. Then, the exchange $\Lambda \leftrightarrow Q^\vee$ can be seen as the exchange of two external lines at $p = \infty$ (for $\mathcal{D}$) or as the exchange of x and p (for $\mathcal{D}_3$) respectively.

Example 3.8. For $d_2 = q^{-m}$, $d_3 = q^{-n}$ with $(m,n) = (1,0)$, the solution $Z = Z(\Lambda, x)$ of Shakirov's equation

$$T_{t,\Lambda} Z(\Lambda, x) = \mathcal{H}_S T_{qtQ,x}^{-1} Z(\Lambda, x) \tag{3.34}$$

is explicitly given by the Heine's series $_2\phi_1\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; t, z\right]$ with base t as

$$Z = {}_2\phi_1 \left[\begin{matrix} \frac{1}{d_1}, \frac{Qt}{d_4} \\ \frac{Qt}{q} \end{matrix}; t, \frac{d_1 d_4 \Lambda}{q^2}\right] - \frac{1-d_1}{1-\frac{q}{Qt}} \; {}_2\phi_1 \left[\begin{matrix} \frac{t}{d_1}, \frac{Qt}{d_4} \\ \frac{Qt^2}{q} \end{matrix}; t, \frac{d_1 d_4 \Lambda}{q^2}\right] x$$

$$= {}_2\phi_1 \left[\begin{matrix} a, z_2 \\ bz_2 \end{matrix}; t, z_1\right] + \frac{bz_2\left(1-\frac{1}{a}\right)}{1-bz_2} \; {}_2\phi_1 \left[\begin{matrix} ta, z_2 \\ tbz_2 \end{matrix}; t, z_1\right] x, \tag{3.35}$$

where we put $\Lambda = \frac{aqz_1}{b}$, $Q = \frac{bqz_2}{t}$, $d_1 = \frac{1}{a}$, $d_4 = bq$ for simplicity. Then the coefficients y_0, y_1 of $Z = y_0 + y_1 x$ satisfy

$$\left(1 - \frac{az_1}{b}\right) T_{t,z_1} Y = Y M(z_1), \tag{3.36}$$

$$\left(1 - \frac{t}{bz_2}\right) T_{t,z_2}^{-1} Y = Y M\left(\frac{t}{z_2}\right), \tag{3.37}$$

where

$$Y = [y_0, y_1], \qquad M(u) = \begin{bmatrix} 1 & \\ & \frac{az_1}{bz_2} \end{bmatrix} \begin{bmatrix} 1 - \frac{u}{b} & 1 - \frac{1}{a} \\ 1 - \frac{1}{b} & 1 - \frac{1}{au} \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{3.38}$$

The equation (3.36) is the truncated form of (3.34), while the equation (3.37) follows from

$$b^{-1}\left(1 - \frac{t}{z_2}\right) T_{t,z_2}^{-1} \tilde{Y} = \tilde{Y} M\left(\frac{t}{z_2}\right), \tag{3.39}$$

where $\tilde{Y} = [\tilde{y}_0, \tilde{y}_1]$, $\tilde{Z} = \tilde{y}_0, +\tilde{y}_1 x$, and

$$\tilde{Z} := {}_2\phi_1 \left[\begin{matrix} b, z_1 \\ az_1 \end{matrix}; t, z_2\right] + \frac{bz_2(1-\frac{1}{a})}{1-az_1} {}_2\phi_1 \left[\begin{matrix} tb, z_1 \\ taz_1 \end{matrix}; t, z_2\right] x \frac{(bz_2, z_1; t)_\infty}{(az_1, z_2; t)_\infty} Z. \tag{3.40}$$

For $x = 0$, the last relation in (3.40) is Heine's transformation. Note that the ratio $\tilde{Z}/Z$ is independent of x, which explains how a single function can satisfy the dual pair of equations (3.36) and (3.37).

$$\left(\lambda^{(1)}, \lambda^{(2)}\right) = \left(\quad , \quad \right), \qquad \left(\lambda^{(1)}\right)^{\vee} = (4, 2, 1, 1), \qquad \left(\lambda^{(2)}\right)^{\vee} = (3, 2).$$

Figure 3. $\mathbb{Z}_2$-coloring of a pair of Young diagrams.

4 Nekrasov partition function as Jackson integral

Let us show that the K-theoretic Nekrasov partition function from the affine Laumon space agrees with the Matsuo bases. When $n = 2$ the general formula (F.28) derived in Appendix F for the orbifolded Nekrasov factor reduces to

$$\mathsf{N}^{(0|2)}_{\lambda,\mu}(u|q,t) = \prod_{i,j=1}^{\infty} \frac{\left[uq^{j-i}t^{1+\lfloor \frac{\mu_i^{\vee}-\lambda_j^{\vee}}{2}\rfloor}; t\right]_{\infty}}{\left[uq^{j-i-1}t^{1+\lfloor \frac{\mu_i^{\vee}-\lambda_j^{\vee}}{2}\rfloor}; t\right]_{\infty}} \frac{\left[uq^{j-i-1}t; t\right]_{\infty}}{\left[uq^{j-i}t; t\right]_{\infty}}, \tag{4.1}$$

$$\mathsf{N}^{(1|2)}_{\lambda,\mu}(u|q,t) = \prod_{i,j=1}^{\infty} \frac{\left[uq^{j-i}t^{\frac{1}{2}+\lfloor \frac{\mu_i^{\vee}-\lambda_j^{\vee}+1}{2}\rfloor}; t\right]_{\infty}}{\left[uq^{j-i-1}t^{\frac{1}{2}+\lfloor \frac{\mu_i^{\vee}-\lambda_j^{\vee}+1}{2}\rfloor}; t\right]_{\infty}} \frac{\left[uq^{j-i-1}t^{\frac{1}{2}}; t\right]_{\infty}}{\left[uq^{j-i}t^{\frac{1}{2}}; t\right]_{\infty}}, \tag{4.2}$$

where $\mu_i^{\vee}$ and $\lambda_j^{\vee}$ denote the transpose of the Young diagrams. For the definition of $[u; t]_{\infty}$, we refer to (F.5). When one of the partitions is empty, the formula simplifies to

$$\mathsf{N}^{(0|2)}_{\lambda,\varnothing}(u|q,\kappa) = \prod_{i\geq 1}\left[uq^{i-1}; \kappa^2\right]_{\lfloor \frac{\lambda_i^{\vee}+1}{2}\rfloor}, \qquad \mathsf{N}^{(1|2)}_{\lambda,\varnothing}(u|q,\kappa) = \prod_{i\geq 1}\left[uq^{i-1}\kappa; \kappa^2\right]_{\lfloor \frac{\lambda_i^{\vee}}{2}\rfloor}, \tag{4.3}$$

$$\mathsf{N}^{(0|2)}_{\varnothing,\mu}(u|q,\kappa) = \prod_{i\geq 1}\left[uq^{-i}\kappa^{-2\lfloor \frac{\mu_i^{\vee}}{2}\rfloor}; \kappa^2\right]_{\lfloor \frac{\mu_i^{\vee}}{2}\rfloor},$$

$$\mathsf{N}^{(1|2)}_{\varnothing,\mu}(u|q,\kappa) = \prod_{i\geq 1}\left[uq^{-i}\kappa^{1-2\lfloor \frac{\mu_i^{\vee}+1}{2}\rfloor}; \kappa^2\right]_{\lfloor \frac{\mu_i^{\vee}+1}{2}\rfloor}, \tag{4.4}$$

where $t = \kappa^{-2}$. Note that $\lfloor \frac{m+1}{2}\rfloor + \lfloor \frac{m}{2}\rfloor = m$. When $\lambda_i^{\vee}$, $\mu_i^{\vee}$ are even, $\mathsf{N}^{(0|2)}$ and $\mathsf{N}^{(1|2)}$ have the same number of factors. But when they are odd, $\lambda_i^{\vee}$ contributes more to $\mathsf{N}^{(0|2)}$, while $\mu_i^{\vee}$ does to $\mathsf{N}^{(1|2)}$. This is due to a difference of the coloring of λ and μ (see Figure 3).

Recall that by the localization formula the Nekrasov partition function with a surface defect is given by

$$\mathcal{Z}_{\mathrm{AL}} = \mathcal{Z}_{\mathrm{AL}}\left(\begin{array}{c} u_1, u_2 \\ v_1, v_2 \\ w_1, w_2 \end{array} \middle| x_1, x_2 \middle| q, t \right) \tag{4.5}$$

$$= \sum_{\left(\lambda^{(1)}, \lambda^{(2)}\right)} \prod_{i,j=1}^{2} \frac{\mathsf{N}^{(j-i|2)}_{\varnothing,\lambda^{(j)}}(u_i/v_j|q,t)\mathsf{N}^{(j-i|2)}_{\lambda^{(i)},\varnothing}(v_i/w_j|q,t)}{\mathsf{N}^{(j-i|2)}_{\lambda^{(i)},\lambda^{(j)}}(v_i/v_j|q,t)} \cdot x_1^{|\lambda^{(1)}|_o+|\lambda^{(2)}|_e} x_2^{|\lambda^{(1)}|_e+|\lambda^{(2)}|_o},$$

where $\left(\lambda^{(1)}, \lambda^{(2)}\right)$ is a fixed point of the toric action on the affine Laumon space. The weight in the summation over the fixed points is $x_1^{|\lambda^{(1)}|_o+|\lambda^{(2)}|_e} x_2^{|\lambda^{(1)}|_e+|\lambda^{(2)}|_o}$, where $|\lambda|_o = \sum_{k\geq 1} \lambda_{2k-1}$

and $|\lambda|_e = \sum_{k\geq1} \lambda_{2k}$. The expansion parameters (x_1, x_2) are related to the physical parameters (Λ, x) by

$$
x_1 = -\frac{\sqrt{Qd_1d_2}}{\kappa}x, \qquad x_2 = -\sqrt{\frac{d_3d_4}{q^2Q}}\frac{\Lambda}{x}. \tag{4.6}
$$

By the relations (2.2) and (2.3), this specialization gives $x_1 = -t^{1/2}T_1^{1/2}T_2^{1/2}x$ and $x_2 = -t^{1/2}T_3^{1/2}T_4^{1/2}\frac{\Lambda}{x}$. We also employ the following specialization of the spectral parameters with $\kappa = t^{-\frac{1}{2}}$:

$$
u_1 = \frac{qQ}{d_3}, \quad u_2 = \frac{\kappa q}{d_1}, \qquad v_1 = 1, \quad v_2 = \frac{Q}{\kappa}, \qquad w_1 = \frac{1}{d_2}, \quad w_2 = \frac{Q}{d_4\kappa}. \tag{4.7}
$$

One can check that these specializations agree those of $\mathcal{F}^{(1)}$ defined in [10, Section 6]. The overall scaling of parameters by $Q^{1/2}$ is necessary for the matching of the parameters v_i of $\mathcal{Z}_{\mathrm{AL}}$ and $\mathcal{F}^{(1)}$. After the same scaling the specialization for the function $\mathcal{F}^{(1)}$ is

$$
u_1 = q^{1/2}\kappa^{-1}QT_3, \quad u_2 = q^{1/2}T_1^{-1}, \qquad w_1 = q^{-1/2}\kappa^{-1}QT_2, \quad w_2 = q^{-1/2}T_4^{-1}. \tag{4.8}
$$

By substituting (2.3), we can see the agreement with (4.7).

There are four possibilities of a specialization of the spectral parameters u_i and w_i due to the symmetry $d_1 \leftrightarrow d_2$, $d_3 \leftrightarrow d_4$ of the Hamiltonian $\mathcal{H}_{\mathrm{S}}$. But the symmetry is broken by choosing the specialization (4.7). The are paired into two SU(2) doublets; (d_1, d_3) and (d_2, d_4) in the case of the affine Laumon space. Such a rearrangement of SU(2) doublets of mass parameters between the five point conformal block with a degenerate field insertion and the four point current block was already observed in the four-dimensional theory (see [9, equation (5.33)]). From the viewpoint of the orbifold coloring of four mass parameters, the decomposition into the pairs (d_1, d_2) and (d_3, d_4) is by the parity of $\mathbb{Z}_2$ coloring. On the other hand, the decomposition into (d_1, d_3) and (d_2, d_4) corresponds to the fundamental and the anti-fundamental representations of SU(2) gauge symmetry.

Let us introduce a notation for the length of the columns of the Young diagrams,

$$
\big(\lambda^{(1)}\big)^\vee = (\ell_1, \ell_2, \dots), \qquad \big(\lambda^{(2)}\big)^\vee = (k_1, k_2, \dots). \tag{4.9}
$$

Omitting the normalization factors, which are the last factors in (4.1) and (4.2), we find the following contributions to the partition function (4.5):

1. Fundamental and anti-fundamental matter contribution

$$
\prod_{i,j=1}^{2} \mathsf{N}^{(j-i|2)}_{\varnothing,\lambda^{(j)}}(u_i/v_j|q,t)\mathsf{N}^{(j-i|2)}_{\lambda^{(i)},\varnothing}(v_i/w_j|q,t) \propto \prod_{i=1}^{\infty} \frac{\big[d_2q^{i-1}t^{1-\lfloor\frac{\ell_i+1}{2}\rfloor};t\big]_\infty}{\big[d_4^{-1}Qq^{1-i}t^{1+\lfloor\frac{\ell_i}{2}\rfloor};t\big]_\infty}
$$

$$
\times \prod_{i=1}^{\infty} \frac{\big[d_3q^{i-1}t^{-\lfloor\frac{k_i-1}{2}\rfloor};t\big]_\infty \big[Q^{-1}d_3q^{i-1}t^{-\lfloor\frac{\ell_i}{2}\rfloor};t\big]_\infty \big[Qd_2q^{i-1}t^{1-\lfloor\frac{k_i}{2}\rfloor};t\big]_\infty}{\big[d_1^{-1}Q^{-1}q^{1-i}t^{\lfloor\frac{k_i}{2}\rfloor};t\big]_\infty \big[d_1^{-1}q^{1-i}t^{\lfloor\frac{\ell_i+1}{2}\rfloor};t\big]_\infty \big[d_4^{-1}q^{1-i}t^{1+\lfloor\frac{k_i-1}{2}\rfloor};t\big]_\infty}. \tag{4.10}
$$

2. Vector multiplet contribution

$$
\Bigg(\prod_{i,j=1}^{2} \mathsf{N}^{(j-i|2)}_{\lambda^{(i)},\lambda^{(j)}}(v_i/v_j|q,t)\Bigg)^{-1} \propto \prod_{i,j=1}^{\infty} \frac{\big[q^{j-i-1}t^{1+\lfloor\frac{\ell_i-\ell_j}{2}\rfloor};t\big]_\infty}{\big[q^{j-i}t^{1+\lfloor\frac{\ell_i-\ell_j}{2}\rfloor};t\big]_\infty} \prod_{i,j=1}^{\infty} \frac{\big[q^{j-i-1}t^{1+\lfloor\frac{k_i-k_j}{2}\rfloor};t\big]_\infty}{\big[q^{j-i}t^{1+\lfloor\frac{k_i-k_j}{2}\rfloor};t\big]_\infty}
$$

$$
\times \prod_{i,j=1}^{\infty} \frac{\big[Q^{-1}q^{j-i-1}t^{1+\lfloor\frac{k_i-\ell_j-1}{2}\rfloor};t\big]_\infty}{\big[Q^{-1}q^{j-i}t^{1+\lfloor\frac{k_i-\ell_j-1}{2}\rfloor};t\big]_\infty} \prod_{i,j=1}^{\infty} \frac{\big[Qq^{j-i-1}t^{1+\lfloor\frac{\ell_i-k_j+1}{2}\rfloor};t\big]_\infty}{\big[Qq^{j-i}t^{1+\lfloor\frac{\ell_i-k_j+1}{2}\rfloor};t\big]_\infty}. \tag{4.11}
$$

We note that for a given pair of Young diagrams $\left(\lambda^{(1)}, \lambda^{(2)}\right)$, the normalization factor reduces the formal infinite products of ratio of $[u; t]_\infty$ in the above formulas to finite products due to the cancellations between the numerator and the denominator for sufficiently large i and j. To see it, note that we have $\ell_i = 0$ for $\mathsf{m} := \left(\lambda^{(1)}\right)_1 < i$ and $k_j = 0$ for $\mathsf{n} := \left(\lambda^{(2)}\right)_1 < j$. Later we will make a tuning of mass parameters so that we have $\mathsf{m} = m$ and $\mathsf{n} = n$ uniformly for all $\left(\lambda^{(1)}, \lambda^{(2)}\right)$ that contribute to the partition function. In terms of the variables[18]

$$z_i = q^{1-i} t^{\lfloor \frac{\ell_i}{2} \rfloor}, \qquad w_j = q^{1-j} t^{\lfloor \frac{k_j}{2} \rfloor}, \tag{4.12}$$

the finite product form of the matter contribution is

$$\prod_{i=1}^{\mathsf{m}} \frac{\left[d_2 z_i^{-1} t^{1-(\ell_i)}; t\right]_\infty \left[Q^{-1} d_3 z_i^{-1}; t\right]_\infty}{\left[d_4^{-1} Q z_i t; t\right]_\infty \left[d_1^{-1} z_i t^{(\ell_i)}; t\right]_\infty} \prod_{j=1}^{\mathsf{n}} \frac{\left[d_3 w_j^{-1} t^{1-(k_j)}; t\right]_\infty \left[Q d_2 w_j^{-1} t; t\right]_\infty}{\left[d_1^{-1} Q^{-1} w_j; t\right]_\infty \left[d_4^{-1} w_j t^{(k_j)}; t\right]_\infty}, \tag{4.13}$$

where we have used the identity

$$\left\lfloor \frac{K-L}{2} \right\rfloor = \left\lfloor \frac{K}{2} \right\rfloor - \left\lfloor \frac{L}{2} \right\rfloor + \{(K) - 1\} \cdot (L), \qquad K, L \in \mathbb{Z} \tag{4.14}$$

by the specialization $L = \pm 1$. Here (K) denotes the parity of an integer K; $(K) = 1$ for an odd integer and $(K) = 0$, otherwise.

For the vector multiplet contribution, we have to make an appropriate decomposition of $(i, j) \in \mathbb{N} \times \mathbb{N}$ into four regions $R_{\mathrm{I}}, \ldots, R_{\mathrm{IV}}$ according to the lengths of the first row of the Young diagrams; $\mathsf{m} = \left(\lambda^{(1)}\right)_1$ and $\mathsf{n} = \left(\lambda^{(2)}\right)_1$. For example, for a pair (ℓ_i, k_j), we define

$$R_{\mathrm{I}} = \{1 \le i \le \mathsf{m}, 1 \le j \le \mathsf{n}\}, \qquad R_{\mathrm{II}} = \{\mathsf{m} + 1 \le i < \infty, 1 \le j \le \mathsf{n}\},$$
$$R_{\mathrm{III}} = \{1 \le i \le \mathsf{m}, \mathsf{n} + 1 \le j < \infty\}, \qquad R_{\mathrm{IV}} = \{\mathsf{m} + 1 \le i < \infty, \mathsf{n} + 1 \le j < \infty\}, \tag{4.15}$$

and for a pair (ℓ_i, ℓ_j) we set $\mathsf{m} = \mathsf{n}$ in the decomposition (4.15). The vector multiplet contribution becomes trivial only for R_{IV}. We find the vector multiplet contributions are

(I) From the region R_{I},

$$\prod_{i \ne j = 1}^{\mathsf{m}} \frac{\left[(t z_i / q z_j) \, t^{\{(\ell_i) - 1\} \cdot (\ell_j)}; t\right]_\infty}{\left[(t z_i / z_j) \, t^{\{(\ell_i) - 1\} \cdot (\ell_j)}; t\right]_\infty} \prod_{i \ne j = 1}^{\mathsf{n}} \frac{\left[(t w_i / q w_j) \, t^{\{(k_i) - 1\} \cdot (k_j)}; t\right]_\infty}{\left[(t w_i / w_j) \, t^{\{(k_i) - 1\} \cdot (k_j)}; t\right]_\infty}$$
$$\times \prod_{i=1}^{\mathsf{m}} \prod_{j=1}^{\mathsf{n}} \frac{\left[(Q^{-1} w_j / q z_i) \, t^{(k_j) \cdot \{1 - (\ell_i)\}}; t\right]_\infty}{\left[(Q^{-1} w_j / z_i) \, t^{(k_j) \cdot \{1 - (\ell_i)\}}; t\right]_\infty} \frac{\left[(Q t z_i / q w_j) \, t^{(\ell_i) \cdot \{1 - (k_j)\}}; t\right]_\infty}{\left[(Q t z_i / w_j) \, t^{(\ell_i) \cdot \{1 - (k_j)\}}; t\right]_\infty}. \tag{4.16}$$

Here and henceforth, we use a shorthand notation $\prod_{i \ne j = 1}^{\mathsf{m}}$ for the sum over $1 \le i, j \le \mathsf{m}$ without the diagonal part $i = j$.

(II) For two semi-infinite regions R_{II} and R_{III}, only the "boundary" contributions remain,

$$\prod_{i=1}^{\mathsf{m}} \frac{\left[q^{\mathsf{m}-i} t^{1+\lfloor \frac{\ell_i}{2} \rfloor}; t\right]_\infty}{\left[q^{i-\mathsf{m}-1} t^{1+\lfloor -\frac{\ell_i}{2} \rfloor}; t\right]_\infty} \frac{\left[Q q^{\mathsf{n}-i} t^{1+\lfloor \frac{\ell_i+1}{2} \rfloor}; t\right]_\infty}{\left[Q^{-1} q^{i-\mathsf{n}-1} t^{\lfloor \frac{1-\ell_i}{2} \rfloor}; t\right]_\infty}$$
$$\times \prod_{j=1}^{\mathsf{n}} \frac{\left[q^{\mathsf{n}-j} t^{1+\lfloor \frac{k_j}{2} \rfloor}; t\right]_\infty}{\left[q^{j-\mathsf{n}-1} t^{1+\lfloor \frac{-k_j}{2} \rfloor}; t\right]_\infty} \frac{\left[Q^{-1} q^{\mathsf{m}-j} t^{1+\lfloor \frac{k_j-1}{2} \rfloor}; t\right]_\infty}{\left[Q q^{j-\mathsf{m}-1} t^{1+\lfloor \frac{1-k_j}{2} \rfloor}; t\right]_\infty}$$

---

[18]Note that $\lfloor \frac{\ell_i}{2} \rfloor$ and $\lfloor \frac{k_j}{2} \rfloor$ count the vertical dominos of length 2 in the Young diagrams λ and μ, respectively.

$$
= \prod_{i=1}^{\mathsf{m}} \frac{\left[z_i q^{\mathsf{m}-1} t; t\right]_\infty}{\left[z_i^{-1} q^{-\mathsf{m}} t^{1-(\ell_i)}; t\right]_\infty} \frac{\left[Q z_i q^{\mathsf{n}-1} t^{1+(\ell_i)}; t\right]_\infty}{\left[Q^{-1} z_i^{-1} q^{-\mathsf{n}}; t\right]_\infty}
$$

$$
\times \prod_{j=1}^{\mathsf{n}} \frac{\left[w_j q^{\mathsf{n}-1} t; t\right]_\infty}{\left[w_j^{-1} q^{-\mathsf{n}} t^{1-(k_j)}; t\right]_\infty} \frac{\left[Q^{-1} w_j q^{\mathsf{m}-1} t^{(k_j)}; t\right]_\infty}{\left[Q w_j^{-1} q^{-\mathsf{m}} t; t\right]_\infty}, \tag{4.17}
$$

where we have used (4.14).

In summary, up to a z_i, w_j independent normalization factor, we can rewrite the Nekrasov partition function as a sum over the positive cone of a lattice with the following weight function:

$$
\begin{aligned}
& W_{\mathsf{m}+\mathsf{n}}(z) \\
&= \prod_{i=1}^{\mathsf{m}} \frac{\left[d_2 z_i^{-1} t^{1-(\ell_i)}; t\right]_\infty \left[Q^{-1} d_3 z_i^{-1}; t\right]_\infty}{\left[d_4^{-1} Q z_i t; t\right]_\infty \left[d_1^{-1} z_i t^{(\ell_i)}; t\right]_\infty} \frac{\left[z_i q^{\mathsf{m}-1} t; t\right]_\infty}{\left[z_i^{-1} q^{-\mathsf{m}} t^{1-(\ell_i)}; t\right]_\infty} \frac{\left[Q z_i q^{\mathsf{n}-1} t^{1+(\ell_i)}; t\right]_\infty}{\left[Q^{-1} z_i^{-1} q^{-\mathsf{n}}; t\right]_\infty} \\
&\quad \times \prod_{j=1}^{\mathsf{n}} \frac{\left[Q^{-1} d_3 z_{\mathsf{m}+j}^{-1}; t\right]_\infty \left[d_2 z_{\mathsf{m}+j}^{-1} t^{(k_j)}; t\right]_\infty}{\left[d_1^{-1} z_{\mathsf{m}+j} t^{1-(k_j)}; t\right]_\infty \left[d_4^{-1} Q z_{\mathsf{m}+j} t; t\right]_\infty} \frac{\left[Q z_{\mathsf{m}+j} q^{\mathsf{n}-1} t^{2-(k_j)}; t\right]_\infty}{\left[Q^{-1} z_{\mathsf{m}+j}^{-1} q^{-\mathsf{n}}; t\right]_\infty} \frac{\left[z_{\mathsf{m}+j} q^{\mathsf{m}-1} t; t\right]_\infty}{\left[z_{\mathsf{m}+j}^{-1} q^{-\mathsf{m}} t^{(k_j)}; t\right]_\infty} \\
&\quad \times \prod_{i\neq j=1}^{\mathsf{m}} \frac{\left[(t z_i/q z_j)\, t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t\right]_\infty}{\left[(t z_i/z_j)\, t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t\right]_\infty} \prod_{i\neq j=1}^{\mathsf{n}} \frac{\left[(t z_{\mathsf{m}+i}/q z_{\mathsf{m}+j})\, t^{(k_i)\cdot\{(k_j)-1\}}; t\right]_\infty}{\left[(t z_{\mathsf{m}+i}/z_{\mathsf{m}+j})\, t^{(k_i)\cdot\{(k_j)-1\}}; t\right]_\infty} \\
&\quad \times \prod_{i=1}^{\mathsf{m}} \prod_{j=1}^{\mathsf{n}} \frac{\left[(t z_{\mathsf{m}+j}/q z_i)\, t^{-(k_j)\cdot(\ell_i)}; t\right]_\infty}{\left[(t z_{\mathsf{m}+j}/z_i)\, t^{-(k_j)\cdot(\ell_i)}; t\right]_\infty} \frac{\left[(t z_i/q z_{\mathsf{m}+j})\, t^{\{(\ell_i)-1\}\cdot\{1-(k_j)\}}; t\right]_\infty}{\left[(t z_i/z_{\mathsf{m}+j})\, t^{\{(\ell_i)-1\}\cdot\{1-(k_j)\}}; t\right]_\infty},
\end{aligned} \tag{4.18}
$$

where we have defined[19] $z_{\mathsf{m}+j} = Q^{-1} w_j t^{(k_j)-1} = Q^{-1} q^{1-j} t^{\lfloor \frac{k_j-1}{2} \rfloor}$.

In the above computations, we have used the fact that that $\ell_i = 0$ for $\mathsf{m} < i$ and $k_j = 0$ for $\mathsf{n} < j$, where m and n depend on $\left(\lambda^{(1)}, \lambda^{(2)}\right)$. But as is shown in Appendix E, by tuning some of mass parameters $d_2 = q^{-m}$ and $d_3 = q^{-n}$, we can assume $\mathsf{m} = m$ and $\mathsf{n} = n$ for any pair of Young diagrams $\left(\lambda^{(1)}, \lambda^{(2)}\right)$, because a pair $\left(\lambda^{(1)}, \lambda^{(2)}\right)$ with $\mathsf{m} > m$ or $\mathsf{n} > n$ does not contribute to the partition function. Furthermore, the substitution of the mass tuning condition $d_2 = q^{-m}$ and $d_3 = q^{-n}$ leads nice cancellation in the weight function (4.18). By defining $(\ell_{m+j}) = 1-(k_j)$, we can make $W_{m,n}(z, w)$ completely symmetric in z and w,

$$
\begin{aligned}
W_{m+n}(z) &= \prod_{I=1}^{m+n} \frac{\left[z_I q^{m-1} t; t\right]_\infty \left[Q z_I q^{n-1} t^{1+(\ell_I)}; t\right]_\infty}{\left[d_4^{-1} Q z_I t; t\right]_\infty \left[d_1^{-1} z_I t^{(\ell_I)}; t\right]_\infty} \\
&\quad \times \prod_{I\neq J=1}^{m+n} \frac{\left[(t z_I/q z_J)\, t^{\{(\ell_I)-1\}\cdot(\ell_J)}; t\right]_\infty}{\left[(t z_I/z_J)\, t^{\{(\ell_I)-1\}\cdot(\ell_J)}; t\right]_\infty}.
\end{aligned} \tag{4.19}
$$

In the expansion of Nekrasov partition function, the contribution from a fixed point $\left(\lambda^{(1)}, \lambda^{(2)}\right)$ has the weight $x_1^{|\lambda^{(1)}|_o + |\lambda^{(2)}|_e} x_2^{|\lambda^{(1)}|_e + |\lambda^{(2)}|_o}$, where

$$
|\lambda|_o = \sum_{k\geq 1} \lambda_{2k-1} = \sum_{k\geq 1} \left\lfloor \frac{\lambda_k^\vee + 1}{2} \right\rfloor, \qquad |\lambda|_e = \sum_{k\geq 1} \lambda_{2k} = \sum_{k\geq 1} \left\lfloor \frac{\lambda_k^\vee}{2} \right\rfloor. \tag{4.20}
$$

Recall that we have specialized x_1 and x_2 as (4.6), which implies that the dependence of the weight on Λ and x is

$$
\Lambda^{|\lambda^{(1)}|_e + |\lambda^{(2)}|_o} x^{|\lambda^{(1)}|_o - |\lambda^{(1)}|_e + |\lambda^{(2)}|_e - |\lambda^{(2)}|_o}. \tag{4.21}
$$

---

[19]See (4.14).

Since $|\lambda|_o - |\lambda|_e = \sum_{k\geq 1}(\lambda_k^\vee)$, the power of x, which is identified with the SU(2) spin variable, is

$$2s = \sum_{i=1}^{m}(\ell_i) - \sum_{j=1}^{n}(k_j) = \sum_{I=1}^{m+n}(\ell_I) - n. \tag{4.22}$$

Hence, only columns with odd length can produce a non-vanishing power of x. We see that the range of the power p of x is $-n \leq p \leq m$ (see also Figure 1). Similarly, the power of Λ, which we identify as the instanton number, is

$$\sum_{i=1}^{m}\left\lfloor\frac{\ell_i}{2}\right\rfloor + \sum_{j=1}^{n}\left\lfloor\frac{k_j+1}{2}\right\rfloor = \sum_{i=1}^{m}\left\lfloor\frac{\ell_i}{2}\right\rfloor + \sum_{j=1}^{n}\left\lfloor\frac{k_j}{2}\right\rfloor + \sum_{j=1}^{n}(k_j). \tag{4.23}$$

Now we are ready to prove the following theorem.

Theorem 4.1. *Under the identification of the parameters*

$$d_1 = \frac{1}{q^{m-1}a_1b_1}, \qquad d_2 = q^{-m}, \qquad d_3 = q^{-n}, \qquad d_4 = \frac{1}{q^{n-1}a_2b_2},$$
$$Q = \frac{q^{m-n}a_1}{ta_2}, \tag{4.24}$$

the affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$ (4.5) coincides with the $\mathbb{C}^{N+1}$-valued function (3.7) with the integration cycle ξ of (3.21). More precisely, the $N+1$ components of $\mathcal{Z}_{\mathrm{AL}}$ with respect to the expansion parameter x are identified as the Jackson integrals $\langle e_k(a_2, b_1), \xi\rangle$ with respect to the Matsuo basis e_k, $0 \leq k \leq N = m + n$.

Proof. Rewriting the dummy indices in (4.19), we consider

$$W_N(z) = \prod_{i=1}^{N} \frac{[z_iq^{m-1}t; t]_\infty [Qz_iq^{n-1}t^{1+(\ell_i)}; t]_\infty}{[d_4^{-1}Qtz_i; t]_\infty [d_1^{-1}z_it^{(\ell_i)}; t]_\infty} \prod_{i\neq j=1}^{N} \frac{[(tz_i/qz_j)\ t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}{[(tz_i/z_j)\ t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}, \tag{4.25}$$

where $N = m + n$. By the parameter relation (4.24), it can be written as

$$W_N(z) = \prod_{i=1}^{N} \frac{[z_iq^{m-1}t; t]_\infty [q^{m-1}\frac{a_1}{a_2}z_it^{(\ell_i)}; t]_\infty}{[q^{m-1}a_1b_2z_i; t]_\infty [q^{m-1}a_1b_1z_it^{(\ell_i)}; t]_\infty}$$
$$\times \prod_{i\neq j=1}^{N} \frac{[(tz_i/qz_j)t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}{[(tz_i/z_j)t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}. \tag{4.26}$$

Rescaling the integration variables as $z_i \to z_i/(q^{m-1}a_1)$, we have

$$W_N(z) = \prod_{i=1}^{N} \frac{[\frac{z_i}{a_1}t; t]_\infty [\frac{z_i}{a_2}t^{(\ell_i)}; t]_\infty}{[b_2z_i; t]_\infty [b_1z_it^{(\ell_i)}; t]_\infty} \prod_{i\neq j=1}^{N} \frac{[(tz_i/qz_j)\ t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}{[(tz_i/z_j)\ t^{\{(\ell_i)-1\}\cdot(\ell_j)}; t]_\infty}. \tag{4.27}$$

We decompose the index set as

$$I \cup J = \{1, 2, \ldots, N\}, \qquad I = \{i|(\ell_i) = 0\}, \quad J = \{j|(\ell_j) = 1\}. \tag{4.28}$$

From (4.22), we see that $2s + n = |J|$. Then

$$W_N(z) = W_N^{(0)}(z)P_J(z), \tag{4.29}$$

where

$$W_N^{(0)}(z) = \prod_{i=1}^{N} \frac{\left[\frac{z_i}{a_1}t;t\right]_\infty \left[\frac{z_i}{a_2}t;t\right]_\infty}{[b_2 z_i;t]_\infty [b_1 z_i;t]_\infty} \prod_{i\neq j=1}^{N} \frac{[tz_i/qz_j;t]_\infty}{[tz_i/z_j;t]_\infty},$$ (4.30)

$$P_J(z) = \prod_{j\in J} \frac{1-b_1 z_j}{1-\frac{z_j}{a_2}} \times \prod_{i\in I}\prod_{j\in J} \frac{z_j - q^{-1}z_i}{z_j - z_i}.$$ (4.31)

Putting $z_i = \xi_i t^{\nu_i}$, $i = 1, \ldots, N$, we can rewrite the sum of $W_N(z)$ over the partitions μ and λ as the sum of $W_N(z) = W_N^{(0)}(z)P_J(z)$ over $\nu_i \in \mathbb{Z}_{\geq 0}^N$ and $J \subset \{1, 2, \ldots, N\}$, where the correspondence is

$$\mu_i^\vee = \begin{cases} 2\nu_i, & i \in J, \\ 2\nu_i - 1, & i \notin J, \end{cases} \qquad i = 1, \ldots, n,$$ (4.32)

$$\lambda_i^\vee = \begin{cases} 2\nu_{i+n} + 1, & i + n \in J, \\ 2\nu_{i+n}, & i + n \notin J, \end{cases} \qquad i = 1, \ldots, m.$$ (4.33)

Then the latter sum can be identified with the Jackson integral studied in Section 3. In fact, from (3.3) and (3.6), we see that $W^{(0)}(z) = \Delta(z,1)\Phi(z)$. The sum of $P_J(z)$ for $|J| = k$ and the k-th cocycle factor $\hat{e}_k$ (3.8) is related by

$$\sum_{\substack{J\subset\{1,2,\ldots,N\} \\ |J|=k}} P_J(z) = c_k \hat{e}_k(z) \prod_{i=1}^{N} \frac{1}{1 - \frac{z_i}{a_2}},$$ (4.34)

where $c_k = q^{(k-m)(k-n)/2} \begin{bmatrix} N \\ k \end{bmatrix}_{q^{-1}} / \begin{bmatrix} N \\ m \end{bmatrix}_{q^{-1}}$. ∎

5 Four-dimensional limit and KZ equation

In [10], we computed the four-dimensional limit of the Hamiltonian $\mathcal{H}_S$ and confirmed that Shakirov's non-stationary difference equation has the correct four-dimensional limit. In Section 2, we have shown the representation matrix of $\mathcal{H}_S$ is nothing but the R-matrix of $U_v(A_1^{(1)})$ with generic spins. Let us work out the four-dimensional limit of the R-matrix.

Proposition 5.1. *Under the limit $q = e^h$, $d_i = q^{m_i}$, $h \to 0$, we have*

$$R = 1 + hR^{(1)} + O(h^2),$$ (5.1)

where $R^{(1)}$ is a tridiagonal matrix, which is generically infinite-dimensional. Furthermore, it can be truncated to a finite matrix of size $m+n+1$, if m_1 (or m_2) = $-m$ and m_3 (or m_4) = $-n$ for $m, n \in \mathbb{Z}_{\geq 0}$.

For example, in case of $m_1 = -2$, $m_3 = -1$, we have

$$R^{(1)} = \begin{bmatrix} -\frac{3(\Lambda m_2 - \Lambda)}{\Lambda - 1} & \frac{3(m_2 - 1)}{\Lambda - 1} & 0 & 0 \\ \frac{\Lambda m_4}{\Lambda - 1} & -\frac{2\Lambda m_2 + \Lambda m_4}{\Lambda - 1} & \frac{2m_2}{\Lambda - 1} & 0 \\ 0 & \frac{2\Lambda(m_4 - 1)}{\Lambda - 1} & -\frac{\Lambda + \Lambda m_2 + 2\Lambda m_4 - 2}{\Lambda - 1} & -\frac{-m_2 - 1}{\Lambda - 1} \\ 0 & 0 & \frac{3(\Lambda m_4 - 2\Lambda)}{\Lambda - 1} & -\frac{3(\Lambda m_4 - 2)}{\Lambda - 1} \end{bmatrix}.$$

Proof. Under the limit $q = e^h$, $h \to 0$, we have

$$\frac{(-q^\alpha x; q)_\infty}{(-q^\beta x; q)_\infty} = (1+x)^{\beta-\alpha}\left\{1 - \frac{h}{2}(\alpha-\beta)(\alpha+\beta-1)\frac{x}{x-1} + O(h^2)\right\}. \tag{5.2}$$

Then the defining relation of the matrix $R = (r_{i,j})$

$$q^{i(i+1)/2}x^i \frac{\left(-q^i d_1 d_2 x; q\right)_\infty}{(-d_2 x; q)_\infty} \frac{\left(-q^{-i} d_3 d_4 \frac{\Lambda}{x}; q\right)_\infty}{\left(-d_4 \frac{\Lambda}{x}; q\right)_\infty}$$

$$= \sum_j r_{i,j} q^{-j(j+1)/2} x^j \frac{(-d_1 x; q)_\infty}{(-q^{-j}x; q)_\infty} \frac{\left(-d_3 \frac{\Lambda}{x}; q\right)_\infty}{\left(-q^j \frac{\Lambda}{x}; q\right)_\infty}, \tag{5.3}$$

can be written as

$$A_i = \sum_j \left(\frac{x+\Lambda}{x+1}\right)^{j-i} B_j r_{i,j}, \tag{5.4}$$

where up to $O(h^2)$,

$$A_i = 1 + \frac{1}{2}h\left\{-\frac{\Lambda(i-m_3)(i-m_3-2m_4+1)}{x+\Lambda}\right.$$
$$\left. - \frac{x(i+m_1)(i+m_1+2m_2-1)}{x+1} + i(i+1)\right\},$$

$$B_j = 1 + \frac{1}{2}h\left\{\frac{\Lambda(j-m_3)(j+m_3-1)}{x+\Lambda} + \frac{x(j-m_1+1)(j+m_1)}{x+1} - j(j+1)\right\}. \tag{5.5}$$

We put $r_{i,j} = r_{i,j}^{(0)} + h r_{i,j}^{(1)} + O(h^2)$. From the leading term, we have

$$\sum_j \left(\frac{x+\Lambda}{x+1}\right)^{j-i} r_{i,j}^{(0)} = 1, \tag{5.6}$$

hence

$$r_{i,j}^{(0)} = \delta_{i,j}. \tag{5.7}$$

Then, from $O(h)$ terms, we obtain

$$\sum_j \left(\frac{x+\Lambda}{x+1}\right)^{j-i} r_{i,j}^{(1)} = i(i+1) + \frac{x(i+m_1)(i+m_2)}{1+x} + \frac{\Lambda(i-m_3)(i-m_4)}{x+\Lambda}, \tag{5.8}$$

and the solution is given by

$$r_{i,i-1}^{(1)} = \frac{\Lambda(i-m_3)(i-m_4)}{\Lambda-1},$$
$$r_{i,i}^{(1)} = -\frac{\Lambda\{(i+m_1)(i+m_2) + (i-m_3)(i-m_4)\}}{\Lambda-1} + i(i+1),$$
$$r_{i,i+1}^{(1)} = \frac{(i+m_1)(i+m_2)}{\Lambda-1},$$
$$r_{i,j}^{(1)} = 0 \qquad \text{otherwise.} \tag{5.9}$$

The truncation can also be seen from this expression. ∎

5.1 Identification with the KZ equation

The result given above is consistent with the four-dimensional limit of the operator $\mathcal{H}_S$ (see [10, Theorem 5.6]):

$$\mathcal{H}_S = 1 + hH_{4d} + O(h^2), \tag{5.10}$$

$$H_{4d} = \vartheta_x(\vartheta_x + 1) + \frac{\Lambda - x}{1 - \Lambda}(\vartheta_x + m_1)(\vartheta_x + m_2) + \frac{\Lambda}{x}\frac{x - 1}{1 - \Lambda}(\vartheta_x - m_3)(\vartheta_x - m_4), \tag{5.11}$$

where $\vartheta_x = x\frac{\partial}{\partial x}$. Using (5.11) and defining $\varkappa$ and a by $t = q^\varkappa$, $Q = q^a$,[20] we can see that the four-dimensional limit of the Shakirov's equation

$$\Psi(\Lambda, x) = \mathcal{H}_S \Psi\left(\frac{\Lambda}{t}, \frac{x}{qtQ}\right), \tag{5.12}$$

takes the form that looks like the Knizhnik–Zamolodchikov equation

$$\varkappa \frac{\partial}{\partial \Lambda} \Psi(\Lambda, x) = \frac{H_{4d} - (\varkappa + 1 + a)\vartheta_x}{\Lambda} \Psi(\Lambda, x) = \left\{\frac{A_0}{\Lambda} + \frac{A_1}{\Lambda - 1}\right\} \Psi(\Lambda, x), \tag{5.13}$$

where A_0, A_1 are operators acting on variable x:

$$A_0 = \vartheta_x(\vartheta_x - \varkappa - a) - x(\vartheta_x + m_1)(\vartheta_x + m_2), \tag{5.14}$$

$$A_1 = (x - 1)(\vartheta_x + m_1)(\vartheta_x + m_2) + \left(\frac{1}{x} - 1\right)(\vartheta_x - m_3)(\vartheta_x - m_4). \tag{5.15}$$

Below, we will show the relation with the standard KZ equation more explicitly.

Following [24], we write the $\widehat{\mathfrak{sl}}_2$ KZ equation for generic complex spins j_a and level k, $\varkappa = k + 2 = \log t / \log q$, in the form

$$\varkappa \frac{\partial}{\partial z_a} \psi = \sum_{b(\neq a)=1}^{n} \frac{\Omega_{a,b}}{z_a - z_b} \psi, \qquad a = 1, \ldots, n, \tag{5.16}$$

$$\Omega_{a,b} = -(x_a - x_b)^2 \frac{\partial^2}{\partial x_a \partial x_b} + 2(x_a - x_b)\left(j_a \frac{\partial}{\partial x_b} - j_b \frac{\partial}{\partial x_a}\right) + 2j_a j_b. \tag{5.17}$$

The number of variables z_a, x_a can be reduced to $2(n - 3)$, thanks to the $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ symmetry for x_a and z_a,

$$\sum_{a=1}^{n} x_a^i \left\{x_a \frac{\partial}{\partial x_a} - j_a(i + 1)\right\} \psi = 0, \qquad \sum_{a=1}^{n} z_a^i \left\{z_a \frac{\partial}{\partial z_a} + h_a(i + 1)\right\} \psi = 0, \tag{5.18}$$

where $i = 0, \pm 1$ and $h_a = j_a(j_a + 1)/\varkappa$.

We will consider the case $n = 4$, where the conditions (5.18) can be solved as

$$\psi = uvf_1(z, x), \qquad z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad x = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \tag{5.19}$$

$$u = z_{12}^{-2h_1} z_{23}^{h_1 - h_2 - h_3 + h_4} z_{24}^{-h_1 - h_2 + h_3 - h_4} z_{34}^{h_1 + h_2 - h_3 - h_4},$$

$$v = x_{12}^{2j_1} x_{23}^{-j_1 + j_2 + j_3 - j_4} x_{24}^{j_1 + j_2 - j_3 + j_4} x_{34}^{-j_1 - j_2 + j_3 + j_4},$$

where $z_{ab} = z_a - z_b$, $x_{ab} = x_a - x_b$. Then the equations (5.16) for $a = 1, \ldots, 4$ are all equivalent and we obtain a single equation for $f_1(z, x)$. Through the gauge transformation

$$f(z, x) = gf_1(z, x), \qquad g = x^{j_1 + j_2} z^{-h_1 - h_2} (z - 1)^{\frac{2j_1 j_4}{\varkappa}}, \tag{5.20}$$

---

[20]We have scaled the Coulomb parameter a by ϵ_1.

the KZ equation for $f(z, x)$ can be written concisely as

$$\left\{ \varkappa\vartheta_z - \vartheta_x(\vartheta_x - 1) + \frac{x - z}{1 - z}(\vartheta_x - j_1 + j_2)(\vartheta_x + j_3 - j_4) \right.$$
$$\left. + \frac{z(1 - x)}{x(1 - z)}(\vartheta_x + j_1 + j_2)(\vartheta_x + j_3 + j_4) \right\} f(z, x) = 0, \tag{5.21}$$

where $\vartheta_z = z\frac{\partial}{\partial z}$.

In order to identify (5.21) with the four-dimensional limit of Shakirov's equation, we change the parameters as[21]

$$(j_1, j_2, j_3, j_4) = \left(\frac{-m_1 - m_3}{2}, \frac{\tilde{a} - 1 + m_1 - m_3}{2}, \frac{\tilde{a} - 1 + m_2 - m_4}{2}, \frac{-m_2 - m_4}{2} \right), \tag{5.22}$$

and make a gauge transformation

$$\Psi = x^{-m_3}(z - 1)^{\frac{(m_1 + m_3)(m_2 + m_4)}{2\varkappa}} z^{\frac{(m_3 - m_1)\tilde{a} - (m_1 - 1)m_1 - (m_3 - 1)m_3}{2\varkappa}} f. \tag{5.23}$$

Then the equation (5.21) exactly agrees with (5.13)

$$\left\{ \varkappa\vartheta_z - \vartheta_x(\vartheta_x - \tilde{a}) + \frac{x - z}{1 - z}(\vartheta_x + m_1)(\vartheta_x + m_2) \right.$$
$$\left. + \frac{z(1 - x)}{x(1 - z)}(\vartheta_x - m_3)(\vartheta_x - m_4) \right\} \Psi = 0 \tag{5.24}$$

by the identification $z = \Lambda$.

There are 2^4 such identifications. This is because the equation (5.24) (= the quantum P_{VI} equation), has $(\mathbb{Z}_2)^4 \subset W\big(D_4^{(1)}\big)$ symmetry

$$\{m_1 \to m_2, m_2 \to m_1\} \times \{m_3 \to m_4, m_4 \to m_3\} \times \{m_1 \to 1 - \tilde{a} - m_2, m_2 \to 1 - \tilde{a} - m_1\}$$
$$\times \{m_3 \to 1 + a - m_4, m_4 \to 1 + a - m_3\}, \tag{5.25}$$

up to a gauge transformation of the form $g = x^{k_1}(x - 1)^{k_2} z^{k_3}(z - 1)^{k_4}(x - z)^{k_5}$. Each $\mathbb{Z}_2$ transform in (5.25) corresponds to the Weyl reflection with respect to the outer nodes of $D_4^{(1)}$ Dynkin diagram.

Remark 5.2. The relation between the KZ equation and the quantum P_{VI} was studied by [42]. The derivation of the KZ equation from the gauge theory is given by [47]. The equation (5.13) was previously obtained from Virasoro five-point functions.

A q-Borel transformation and refined Chern–Simons theory

In this appendix, we would like to explain a part of motivation for the original non-stationary difference equation of [51]. Namely, we would like to elucidate the reason for introduction of the operator $\mathcal{B}$ in the form that it was introduced.

A part of motivation to study non-stationary difference equations comes from the most well-studied example, which is the non-stationary Ruijsenaars–Schneider equation suggested by [34]. This non-stationary equation is satisfied by the K-theoretic character of the affine Laumon space of type $\widehat{\mathfrak{gl}}_N$. The character in question is a special function generalizing the Macdonald polynomial.

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[21]Here $\tilde{a} = a + \varkappa$.

For Macdonald polynomials, a relation to Chern–Simons theory and knot invariants is well-known [2]. The relation is obtained through so-called refined Chern–Simons theory. This theory provides some of the most prominent equations satisfied by Macdonald polynomials, among them a key role is played by the integral identity,

$$
\oint_{|x_1|=1} \frac{\mathrm{d}x_1}{x_1} \cdots \oint_{|x_N|=1} \frac{\mathrm{d}x_N}{x_N} \exp\left(-\sum_{i=1}^{N} \frac{x_i^2}{2g}\right) P_\lambda(x_1,\ldots,x_N) \prod_{i\neq j=1}^{N} \prod_{m=0}^{\infty} \frac{(1-q^m x_i/x_j)}{(1-tq^m x_i/x_j)}
$$
$$
= \mathrm{const}\, T_\lambda \, P_\lambda(t^{\rho_1},\ldots,t^{\rho_N}). \tag{A.1}
$$

Here $\rho_i = (N+1)/2 - i$ is the Weyl vector of $\mathfrak{gl}_N$, the partition λ is arbitrary, and T_λ is a certain product of q, t to the power of quadratic form of λ.

In [2, Appendix B], it is explained that changing the variables via $u_i = \log(x_i)$, it is possible to trade the Gaussian exponents $\mathrm{e}^{-x_i^2/2g}$ under the integration sign for theta functions $\sum_n q^{n^2/2} u_i^n$. This can be understood simply as the result of action on the rest of the integrand by a product of operators

$$
\mathcal{B}' u^n = q^{n^2/2} u^n, \tag{A.2}
$$

or equivalently (see (1.4)),

$$
\mathcal{B} u^n = q^{n^2/2 + n/2} u^n \tag{A.3}
$$

in each variable u_i, and setting all $u_1 = \cdots = u_N = 1$ afterwards. The linear difference between operators $\mathcal{B}'$ and $\mathcal{B}$ is inessential and amounts to a shift of variable.

In our view, equations for more complicated functions generalizing the Macdonald polynomials (such as the K-theoretic character of the affine Laumon space) are most likely far-going generalizations of the elementary equations for Macdonald polynomials. For this reason, when considering the non-stationary difference equation for degenerate five-point conformal blocks of q-Virasoro algebra in [51], we paid attention to the following interesting fact. In [34], the main part of the non-stationary difference equation is the operator $q^{\Delta/2}$, where Δ is a certain second order differential operator. For the case of rank 1, i.e., $\mathfrak{gl}_2$, the action of $q^{\Delta/2}$ on a monomial of the form $(x_1/x_2)^n$ is proportional (up to factors of the form q, t to the linear power in n) to q^{n^2}.

From the perspective described above, it is natural to expect that this operator should be understood as a product of two operators, each of which contributes $q^{n^2/2}$. This is natural both from the perspective of Macdonald theory, where this operator ($\mathcal{B}'$ or $\mathcal{B}$, equivalently) arises in the integral identities, as well as from the perspective of Ruijsenaars–Schneider system, where Δ is a sum of N contributions, i.e., 2 contributions in the $\mathfrak{gl}_2$ case. This was the principal reason why in [51] we chose to look for the Hamiltonian operator on the right-hand side of the non-stationary equation in the form (see (1.3))

$$
\mathcal{A}_1 \mathcal{B} \mathcal{A}_2 \mathcal{B} \mathcal{A}_3,
$$

where the two operators $\mathcal{B}$ are not joined together but separate. It eventually turned out to be the correct ansatz.

B   Lemma on the q-Borel transformation

A proof of the following lemma is given in [10, Appendix B]. Here we provide another more direct proof. In the following we use a shorthand notation for the infinite product $\varphi(z) := (z;q)_\infty$.

Lemma B.1 ([51]). *For $n \in \mathbb{Z}$, we have*

$$\mathcal{B} \cdot \frac{1}{\varphi(\alpha x)\varphi(\beta \Lambda/x)}x^n = \frac{\varphi(-q^{1+n}\alpha x)\varphi(-q^{-n}\beta \Lambda/x)}{\varphi(\alpha\beta\Lambda)}q^{n(n+1)/2}x^n, \tag{B.1}$$

$$\mathcal{B}^{-1} \cdot \varphi(\alpha x)\varphi(\beta \Lambda/x)x^n = \frac{\varphi(q^{-1}\alpha\beta\Lambda)}{\varphi(-q^{-1-n}\alpha x)\varphi(-q^n\beta \Lambda/x)}q^{-n(n+1)/2}x^n. \tag{B.2}$$

Proof. We will show the first relation (the second relation follows from the first one). By a rescaling of Λ, x, it is enough to show

$$f(\Lambda, x) := \frac{1}{\varphi(-qx)\varphi\left(-\frac{\Lambda}{x}\right)}\mathcal{B}\frac{1}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)} = \frac{1}{\varphi(\Lambda)}. \tag{B.3}$$

Recall that $\vartheta_x = x\frac{\partial}{\partial x}$, $\mathcal{B} = q^{\vartheta_x(\vartheta_x+1)/2}$ and $p = q^{\vartheta_x}$. First we note that, for any function $F(x)$,

$$\mathcal{B} \cdot x^n F(x) = (px)^n F(px) = q^{n(n+1)/2}x^n p^n F(px)$$
$$= q^{n(n+1)/2}x^n F(pq^n x) = q^{n(n+1)/2}x^n \mathcal{B} \cdot F(q^n x). \tag{B.4}$$

Using this identity, we obtain the following difference equation:

$$f(q\Lambda, x) = \frac{1}{\varphi(-qx)\varphi\left(-\frac{q\Lambda}{x}\right)}\mathcal{B}\frac{1 - \frac{\Lambda}{x}}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)}$$

$$= \frac{1}{\varphi(-qx)\varphi\left(-\frac{q\Lambda}{x}\right)}\mathcal{B}\frac{1}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)} - \frac{1}{\varphi(-qx)\varphi\left(-\frac{q\Lambda}{x}\right)}\mathcal{B}\frac{\frac{\Lambda}{x}}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)}$$

$$= \left(1 + \frac{\Lambda}{x}\right)f(\Lambda, x) - \frac{1 + x}{\varphi(-x)\varphi\left(-\frac{q\Lambda}{x}\right)}\frac{\Lambda}{x}\mathcal{B}\frac{1}{\varphi(q^{-1}x)\varphi\left(\frac{q\Lambda}{x}\right)}$$

$$= \left(1 + \frac{\Lambda}{x}\right)f(\Lambda, x) - (1 + x)\frac{\Lambda}{x}f\left(\Lambda, q^{-1}x\right). \tag{B.5}$$

Similarly, we have

$$f(q\Lambda, qx) = \frac{1}{\varphi(-q^2x)\varphi\left(-\frac{\Lambda}{x}\right)}\mathcal{B}\frac{1 - x}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)}$$

$$= \frac{1}{\varphi(-q^2x)\varphi\left(-\frac{\Lambda}{x}\right)}\mathcal{B}\frac{1}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)} - \frac{1}{\varphi(-q^2x)\varphi\left(-\frac{\Lambda}{x}\right)}\mathcal{B}\frac{x}{\varphi(x)\varphi\left(\frac{\Lambda}{x}\right)}$$

$$= (1 + qx)f(\Lambda, x) - \frac{1 + \frac{\Lambda}{qx}}{\varphi(-q^2x)\varphi\left(-\frac{\Lambda}{qx}\right)}qx\mathcal{B}\frac{1}{\varphi(qx)\varphi\left(\frac{\Lambda}{qx}\right)}$$

$$= (1 + qx)f(\Lambda, x) - (qx + \Lambda)f(\Lambda, qx). \tag{B.6}$$

Comparing the $f(q\Lambda, qx)$ given by (B.5) $|_{x \to qx}$ and (B.6), we have

$$f(\Lambda, qx) = f(\Lambda, x). \tag{B.7}$$

Hence, from (B.5) we have

$$f(q\Lambda, x) = (1 - \Lambda)f(\Lambda, x). \tag{B.8}$$

Since $f(0, x) = 1$, we obtain

$$f(\Lambda, x) = \varphi(\Lambda)^{-1}, \tag{B.9}$$

as desired. ∎

B.1   Relation to the pentagon identity

There is yet another proof of Lemma B.1,

$$\mathcal{B}\frac{1}{\varphi(\alpha x)\varphi\left(\beta\frac{\Lambda}{x}\right)}x^n = \frac{\varphi\left(-q^{n+1}\alpha x\right)\varphi\left(-q^{-n}\beta\frac{\Lambda}{x}\right)}{\varphi(\alpha\beta\Lambda)}q^{n(n+1)/2}x^n, \tag{B.10}$$

and show its relation to the pentagon (quantum dilogarithm) identity.

   (i) First we note that (B.10) follows from its special case $n=0$:

$$\mathcal{B}\frac{1}{\varphi(\alpha x)\varphi\left(\beta\frac{\Lambda}{x}\right)} = \frac{\varphi(-q\alpha x)\varphi\left(-\beta\frac{\Lambda}{x}\right)}{\varphi(\alpha\beta\Lambda)}. \tag{B.11}$$

In fact, we have

$$\mathcal{B}f(x)x^n = (px)^n\mathcal{B}f(x) = q^{n(n+1)/2}x^n p^n\mathcal{B}f(x) = q^{n(n+1)/2}x^n\mathcal{B}f(q^n x).$$

Apply this to $f(x) = \frac{1}{\varphi(\alpha x)\varphi(\beta\frac{\Lambda}{x})}$ and using (B.11) we obtain (B.10),

$$\mathcal{B}\frac{1}{\varphi(\alpha x)\varphi\left(\beta\frac{\Lambda}{x}\right)}x^n = q^{n(n+1)/2}x^n\mathcal{B}\frac{1}{\varphi(\alpha q^n x)\varphi\left(\beta\frac{q^{-n}\Lambda}{x}\right)}$$

$$= q^{n(n+1)/2}x^n\frac{\varphi\left(-q^{n+1}\alpha x\right)\varphi\left(-q^{-n}\beta\frac{\Lambda}{x}\right)}{\varphi(\alpha\beta\Lambda)}.$$

   (ii) The equation (B.11) can be written as

$$\frac{\varphi(\alpha\beta\Lambda)}{\varphi(\alpha x)\varphi\left(\beta\frac{\Lambda}{x}\right)} = \mathcal{B}^{-1}\varphi(-q\alpha x)\varphi\left(-\beta\frac{\Lambda}{x}\right)$$

$$= \mathcal{B}^{-1}\sum_{k,l=0}^{\infty}\frac{q^{k(k+1)/2}(\alpha x)^k}{(q)_k}\frac{q^{l(l-1)/2}\left(\beta\frac{\Lambda}{x}\right)^l}{(q)_l} = \sum_{k,l=0}^{\infty}\frac{(\alpha x)^k}{(q)_k}\frac{\left(\beta\frac{\Lambda}{x}\right)^l}{(q)_l}q^{kl}, \tag{B.12}$$

hence it is enough to show

$$\frac{\varphi(\alpha\beta\Lambda)}{\varphi(\alpha x)\varphi\left(\beta\frac{\Lambda}{x}\right)} = \sum_{k,l=0}^{\infty}\frac{(\alpha x)^k}{(q)_k}\frac{\left(\beta\frac{\Lambda}{x}\right)^l}{(q)_l}q^{kl}. \tag{B.13}$$

In terms of q-commutative variables $ba = qab$ and q-exponential function $e_q(x) = \varphi(x)^{-1}$, the equation (B.13) is equivalent to the quantum dilogarithm (pentagon) identity

$$e_q(b)e_q(a) = e_q(a)e_q(-ab)e_q(b). \tag{B.14}$$

The identity (B.14) can be easily derived as follows:

$$e_q(a)e_q(b) = e_q(a+b), \tag{B.15}$$

$$e_q(b)e_q(a)e_q(b)^{-1} = e_q\left(e_q(b)ae_q(b)^{-1}\right) = e_q\left(ae_q(qb)e_q(b)^{-1}\right)$$

$$= e_q(a(1-b)) = e_q(a-ab) = e_q(a)e_q(-ab). \tag{B.16}$$

C List of various R matrices and their characterization

Since various R-matrices are used in the main text, we summarize them along with its characterization.

(1) The matrix $R^{\mathrm{Sh}} = (r_{i,j}(\Lambda))_{i,j=-n}^{m}$ arising from Shakirov's operator in (2.6):

$$q^{\frac{1}{2}i(i+1)}x^i\left(-d_1 q^{i-m}x\right)_{m-i}\left(-d_4 q^{-i-n}\frac{\Lambda}{x}\right)_{i+n}$$
$$= \sum_{j=-n}^{m} q^{-\frac{1}{2}j(j+1)}x^j(-q^{-m}x)_{m-j}\left(-q^{-n}\frac{\Lambda}{x}\right)_{j+n} r_{i,j}(\Lambda). \tag{C.1}$$

(2) The matrix $R^{\mathrm{HG}} = \left(R^{\mathrm{HG}}_{i,j}\right)_{i,j=0}^{\ell}$ defined by $_4\phi_3$ series in (2.13):

$$\left(q^{\ell-1}\frac{x}{\beta};q^{-1}\right)_i\left(\frac{\alpha x}{z};q\right)_{\ell-i} = \sum_{j=0}^{\ell}\left(q^{\ell-1}\frac{x}{z};q^{-1}\right)_{\ell-j}(x;q)_j\, R^{\mathrm{HG}}_{i,j}. \tag{C.2}$$

(3) The matrix $R^{\mathrm{Ito}} = \left(R^{\mathrm{Ito}}_{i,j}\right)_{i,j=0}^{n}$ in Theorem 3.2:

$$\left(q^{n-1}b_1 x;q^{-1}\right)_j\left(\frac{x}{a_2};q\right)_{n-j} = \sum_{i=0}^{n}(q^{n-1}b_2 x;q^{-1})_{n-i}\left(\frac{x}{a_1};q\right)_i R^{\mathrm{Ito}}_{i,j}. \tag{C.3}$$

Actually, these R matrices are not unrelated. When $\ell = n+m$, the relation of R^{Sh} and R^{HG} is already proved in Section 2.1, namely we have (2.14). One may confirm it by making a change of the "dummy" variable x in (C.2) to $-q^{-n}\Lambda/x$, which implies (C.1) up to a scaling constant. On the other hand, with the identification

$$\ell = n; \qquad a_1 = 1, \quad a_2 = \frac{z}{\alpha}, \quad b_1 = \frac{1}{\beta}, \quad b_2 = \frac{1}{z}, \tag{C.4}$$

we have $R^{\mathrm{HG}}_{i,j} = R^{\mathrm{Ito}}_{j,i}$.

(4) The matrix $A = A^{\mathrm{Ito}} = \left(A^{\mathrm{Ito}}_{i,j}\right)_{i,j=0}^{n}$ in Theorem 3.2:

$$\begin{bmatrix} n \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}\left(-\frac{q^{1-n}}{z}\right)^i\left(\frac{x}{z};q\right)_{n-i}(q^{n-i}x;q)_i$$
$$= \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{(n-j)(n-j-1)}{2}}b_1^n\left(-\frac{q^{1-n}}{a_2 b_1 z}\right)^j (q^{n-1}a_2 b_2 x;q)_j\left(\frac{q^{j-n+1}x}{a_1 b_1 z};q\right)_{n-j} A^I_{i,j}. \tag{C.5}$$

(5) The matrix $\tilde{A} = \left(\tilde{A}_{i,j}\right)_{i,j=0}^{N}$ for the normalized solution in Section 3.2:

$$\left(q^{N-j}a_2 b_2 x;q\right)_j\left(\frac{q^{1-N}}{a_1 b_1}\frac{x}{\Lambda};q\right)_{N-j} = \sum_{i=0}^{N}(x;q)_i\left(q^{i+1-N}\frac{x}{\Lambda};q\right)_{N-i}\tilde{A}_{i,j}. \tag{C.6}$$

(6) The matrix $\mathbf{R}_{\mu,\nu} = \left(\delta_{i+j,k+l}\mathbf{R}_{\mu,\nu}{}^{k,l}_{i,j}\right)_{i,j,k,l\geq 0}$ for $U_{\mathsf{v}}\!\left(A_1^{(1)}\right)$:

$$\left(\frac{b_\nu x}{z_\nu};q\right)_i\left(\frac{q^i x}{z_\mu};q\right)_j = \sum_{k=0}^{n}\left(\frac{b_\mu x}{z_\mu};q\right)_k\left(\frac{q^k x}{z_\nu};q\right)_{n-k}\mathbf{R}_{\mu,\nu}{}^{k,n-k}_{i,j}. \tag{C.7}$$

This matrix $\mathbf{R}_{\mu,\nu}$ is related to the matrix R^{HG} by

$$\mathbf{R}_{\mu,\nu}{}^{j,n-j}_{i,n-i} = \left[R^{\mathrm{HG}}\right]_{\ell\to n,z\to b_\mu\frac{z_\nu}{z_\mu},\alpha\to b_\nu,\beta\to b_\mu}J, \qquad J = (\delta_{i+j,n})_{i,j=0}^{n}.$$

A significance of the matrix $\mathbf{R}$ is that it satisfies the Yang–Baxter relation. The following direct proof using (C.7) may be instructive.

Proposition C.1. *The matrix $\mathbf{R}_{\mu,\nu}$ satisfies the Yang–Baxter equation, namely for any (i,j,k), $(i'',j'',k'') \in \mathbb{Z}^3$ such that $i+j+k = i''+j''+k''$, we have*

$$\sum_{i',j',k'} \mathbf{R}_{1,2}{}^{i'',j''}_{i',j'} \mathbf{R}_{1,3}{}^{i',k''}_{i,k'} \mathbf{R}_{2,3}{}^{j',k'}_{j,k} = \sum_{i',j',k'} \mathbf{R}_{2,3}{}^{j'',k''}_{j',k''} \mathbf{R}_{1,3}{}^{i'',k'}_{i',k} \mathbf{R}_{1,2}{}^{i',j'}_{i,j}. \tag{C.8}$$

Proof. For $k_1, k_2, \ldots, k_s \in \mathbb{Z}_{\geq 0}$ with fixed $n = k_1 + k_2 + \cdots + k_s$, define a rational function in $t_1, \ldots, t_n$ as

$$B_{k_1,k_2,\ldots,k_s} = \mathcal{S}\left(\prod_{1 \leq i < j \leq n} \frac{qt_i - t_j}{t_i - t_j} \cdot \prod_{a=1}^{s} \prod_{i_a = k_1 + \cdots + k_{a-1}+1}^{k_1 + \cdots + k_a} f_a(t_{i_a}) \right), \tag{C.9}$$

where

$$f_a(t) = \prod_{i=1}^{a-1} \frac{1 - \frac{\beta_i}{z_i} t}{1 - \frac{t}{z_i}} \frac{1}{1 - \frac{t}{z_a}},$$

and $\mathcal{S}$ is the symmetrization on variables $t_1, \ldots, t_n$. The denominator of $B_{k_1,k_2,\ldots,k_s}$ is

$$\prod_{i=1}^{n} \prod_{\mu=1}^{s} \left(1 - \frac{t_i}{z_\mu} \right)$$

and the numerator is a symmetric polynomial in $t_1, \ldots, t_n$ whose degree in each t_i is $s-1$. The functions $B_{k_1,\ldots,k_s}$ form a basis of the linear space of such rational functions of dimension $\binom{n+s-1}{s-1}$. Define the action of a permutation σ of $\{1,\ldots,s\}$ on $\{z_a, \beta_a\}$ as

$$s_\sigma = \{z_a \to z_{\sigma(a)}, \ \beta_a \to \beta_{\sigma(a)}\},$$

then we have the connection relation

$$s_\sigma(B_{k_1,\ldots,k_s}) = \sum_{\{l\}} B_{l_1,\ldots,l_s} C_\sigma{}^{\{l\}}_{\{k\}}.$$

Note that the matrix $C_{(a,a+1)}$ is local in index a, i.e., it depends only on k_a, k_{a+1}, l_a, l_{a+1} and z_a, z_{a+1}, β_a, β_{a+1}, since $s_{(a,a+1)}(f_b(t)) = f_b(t)$ for $a \neq b$, $b+1$. Then, due to the relation $s_{(2,3)} s_{(1,2)} s_{(1,2)} B_{\{k\}} = s_{(1,2)} s_{(2,3)} s_{(1,2)} B_{\{k\}}$, the matrices C_σ satisfy the Yang–Baxter relation

$$C_{(2,3)}\big(s_{(2,3)} C_{(1,2)}\big)\big(s_{(2,3)} s_{(1,2)} C_{(2,3)}\big) = C_{(1,2)}\big(s_{(1,2)} C_{(2,3)}\big)\big(s_{(1,2)} s_{(2,3)} C_{(1,2)}\big). \tag{C.10}$$

Furthermore, by the locality, the matrix $C_{(a,a+1)}$ reduces to $C_{(1,2)}$, and we have

$$C_{(a,a+1)} = C_{(1,2)}\big|_{z_1 \to z_a, z_2 \to z_{a+1}, \beta_1 \to \beta_a, \beta_2 \to \beta_{a+1}}. \tag{C.11}$$

Thanks to the specialization

$$B_{k_1,k_2}\big|_{t_i \to x q^{i-1}} = \frac{(q;q)_n}{(1-q)^{-n}} \frac{\left(\frac{b_1 x}{z_1}; q \right)_{k_2}}{\left(\frac{x}{z_1}; q \right)_{k_1+k_2} \left(\frac{x}{z_2}; q \right)_{k_2}},$$

the coefficient $C_{(1,2)}{}^{l_1,l_2}_{k_1,k_2}$ $(k_1 + k_2 = l_1 + l_2)$ is obtained from

$$\frac{\left(\frac{b_2 x}{z_2}; q \right)_{k_2}}{\left(\frac{x}{z_2}; q \right)_{k_1+k_2} \left(\frac{x}{z_1}; q \right)_{k_2}} = \sum_{l_1,l_2} \frac{\left(\frac{b_1 x}{z_1}; q \right)_{l_2}}{\left(\frac{x}{z_1}; q \right)_{l_1+l_2} \left(\frac{x}{z_2}; q \right)_{l_2}} (C_{1,2})^{l_1,l_2}_{k_1,k_2},$$

i.e.,

$$\left(\frac{b_2 x}{z_2}; q \right)_{k_2} \left(q^{k_2} \frac{x}{z_1}; q \right)_{k_1} = \sum_{l_1,l_2} \left(\frac{b_1 x}{z_1}; q \right)_{l_2} \left(q^{l_2} \frac{x}{z_2}; q \right)_{l_1} (C_{1,2})^{l_1,l_2}_{k_1,k_2}. \tag{C.12}$$

Equations (C.10)–(C.12) give the desired Yang–Baxter relation and defining equation for the matrix C (= $\mathbf{R}$). ∎

D   Proof of $RD_2A = ARD_2$

D.1   Recollection of Ito's R and A matrices

Let $n \in \mathbb{Z}_{\geq 0}$,[22] and $a_1, a_2, b_1, b_2, q, \Lambda \in \mathbb{C}$ be generic parameters. Ito has introduced the matrices R and A (in [29, Theorems 1.4 and 1.7]) given as follows. Remark that to have a consistent notation with the main body of our paper, we change the notation as $t_{\mathrm{Ito}} \to q$, $(q^\alpha)_{\mathrm{Ito}} \to \Lambda$.

Definition D.1. Let R be the $(n+1)$-dimensional square matrix given in the Gauss decomposed forms

$$
\begin{aligned}
R &= L_R D_R U_R = U'_R D'_R L'_R, \\
L_R &= \left(l^R_{ij}\right)_{0 \leq i,j \leq n}, \qquad D_R = \left(\delta_{ij} d^R_j\right)_{0 \leq i,j \leq n}, \qquad U_R = \left(u^R_{ij}\right)_{0 \leq i,j \leq n}, \\
U'_R &= \left(u'^R_{ij}\right)_{0 \leq i,j \leq n}, \qquad D'_R = \left(\delta_{ij} d'^R_j\right)_{0 \leq i,j \leq n}, \qquad L'_R = \left(l'^R_{ij}\right)_{0 \leq i,j \leq n},
\end{aligned}
$$

where

$$
\begin{aligned}
l^R_{ij} &= \begin{bmatrix} n-j \\ n-i \end{bmatrix}_{q^{-1}} \frac{(-1)^{i-j} q^{-\binom{i-j}{2}} \left(a_2 b_2 q^j; q\right)_{i-j}}{\left(a_1^{-1} a_2 q^{-(n-2j-1)}; q\right)_{i-j}}, \\
d^R_j &= \frac{\left(a_1 a_2^{-1} q^{-j}; q\right)_{n-j} (a_2 b_1; q)_j}{(a_1 b_2; q)_{n-j} \left(a_1^{-1} a_2 q^{-(n-j)}; q\right)_j}, \\
u^R_{ij} &= \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-1}} \frac{\left(a_1 b_1 q^{n-j}; q\right)_{j-i}}{\left(a_1 a_2^{-1} q^{n-i-j}; q\right)_{j-i}}, \\
u'^R_{ij} &= \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{(-1)^{j-i} q^{\binom{j-i}{2}} \left(a_1^{-1} b_1^{-1} q^{-(n-i-1)}; q\right)_{j-i}}{\left(b_1^{-1} b_2 q^{i+j-n}; q\right)_{j-i}}, \\
d'^R_j &= \frac{\left(b_1 b_2^{-1} q^{n-2j+1}; q\right)_j \left(a_2^{-1} b_1^{-1} q^{-(n-j-1)}; q\right)_{n-j}}{\left(a_1^{-1} b_2^{-1} q^{-(j-1)}; q\right)_j \left(b_1^{-1} b_2 q^{-(n-2j-1)}; q\right)_{n-j}}, \\
l'^R_{ij} &= \begin{bmatrix} n-j \\ n-i \end{bmatrix}_q \frac{\left(a_2^{-1} b_2^{-1} q^{-(i-1)}; q\right)_{i-j}}{\left(b_1 b_2^{-1} q^{n-2i+1}; q\right)_{i-j}}.
\end{aligned}
\tag{D.1}
$$

Definition D.2. Let A be the $(n+1)$-dimensional square matrix given in the Gauss decomposed forms

$$
\begin{aligned}
A &= L_A D_A U_A = U'_A D'_A L'_A, \\
L_A &= \left(l^A_{ij}\right)_{0 \leq i,j \leq n}, \qquad D_A = \left(\delta_{ij} d^A_j\right)_{0 \leq i,j \leq n}, \qquad U_A = \left(u^A_{ij}\right)_{0 \leq i,j \leq n}, \\
U'_A &= \left(u'^A_{ij}\right)_{0 \leq i,j \leq n}, \qquad D'_A = \left(\delta_{ij} d'^A_j\right)_{0 \leq i,j \leq n}, \qquad L'_A = \left(l'^A_{ij}\right)_{0 \leq i,j \leq n},
\end{aligned}
$$

where

$$
\begin{aligned}
l^A_{ij} &= (-1)^{i-j} q^{\binom{n-i}{2} - \binom{n-j}{2}} \begin{bmatrix} n-j \\ n-i \end{bmatrix}_q \frac{\left(a_2 b_2 q^j; q\right)_{i-j}}{\left(\Lambda a_2 b_2 q^{2j}; q\right)_{i-j}}, \\
d^A_j &= a_1^{n-j} a_2^j q^{\binom{j}{2} + \binom{n-j}{2}} \frac{(\Lambda; q)_j \left(\Lambda a_2 b_2 q^{2j}; q\right)_{n-j}}{\left(\Lambda a_2 b_2 q^{j-1}; q\right)_j \left(\Lambda a_1 a_2 b_1 b_2 q^{n+j-1}; q\right)_{n-j}},
\end{aligned}
$$

---

[22]The non-negative integer n is denoted by N in the main text.

$$u_{ij}^A = \left(-\Lambda a_1^{-1} a_2\right)^{j-i} q^{\binom{j}{2}-\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{\left(a_1 b_1 q^{n-j}; q\right)_{j-i}}{\left(\Lambda a_2 b_2 q^{2i}; q\right)_{j-i}},$$

$$u'_{ij}^A = (-\Lambda)^{j-i} q^{\binom{n-i}{2}-\binom{n-j}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{\left(a_1 b_1 q^{n-j}; q\right)_{j-i}}{\left(\Lambda a_1 b_1 q^{2(n-j)}; q\right)_{j-i}},$$

$$d'_j^A = a_1^{n-j} a_2^j q^{\binom{j}{2}+\binom{n-j}{2}} \frac{\left(\Lambda a_1 b_1 q^{2(n-j)}; q\right)_j (\Lambda; q)_{n-j}}{\left(\Lambda a_1 a_2 b_1 b_2 q^{2n-j-1}; q\right)_j \left(\Lambda a_1 b_1 q^{n-j-1}; q\right)_{n-j}},$$

$$l'_{ij}^A = \left(-a_1 a_2^{-1}\right)^{i-j} q^{\binom{j}{2}-\binom{i}{2}} \begin{bmatrix} n-j \\ n-i \end{bmatrix}_q \frac{\left(a_2 b_2 q^j; q\right)_{i-j}}{\left(\Lambda a_1 b_1 q^{2(n-i)}; q\right)_{i-j}}. \tag{D.2}$$

It was shown that a certain system of Jackson integrals satisfies two different types of q-KZ equations described by two matrices R and A (see [29, Proposition 1.2 (Matsuo) and Theorem 1.7]), which one may regard as a consequence of the so-called base-fiber duality of the gauge theory. Hence, the compatibility of the two dual systems guarantees the commutativity of the pertinent matrices.

Theorem D.3. *Let R and A be Ito's matrices given as above. Set $D_2 = \left(\left(\Lambda q^{n-1}\right)^i \delta_{ij}\right)_{i,j=0}^n$. Then we have $RD_2 A = ARD_2$.*

In this appendix, we give an alternative direct proof of this commutativity.

D.2 Notations

We use the standard notations for the basic hypergeometric $r+1\phi_r$ series

$$_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_{r+1}; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_r; q)_n} z^n.$$

A $_{r+1}\phi_r$ series is called balanced (or Saalschützian) if $b_1 b_2 \cdots b_r = q a_1 a_2 \cdots a_{r+1}$ and $z = q$. We use the shorthand notation $_{r+1}W_r$ for the very-well-poised $_{r+1}\phi_r$ series

$$_{r+1}W_r(a_1; a_4, a_5, \ldots, a_{r+1}; q, z) = {}_{r+1}\phi_r \left[\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \ldots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{matrix}; q, z \right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_4; q)_n \cdots (a_{r+1}; q)_n}{(q; q)_n (qa_1/a_4; q)_n \cdots (qa_1/a_{r+1}; q)_n} \frac{1 - a_1 q^{2n}}{1 - a_1} z^n.$$

As for the detail, see [28].

D.3 Andrews' and Bressoud's matrix inversion formulas

Let $\mathcal{A}(a) = (\mathcal{A}_{ij}(a))_{i,j=0}^\infty$ be Andrews' infinite-dimensional lower triangular matrix [7],

$$\mathcal{A}_{ij}(a) = \frac{1}{(q; q)_{i-j}(aq; q)_{i+j}}. \tag{D.3}$$

Then the inverse matrix $\mathcal{A}^{-1}(a) = \left(\mathcal{A}_{ij}^{-1}(a)\right)_{i,j=0}^\infty$ is given by

$$\mathcal{A}_{ij}^{-1}(a) = \frac{\left(1 - aq^{2i}\right)(a; q)_{i+j}(-1)^{i-j} q^{\binom{i-j}{2}}}{(1 - a)(q; q)_{i-j}}. \tag{D.4}$$

Let $\mathcal{D}(a,b) = (\mathcal{D}_{ij}(a,b))_{i,j=0}^{\infty}$ be Bressoud's infinite-dimensional lower triangular matrix [18] in the original form,

$$\mathcal{D}_{ij}(a,b) = \frac{(1-aq^{2j})(b;q)_{i+j}(ba^{-1};q)_{i-j}(ba^{-1})^{j}}{(1-a)(aq;q)_{i+j}(q;q)_{i-j}}. \tag{D.5}$$

Then the inverse matrix is given by $\mathcal{D}^{-1}(a,b) = (\mathcal{D}_{ij}(b,a))_{i,j=0}^{\infty}$.

Closely following the line in [3], one can state the interrelations between Andrews' and Bressoud's matrices in the following manner, aiming at (D.8) and (D.9) in Corollary D.6 below. Consider the gauge transformation $\mathcal{B}(a,b) = (\mathcal{B}_{ij}(a,b))_{i,j=0}^{\infty}$ of Bressoud's matrix

$$\mathcal{B}_{ij}(a,b) = \frac{(aq;q)_{2i}}{(a;q)_{2i}}b^{i} \cdot \mathcal{D}_{ij}(b,a) \cdot \frac{(b;q)_{2j}}{(bq;q)_{2j}}a^{-j} = \frac{(1-aq^{2i})(a;q)_{i+j}(a/b;q)_{i-j}}{(1-a)(bq;q)_{i+j}(q;q)_{i-j}}b^{i-j}. \tag{D.6}$$

Remark D.4. Our choice of the gauge transformation is different from the one in [3].

Proposition D.5 ([3]). *We have the initial condition $\mathcal{B}(a,a) = (\delta_{ij})_{i,j=0}^{\infty}$ and the transition property*

$$\mathcal{B}(a,b)\mathcal{B}(b,c) = \mathcal{B}(a,c), \tag{D.7}$$

which in the particular case $a = c$ means Bressoud's matrix inversion $\mathcal{B}^{-1}(a,b) = \mathcal{B}(b,a)$, or $\mathcal{D}^{-1}(a,b) = \mathcal{D}(b,a)$.

For the readers' convenience, we reproduce the proof in [3].

Proof. Let $i \geq j$. We need to calculate $\sum_{k=j}^{i} \mathcal{B}_{ik}(a,b)\mathcal{B}_{kj}(b,c)$. In the following summation, by using the definition (D.6), one finds the summable very-well-poised $_6W_5$ series [28, formula (2.4.2)] as

$$\sum_{l=0}^{i-j} \frac{\mathcal{B}_{i,j+l}(a,b)\mathcal{B}_{j+l,j}(b,c)}{\mathcal{B}_{ij}(a,b)\mathcal{B}_{jj}(b,c)} = {}_6W_5\big(bq^{2j};aq^{i+j},b/c,q^{-i+j};q,cq/a\big)$$

$$= \frac{(a/c,bq^{2j+1};q)_{i-j}}{(a/b,cq^{2j+1};q)_{i-j}}(c/b)^{i-j}.$$

Then simplification of the factors shows that

$$\mathcal{B}_{ij}(a,b)\mathcal{B}_{jj}(b,c)\frac{(a/c,bq^{2j+1};q)_{i-j}}{(a/b,cq^{2j+1};q)_{i-j}}(c/b)^{i-j} = \mathcal{B}_{ij}(a,c).$$

Hence, we have (D.7). ∎

Corollary D.6. *By taking the limits $b \to 0$ or ∞ in (D.7), we have*

$$\sum_{k} a^{i}\mathcal{A}_{ik}^{-1}(a)a^{-k}c^{k}\mathcal{A}_{kj}(c)c^{-j} = \mathcal{B}_{ij}(a,c), \tag{D.8}$$

$$\sum_{k} q^{-i^2}\mathcal{A}_{ik}^{-1}(a)\mathcal{A}_{kj}(c)q^{j^2} = \mathcal{B}_{ij}(a,c), \tag{D.9}$$

which reproduce Andrews' matrix inversion by setting $a = c$.

Proof. It follows from

$$\lim_{b\to 0}\mathcal{B}_{ik}(a,b) = a^{i-k}\mathcal{A}_{ik}^{-1}(a), \qquad\qquad \lim_{b\to 0}\mathcal{B}_{kj}(b,c) = c^{k-j}\mathcal{A}_{kj}(c),$$

$$\lim_{b\to\infty} b^{2k}\mathcal{B}_{ik}(a,b) = q^{-i^2-k(k+1)}\mathcal{A}_{ik}^{-1}(a), \qquad \lim_{b\to\infty} b^{-2k}\mathcal{B}_{kj}(b,c) = q^{k(k+1)+j^2}\mathcal{A}_{kj}(c). \qquad ∎$$

D.4 Expressions for L_X, U_X, U'_X, L'_X ($X = R, A$) in terms of Andrews' triangular matrices $\mathcal{A}^{\pm 1}(a)$

Definition D.7. Set

$$\alpha = 1/a_2 b_2, \qquad \beta = 1/a_1 b_1, \qquad z = 1/a_1 b_2, \qquad w = \Lambda q^{n-1}. \tag{D.10}$$

Definition D.8. Define the gauge factors g_i^L, g_i^U, $g_i'^L$, $g_i'^U$ and h_i^R, h_i^A, $h_i'^R$, $h_i'^A$ by

$$
\begin{aligned}
g_i^L &= q^i (1/\alpha; q)_i (q^{-n}; q)_i, & g_i^U &= (z/\alpha\beta)^i (q^{-n+1}\beta; q)_i (q; q)_i, \\
g_i'^L &= q^i (z/\alpha\beta)^{-i} (1/\alpha; q)_i (q^{-n}; q)_i, & g_i'^U &= (q^{-n+1}\beta; q)_i (q; q)_i, \\
h_i^R &= (q^{-n+1}z/\alpha; q)_{2i}, & h_i^A &= (q^{-n+1}w/\alpha; q)_{2i}, \\
h_i'^R &= (q^{-n+1}\beta/z; q; q)_{2i}, & h_i'^A &= (q^{-n+1}\beta/w; q)_{2i}.
\end{aligned}
\tag{D.11}
$$

Proposition D.9. *We have*

$$l_{ij}^R = g_i^L \mathcal{A}_{ij}(q^{-n}z/\alpha) \frac{h_j^R}{g_j^L}, \qquad l_{ij}^A = g_i^L \mathcal{A}_{ij}(q^{-n}w/\alpha) \frac{h_j^A}{g_j^L}, \tag{D.12}$$

$$u_{ij}^R = \frac{h_i^R}{g_i^U} \mathcal{A}_{ji}(q^{-n}z/\alpha) g_j^U, \qquad u_{ij}^A = \frac{h_i^A}{w^i g_i^U} \mathcal{A}_{ji}(q^{-n}w/\alpha) w^j g_j^U, \tag{D.13}$$

$$u'_{ij}^R = \frac{1}{g_i'^U} \mathcal{A}_{ji}^{-1}(q^{-n}\beta/z) \frac{g_j'^U}{h_j'^R}, \qquad u'_{ij}^A = \frac{1}{g_i'^U} \mathcal{A}_{ji}^{-1}(q^{-n}\beta/w) \frac{g_j'^U}{h_j'^A}, \tag{D.14}$$

$$l'_{ij}^R = \frac{g_i'^L}{h_i'^R} \mathcal{A}_{ij}^{-1}(q^{-n}\beta/z) \frac{1}{g_j'^L}, \qquad l'_{ij}^A = \frac{g_i'^L}{w^i h_i'^A} \mathcal{A}_{ij}^{-1}(q^{-n}\beta/w) \frac{w^j}{g_j'^L}. \tag{D.15}$$

Proof. Straightforward calculation by using the definitions in (D.1) and (D.2), the parametrization (D.10), the gauge factors (D.11), and Andrews' matrices (D.3) and (D.4). ∎

D.5 Matrices R and A and Watson's transformation formula for terminating very-well-poised $_8\phi_7$ series

Proposition D.10. *We have*

$$d_j^R = \frac{(\alpha/z; q)_n}{(1/z; q)_n} \cdot q^{j^2}(q^n\alpha)^{-j} \frac{(z/\alpha\beta; q)_j (qz/\alpha; q)_j (q^{-n}z/\alpha; q)_j (q^{-n+1}z; q)_j}{(q^{-n}z/\alpha; q)_{2j} (q^{-n+1}z/\alpha; q)_{2j}}, \tag{D.16}$$

$$d_j'^R = \alpha^n \frac{(z/\alpha\beta; q)_n}{(z/\beta; q)_n} \cdot q^{-j^2}(q^n/\beta)^j \frac{(q^{-n}\beta/z; q)_{2j} (q^{-n+1}\beta/z; q)_{2j}}{(1/z; q)_j (q\beta/z; q)_j (q^{-n}\beta/z; q)_j (q^{-n+1}\alpha\beta/z; q)_j}, \tag{D.17}$$

$$
\begin{aligned}
d_j^A = {}& (-a_1 w/\alpha)^n \frac{(1/w; q)_n}{(w/\alpha\beta; q)_n} \cdot \frac{(\alpha/w; q)_n}{(1/w; q)_n} \cdot z^j q^{j^2}(q^n\alpha)^{-j} \\
& \times \frac{(w/\alpha\beta; q)_j (qw/\alpha; q)_j (q^{-n}w/\alpha; q)_j (q^{-n+1}w; q)_j}{(q^{-n}w/\alpha; q)_{2j} (q^{-n+1}w/\alpha; q)_{2j}},
\end{aligned}
\tag{D.18}
$$

$$
\begin{aligned}
d_j'^A = {}& (-a_1 w/\alpha)^n \frac{(1/w; q)_n}{(w/\alpha\beta; q)_n} \cdot \alpha^n \frac{(w/\alpha\beta; q)_n}{(w/\beta; q)_n} \cdot z^j q^{-j^2}(q^n/\beta)^j \\
& \times \frac{(q^{-n}\beta/w; q)_{2j} (q^{-n+1}\beta/w; q)_{2j}}{(1/w; q)_j (q\beta/w; q)_j (q^{-n}\beta/w; q)_j (q^{-n+1}\alpha\beta/w; q)_j}.
\end{aligned}
\tag{D.19}
$$

Proof. Straightforward calculation by using the definitions in (D.1) and (D.2), and the parametrization (D.10). ∎

Definition D.11. Set

$$\mathsf{R}_{ij}^n(\alpha,\beta|z) = \beta^{-i}\frac{(q;q)_n(\alpha/z;q)_{n-j}(1/\beta;q)_{n-i}(\beta/z;q)_i}{(q;q)_i(q;q)_{n-i}(1/z;q)_n(1/\beta;q)_{n-j}}$$

$$\times \, {}_4\phi_3\!\left[\begin{matrix} q^{-i}, q^{-n+j}, z/\alpha\beta, q^{-n+1}z \\ q^{-n}, q^{-i+1}z/\beta, q^{-n+j+1}z/\alpha \end{matrix}; q, q\right]. \tag{D.20}$$

Proposition D.12. *We have*

$$R_{ij} = \mathsf{R}_{ij}^n(\alpha,\beta|z), \tag{D.21}$$

$$A_{ij} = (-a_1 w/\alpha)^n \frac{(1/w;q)_n}{(w/\alpha\beta;q)_n} \cdot \mathsf{R}_{ij}^n(\alpha,\beta|w) \cdot z^j. \tag{D.22}$$

Recalling that $D_2 = \left(w^i \delta_{ij}\right)_{i,j=0}^n$, the commutativity $RD_2A = ARD_2$ (in Theorem D.3) is recast as follows.

Theorem D.13. *We have*

$$\sum_{k=0}^n \mathsf{R}_{ik}^n(\alpha,\beta|z) w^k \mathsf{R}_{kj}^n(\alpha,\beta|w) z^j = \sum_{k=0}^n \mathsf{R}_{ik}^n(\alpha,\beta|w) z^k \mathsf{R}_{kj}^n(\alpha,\beta|z) w^j. \tag{D.23}$$

Proof of Proposition D.12. Starting from the Gauss decomposition $R = L_R D_R U_R$, one can proceed as follows. We have

$$R_{ij} = \sum_k l_{ik}^R d_k^R u_{kj}^R = l_{i0}^R d_0^R u_{0j}^R \sum_{k=0}^{\min(i,j)} \frac{l_{ik}^R}{l_{i0}^R} \frac{d_k^R}{d_0^R} \frac{u_{kj}^R}{u_{0j}^R} = l_{i0}^R d_0^R u_{0j}^R \cdot X,$$

where the summation X can be calculated by using (D.12), (D.13), and (D.25), (D.28), (D.31) below as

$$X = \sum_{k=0}^{\min(i,j)} \frac{\mathcal{A}_{ik}(q^{-n}z/\alpha)}{\mathcal{A}_{i0}(q^{-n}z/\alpha)} \frac{h_k^R}{g_k^L} \frac{d_k^R}{d_0^R} \frac{h_k^R}{g_k^U} \frac{\mathcal{A}_{jk}(q^{-n}z/\alpha)}{\mathcal{A}_{j0}(q^{-n}z/\alpha)}$$

$$= {}_8W_7\!\left(q^{-n}z/\alpha; q^{-i}, q^{-j}, z/\alpha\beta, qz/\alpha, q^{-n+1}z; q, q^{-n+i+j}\beta/z\right)$$

$$= \frac{\left(q^{-n+1}z/\alpha; q\right)_i(\beta/z;q)_i}{\left(q^{-n+1}\beta; q\right)_i(1/\alpha;q)_i} \, {}_4\phi_3\!\left[\begin{matrix} q^{-i}, q^{-n+j}, z/\alpha\beta, q^{-n+1}z \\ q^{-n}, q^{-i+1}z/\beta, q^{-n+j+1}z/\alpha \end{matrix}; q, q\right].$$

Here, we have used Watson's transformation formula for terminating very-well-poised ${}_8\phi_7$ series [28, formula (2.5.1)].

From (D.11), (D.12), (D.13), (D.24), and (D.30) below, we have

$$l_{i0}^R d_0^R u_{0j}^R = g_i^L \mathcal{A}_{i0}(q^{-n}z/\alpha) d_0^R \mathcal{A}_{j0}(q^{-n}z/\alpha) g_j^U$$

$$= q^i(z/\alpha\beta)^j \frac{(1/\alpha;q)_i(q^{-n};q)_i}{\left(q^{-n+1}z/\alpha;q\right)_i(q;q)_i} \frac{\left(q^{-n+1}\beta;q\right)_j}{\left(q^{-n+1}z/\alpha;q\right)_j} \frac{(\alpha/z;q)_n}{(1/z;q)_n}.$$

Hence, it holds that

$$R_{ij} = q^i(z/\alpha\beta)^j \frac{(\beta/z;q)_i(q^{-n};q)_i}{\left(q^{-n+1}\beta;q\right)_i(q;q)_i} \frac{\left(q^{-n+1}\beta;q\right)_j}{\left(q^{-n+1}z/\alpha;q\right)_j} \frac{(\alpha/z;q)_n}{(1/z;q)_n}$$

$$\times \, {}_4\phi_3\!\left[\begin{matrix} q^{-i}, q^{-n+j}, z/\alpha\beta, q^{-n+1}z \\ q^{-n}, q^{-i+1}z/\beta, q^{-n+j+1}z/\alpha \end{matrix}; q, q\right].$$

Simplifying the prefactor, we have (D.21).

The formula (D.22) for the matrix A immediately follows from the one (D.21) for R, by making a comparison of the matrix elements given in (D.12)–(D.15), and (D.16)–(D.19). ∎

Remark D.14. If we use the opposite Gauss decomposition $R = U'_R D'_R L'_R$, we have

$$R_{ij} = \sum_k u'^R_{ik} d'^R_k l'^R_{kj} = u'^R_{in} d'^R_n l'^R_{nj} \sum_{k=\max(i,j)}^{n} \frac{u'^R_{ik}}{u'^R_{in}} \frac{d'^R_k}{d'^R_n} \frac{l'^R_{kj}}{l'^R_{nj}} = u'^R_{in} d'^R_n l'^R_{nj} \cdot X',$$

where X can be calculated by using (D.14), (D.15), and (D.27), (D.29) and (D.33) below,

$$
\begin{aligned}
X' &= \sum_{l=0}^{\min(i,j)} \frac{\mathcal{A}^{-1}_{n-l,i}(q^{-n}\beta/z)}{\mathcal{A}^{-1}_{n,i}(q^{-n}\beta/z)} \frac{g'^U_{n-l} h'^R_n}{g'^U_n h'^R_{n-l}} \frac{d'^R_{n-l}}{d'^R_n} \frac{g'^L_{n-l} h'^R_n}{g'^L_n h'^R_{n-l}} \frac{\mathcal{A}^{-1}_{n-l,j}(q^{-n}\beta/z)}{\mathcal{A}^{-1}_{n,j}(q^{-n}\beta/z)} \\
&= {}_8W_7\big(q^{-n}z/\beta; q^{-n+i}, q^{-n+j}, z/\alpha\beta, qz/\beta, q^{-n+1}z; q, q^{-n+i+j}\alpha/z\big) \\
&= \frac{\big(q^{-n+1}z/\beta; q\big)_{n-j} (\alpha/z; q)_{n-j}}{\big(q^{-n+1}\alpha; q\big)_{n-j} (1/\beta; q)_{n-j}} {}_4\phi_3 \left[\begin{matrix} q^{-i}, q^{-n+j}, z/\alpha\beta, q^{-n+1}z \\ q^{-n}, q^{-i+1}z/\beta, q^{-n+j+1}z/\alpha \end{matrix} ; q, q \right].
\end{aligned}
$$

Note that from (D.11), (D.14), (D.15), and (D.26), (D.32) below,

$$
\begin{aligned}
u'^R_{in} d'^R_n l'^R_{nj} &= \frac{1}{g'^U_i} \mathcal{A}^{-1}_{n,i}(q^{-n}\beta/z) \frac{g'^U_n}{h'^R_n} d'^R_n \frac{g'^L_n}{h'^R_n} \mathcal{A}^{-1}_{n,j}(q^{-n}\beta/z) \frac{1}{g'^L_j} \\
&= q^i (z/\alpha\beta)^{j-n} \frac{(\beta/z; q)_i (q^{-n}; q)_i}{\big(q^{-n+1}\beta; q\big)_i (q; q)_i} \frac{(\beta/z; q)_j}{(1/\alpha; q)_j} \frac{(1/\beta; q)_n}{(1/z; q)_n}.
\end{aligned}
$$

Simplifying the prefactor, we also have (D.21).

Lemma D.15. *We have*

$$\mathcal{A}_{i0}(a) = \frac{1}{(aq; q)_i (q; q)_i}, \tag{D.24}$$

$$\frac{\mathcal{A}_{ik}(a)}{\mathcal{A}_{i0}(a)} = q^{-\binom{k}{2}} \big(-q^i\big)^k \frac{\big(q^{-i}; q\big)_k}{(aq^{i+1}; q)_k}, \tag{D.25}$$

$$\mathcal{A}^{-1}_{n,i}(a) = (-1)^n q^{\binom{n}{2}} \frac{(aq; q)_{2n}}{(a; q)_{2n}} \frac{(a; q)_n}{(q; q)_n} \cdot q^i (aq^n; q)_i (q^{-n}; q)_i, \tag{D.26}$$

$$\frac{\mathcal{A}^{-1}_{n-l,i}(a)}{\mathcal{A}^{-1}_{n,i}(a)} = q^{\binom{l}{2}} \big(-q^{i+n}a\big)^{-l} \frac{1 - a^{-1}q^{-2n+2l}}{1 - a^{-1}q^{-2n}} \frac{\big(q^{-n+i}; q\big)_l}{\big(a^{-1}q^{1-n-i}; q\big)_l}. \tag{D.27}$$

Lemma D.16. *We have*

$$\frac{h^R_k}{g^L_k} \frac{h^R_k}{g^U_k} = (qz/\alpha\beta)^{-k} \frac{\big(q^{-n+1}z/\alpha; q\big)_{2k} \big(q^{-n+1}z/\alpha; q\big)_{2k}}{(1/\alpha; q)_k (q^{-n}; q)_k \big(q^{-n+1}\beta; q\big)_k (q; q)_k}, \tag{D.28}$$

$$\frac{g'^U_{n-l} h'^R_n}{g'^U_n h'^R_{n-l}} \frac{g'^L_{n-l} h'^R_n}{g'^L_n h'^R_{n-l}} = q^{-2l^2} \big(q^{2n+1}\beta^2/z^3\big)^l \frac{(q^{-n}z/\beta; q)_{2l} (q^{-n}z/\beta; q)_{2l}}{(1/\beta; q)_l (q^{-n}; q)_l \big(q^{-n+1}\alpha; q\big)_l (q; q)_l}. \tag{D.29}$$

Lemma D.17. *We have*

$$d^R_0 = \frac{(\alpha/z; q)_n}{(1/z; q)_n}, \tag{D.30}$$

$$\frac{d^R_k}{d^R_0} = q^{k^2} (q^n \alpha)^{-k} \frac{(qz/\alpha; q)_k \big(q^{-n+1}z; q\big)_k (z/\alpha\beta; q)_k (q^{-n}z/\alpha; q)_k}{\big(q^{-n+1}z/\alpha; q\big)_{2k} (q^{-n}z/\alpha; q)_{2k}}, \tag{D.31}$$

$$d'^R_n = \beta^{-n} \frac{(\beta/z; q)_n}{(1/z; q)_n}, \tag{D.32}$$

$$\frac{d'^R_{n-l}}{d'^R_n} = q^{l^2} (q^{-n}\alpha)^l \frac{(qz/\beta; q)_l \big(q^{-n+1}z; q\big)_l (z/\alpha\beta; q)_l (q^{-n}z/\beta; q)_l}{\big(q^{-n+1}z/\beta; q\big)_{2l} (q^{-n}z/\beta; q)_{2l}}. \tag{D.33}$$

D.6 Products of triangular matrices $L_A^{-1}L_R$, $U_RD_2U'_A$, $U_AU'_R$, and $L'_RD_2L'^{-1}_A$ in terms of Bressoud's triangular matrix $\mathcal{B}(a,b)$

For the convenience of our study on the commutativity $RD_2A = ARD_2$, we write the matrices R and A in Gauss decomposed forms and investigate

$$(L_RD_RU_R)D_2(U'_AD'_AL'_A) = (L_AD_AU_A)(U'_RD'_RL'_R)D_2, \tag{D.34}$$

which is equivalent to

$$\mathsf{L}^1 \cdot D_R \cdot \mathsf{U}^1 = D_A \cdot \mathsf{U}^2 \cdot D'_R \cdot \mathsf{L}^2 \cdot D'^{-1}_A, \tag{D.35}$$

where

$$\mathsf{L}^1 = L_A^{-1}L_R, \qquad \mathsf{U}^1 = U_RD_2U'_A, \qquad \mathsf{U}^2 = U_AU'_R, \qquad \mathsf{L}^2 = L'_RD_2L'^{-1}_A. \tag{D.36}$$

Note from (D.12) and (D.15), we have that

$$\left(L_A^{-1}\right)_{ij} = \frac{g_i^L}{h_i^A}\mathcal{A}_{ij}^{-1}(q^{-n}w/\alpha)\frac{1}{g_j^L}, \tag{D.37}$$

$$\left(L'^{-1}_A\right)_{ij} = \frac{g_i'^L}{w^i}\mathcal{A}_{ij}(q^{-n}\beta/w)\frac{w^j h_j'^A}{g_j'^L}. \tag{D.38}$$

One finds that all the products L^1, U^1, U^2 and L^2 in (D.36) can be treated by the summation formulas (D.8) and (D.9) in Corollary D.6, which are recast as

$$\sum_k \mathcal{A}_{ik}^{-1}(a)(b/a)^k\mathcal{A}_{kj}(b) = a^{-i}\mathcal{B}_{ij}(a,b)b^j, \qquad \sum_k \mathcal{A}_{ik}^{-1}(a)\mathcal{A}_{kj}(b) = q^{i^2}\mathcal{B}_{ij}(a,b)q^{-j^2}. \tag{D.39}$$

Proposition D.18. *We have*

$$\mathsf{L}_{ij}^1 = \frac{q^{i^2}g_i^L}{h_i^A}\mathcal{B}_{ij}(q^{-n}w/\alpha, q^{-n}z/\alpha)\frac{h_j^R}{q^{j^2}g_j^L}, \tag{D.40}$$

$$\mathsf{U}_{ij}^1 = \frac{(q^{-n}z/\alpha)^i h_i^R}{g_i^U}\mathcal{B}_{ji}(q^{-n}\beta/w, q^{-n}z/\alpha)\frac{g_j'^U}{(q^{-n}\beta/w)^j h_j'^A}, \tag{D.41}$$

$$\mathsf{U}_{ij}^2 = \frac{(q^{-n}w/\alpha)^i h_i^A}{w^i g_i^U}\mathcal{B}_{ji}(q^{-n}\beta/z, q^{-n}w/\alpha)\frac{g_j'^U}{(q^{-n}\beta/z)^j h_j'^R}, \tag{D.42}$$

$$\mathsf{L}_{ij}^2 = \frac{q^{i^2}g_i'^L}{h_i'^R}\mathcal{B}_{ij}(q^{-n}\beta/z, q^{-n}\beta/w)\frac{w^j h_j'^A}{q^{j^2}g_j'^L}. \tag{D.43}$$

Proof. Straightforward calculation by using $D_2 = \left(w^i\delta_{ij}\right)_{i,j=0}^n$, (D.11), (D.12)–(D.15), (D.37), (D.38), and (D.39). ∎

D.7 Products $\mathsf{L}^1 \cdot D_R \cdot \mathsf{U}^1$ and $\mathsf{U}^2 \cdot D'_R \cdot \mathsf{L}^2$ in terms of terminating very-well-poised balanced $_{10}\phi_9$ series

Proposition D.19. *We have*

$$\frac{1}{\mathsf{L}_{i0}^1 \cdot d_0^R \cdot \mathsf{U}_{0j}^1}\left(\mathsf{L}^1 \cdot D_R \cdot \mathsf{U}^1\right)_{ij}$$
$$= {}_{10}W_9\left(q^{-n}z/\alpha; q^{-i}, q^{-j}, q^{-n+i}w/\alpha, q^{-n+j}\beta/w, z/\alpha\beta, qz/\alpha, q^{-n+1}z; q, q\right), \tag{D.44}$$

$$\frac{1}{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}}\big(\mathsf{U}^2\cdot D'_R\cdot \mathsf{L}^2\big)_{ij}$$

$$= {}_{10}W_9\big(q^{-n}z/\beta; q^{-n+i}, q^{-n+j}, q^{-i}\alpha/w, q^{-j}w/\beta, z/\alpha\beta, qz/\beta, q^{-n+1}z; q, q\big), \qquad (D.45)$$

and

$$\frac{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}}{\mathsf{L}^1_{i0}\cdot d^R_0\cdot \mathsf{U}^1_{0j}}\cdot \frac{d^A_i}{d'^A_j} = \frac{(1/\alpha, 1/\beta, w/z, \alpha\beta/zw; q)_n}{(\alpha/z, \beta/z, 1/w, w/\alpha\beta; q)_n}\frac{(\beta/z, w/\alpha\beta, q^{-n+1}z/\alpha, q^{-n+1}w; q)_i}{(1/\alpha, w/z, q^{-n+1}\beta, q^{-n+1}zw/\alpha\beta; q)_i}$$

$$\times \frac{(\beta/z, 1/w, q^{-n+1}z/\alpha, q^{-n+1}\alpha\beta/w; q)_j}{(1/\alpha, \alpha\beta/zw, q^{-n+1}\beta, q^{-n+1}z/w; q)_j}. \qquad (D.46)$$

Proof. For (D.44), we have from (D.31), (D.40), (D.41), and (D.47), (D.51) below that

$$\frac{1}{\mathsf{L}^1_{i0}\cdot d^R_0\cdot \mathsf{U}^1_{0j}}\big(\mathsf{L}^1\cdot D_R\cdot \mathsf{U}^1\big)_{ij} = \sum_{k=0}^{\min(i,j)}\frac{\mathsf{L}^1_{ik}\cdot d^R_k\cdot \mathsf{U}^1_{kj}}{\mathsf{L}^1_{i0}\cdot d^R_0\cdot \mathsf{U}^1_{0j}}$$

$$= \sum_{k=0}^{\min(i,j)}\frac{\mathcal{B}_{ik}(q^{-n}w/\alpha, q^{-n}z/\alpha)}{\mathcal{B}_{i0}(q^{-n}w/\alpha, q^{-n}z/\alpha)}\frac{h^R_k}{q^{k^2}g^L_k}\frac{d^R_k}{d^R_0}\frac{(q^{-n}z/\alpha)^k h^R_k}{g^U_k}\frac{\mathcal{B}_{jk}(q^{-n}\beta/w, q^{-n}z/\alpha)}{\mathcal{B}_{j0}(q^{-n}\beta/w, q^{-n}z/\alpha)}$$

$$= {}_{10}W_9\big(q^{-n}z/\alpha; q^{-i}, q^{-j}, q^{-n+i}w/\alpha, q^{-n+j}\beta/w, z/\alpha\beta, qz/\alpha, q^{-n+1}z; q, q\big).$$

For (D.45), in a similar manner, we have from (D.33), (D.42), (D.43), and (D.48), (D.52) below that

$$\frac{1}{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}}\big(\mathsf{U}^2\cdot D'_R\cdot \mathsf{L}^2\big)_{ij} = \sum_{k=\max(i,j)}^{n}\frac{\mathsf{U}^2_{ik}\cdot d'^R_k\cdot \mathsf{L}^2_{kj}}{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}} = \sum_{l=0}^{\min(i,j)}\frac{\mathsf{U}^2_{i,n-l}\cdot d'^R_{n-l}\cdot \mathsf{L}^2_{n-l,j}}{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}}$$

$$= \sum_{l=0}^{\min(i,j)}\frac{\mathcal{B}_{n-l,i}(q^{-n}\beta/z, q^{-n}w/\alpha)}{\mathcal{B}_{ni}(q^{-n}\beta/z, q^{-n}w/\alpha)}\frac{g'^U_{n-l}}{g'^U_n}\frac{(q^{-n}\beta/z)^n h'^R_n}{(q^{-n}\beta/z)^{n-l}h'^R_{n-l}}\frac{d'^R_{n-l}}{d'^R_n}$$

$$\times \frac{q^{(n-l)^2}g'^L_{n-l}}{q^{n^2}g'^L_n}\frac{h'^R_n}{h'^R_{n-l}}\frac{\mathcal{B}_{n-l,j}(q^{-n}\beta/z, q^{-n}\beta/w)}{\mathcal{B}_{nj}(q^{-n}\beta/z, q^{-n}\beta/w)}$$

$$= {}_{10}W_9\big(q^{-n}z/\beta; q^{-n+i}, q^{-n+j}, q^{-i}\alpha/w, q^{-j}w/\beta, z/\alpha\beta, qz/\beta, q^{-n+1}z; q, q\big).$$

Concerning (D.46), one can proceed as

$$\frac{\mathsf{U}^2_{in}\cdot d'^R_n\cdot \mathsf{L}^2_{nj}}{\mathsf{L}^1_{i0}\cdot d^R_0\cdot \mathsf{U}^1_{0j}}\cdot \frac{d^A_i}{d'^A_0}$$

$$= \mathcal{B}_{n0}(q^{-n}\beta/z, q^{-n}w/\alpha)\mathcal{B}_{n0}(q^{-n}\beta/z, q^{-n}\beta/w)\frac{d^A_0}{d'^A_0}\frac{d'^R_n}{d^R_0}\frac{g'^U_n}{(q^{-n}\beta/z)^n h'^R_n}\frac{q^{n^2}g'^L_n}{h'^R_n}$$

$$\times \frac{\mathcal{B}_{ni}(q^{-n}\beta/z, q^{-n}w/\alpha)}{\mathcal{B}_{n0}(q^{-n}\beta/z, q^{-n}w/\alpha)}\frac{1}{\mathcal{B}_{i0}(q^{-n}w/\alpha, q^{-n}z/\alpha)}\frac{(q^{-n}w/\alpha)^i h^A_i h^A_i}{q^{i^2}g^L_i w^i g^U_i}\frac{d^A_i}{d^A_0}$$

$$\times \frac{\mathcal{B}_{nj}(q^{-n}\beta/z, q^{-n}\beta/w)}{\mathcal{B}_{n0}(q^{-n}\beta/z, q^{-n}\beta/w)}\frac{1}{\mathcal{B}_{j0}(q^{-n}\beta/w, q^{-n}z/\alpha)}\frac{(q^{-n}\beta/w)^j h'^A_j w^j h'^A_j}{g'^U_j q^{j^2}g'^L_j}\frac{d'^A_0}{d'^A_j}.$$

Then from (D.53), (D.54), and (D.55) below, we obtain (D.46). ∎

Lemma D.20. *We have*

$$\frac{\mathcal{B}_{ik}(a,b)}{\mathcal{B}_{i0}(a,b)} = \big(a^{-1}q\big)^k\frac{\big(aq^i; q\big)_k\big(q^{-i}; q\big)_k}{\big(bq^{i+1}; q\big)_k\big(a^{-1}bq^{1-i}; q\big)_k}, \qquad (D.47)$$

$$\frac{\mathcal{B}_{n-l,i}(a,b)}{\mathcal{B}_{n,i}(a,b)} = \left(a^{-2}b\right)^l \frac{\left(a^{-1}q^{1-2n};q\right)_{2l}}{\left(a^{-1}q^{-2n};q\right)_{2l}} \frac{\left(b^{-1}q^{-n-i};q\right)_l\left(q^{-n+i};q\right)_l}{\left(a^{-1}q^{1-n-i};q\right)_l\left(a^{-1}bq^{1-n+i};q\right)_l},$$
(D.48)

$$\mathcal{B}_{i0}(a,b) = b^i \frac{(aq;q)_{2i}}{(a;q)_{2i}} \frac{(a;q)_i(a/b;q)_i}{(bq;q)_i(q;q)_i},$$
(D.49)

$$\frac{\mathcal{B}_{n,i}(a,b)}{\mathcal{B}_{n,0}(a,b)} = \left(a^{-1}q\right)^i \frac{(aq^n;q)_i(q^{-n};q)_i}{\left(bq^{1+n};q\right)_i\left(a^{-1}bq^{1-n};q\right)_i}.$$
(D.50)

Lemma D.21. *We have*

$$\frac{h_k^R}{q^{k^2}g_k^L} \frac{(q^{-n}z/\alpha)^k h_k^R}{g_k^U} = q^{-k^2}\left(q^{-n-1}\beta\right)^k \frac{(q^{-n+1}z/\alpha;q)_{2k}\left(q^{-n+1}z/\alpha;q\right)_{2k}}{(1/\alpha;q)_k(q^{-n};q)_k\left(q^{-n+1}\beta;q\right)_k(q;q)_k},$$
(D.51)

$$\frac{g_{n-l}^{\prime U}}{g_n^{\prime U}} \frac{(q^{-n}\beta/z)^n h_n^{\prime R}}{(q^{-n}\beta/z)^{n-l}h_{n-l}^{\prime R}} \frac{q^{(n-l)^2}g_{n-l}^{\prime L}}{q^{n^2}g_n^{\prime L}} \frac{h_n^{\prime R}}{h_{n-l}^{\prime R}}$$

$$= q^{-l^2}\left(q^{-n+1}\beta^3/z^4\right)^l \frac{(q^{-n}z/\beta;q)_{2l}(q^{-n}z/\beta;q)_{2l}}{(1/\beta;q)_l(q^{-n};q)_l\left(q^{-n+1}\alpha;q\right)_l(q;q)_l}.$$
(D.52)

Lemma D.22. *We have*

$$\mathcal{B}_{n0}(q^{-n}\beta/z,q^{-n}w/\alpha)\mathcal{B}_{n0}(q^{-n}\beta/z,q^{-n}\beta/w)\frac{d_0^A}{d_0^{\prime A}} \frac{d_n^{\prime R}}{d_n^R} \frac{g_n^{\prime U}}{(q^{-n}\beta/z)^n h_n^{\prime R}} \frac{q^{n^2}g_n^{\prime L}}{h_n^{\prime R}}$$

$$= \frac{(1/\alpha,1/\beta,w/z,\alpha\beta/zw;q)_n}{(\alpha/z,\beta/z,1/w,w/\alpha\beta;q)_n},$$
(D.53)

$$\frac{\mathcal{B}_{ni}(q^{-n}\beta/z,q^{-n}w/\alpha)}{\mathcal{B}_{n0}(q^{-n}\beta/z,q^{-n}w/\alpha)} \frac{1}{\mathcal{B}_{i0}(q^{-n}w/\alpha,q^{-n}z/\alpha)} \frac{(q^{-n}w/\alpha)^i h_i^A h_i^A}{q^{i^2}g_i^L w^i g_i^U} \frac{d_i^A}{d_0^A}$$

$$= \frac{\left(\beta/z,w/\alpha\beta,q^{-n+1}z/\alpha,q^{-n+1}w;q\right)_i}{\left(1/\alpha,w/z,q^{-n+1}\beta,q^{-n+1}zw/\alpha\beta;q\right)_i},$$
(D.54)

$$\frac{\mathcal{B}_{nj}(q^{-n}\beta/z,q^{-n}\beta/w)}{\mathcal{B}_{n0}(q^{-n}\beta/z,q^{-n}\beta/w)} \frac{1}{\mathcal{B}_{j0}(q^{-n}\beta/w,q^{-n}z/\alpha)} \frac{(q^{-n}\beta/w)^j h_j^{\prime A} w^j h_j^{\prime A}}{g_j^{\prime U} q^{j^2} g_j^{\prime L}} \frac{d_0^{\prime A}}{d_j^{\prime A}}$$

$$= \frac{\left(\beta/z,1/w,q^{-n+1}z/\alpha,q^{-n+1}\alpha\beta/w;q\right)_j}{\left(1/\alpha,\alpha\beta/zw,q^{-n+1}\beta,q^{-n+1}z/w;q\right)_j}.$$
(D.55)

D.8 Final step of proof with Bailey's transformation formula for terminating very-well-poised balanced $_{10}\phi_9$ series

Recall that Bailey's transformation formula for terminating very-well-poised balanced $_{10}\phi_9$ series [28, formulas (2.9.4) and (2.9.5)] reads

$$_{10}W_9(a;b,c,d,e,f,g,h;q,q)$$

$$= \frac{(aq,aq/ef,aq/eg,aq/eh,aq/fg,aq/fh,aq/gh,aq/efgh;q)_\infty}{(aq/e,aq/f,aq/g,aq/h,aq/efg,aq/efh,aq/egh,aq/fgh;q)_\infty}$$

$$\times {}_{10}W_9\big(qa^2/bcd;aq/bc,aq/bd,aq/cd,e,f,g,h;q,q\big),$$
(D.56)

where at least one of the parameters e, f, g, h is of the form q^{-n}, $n = 0,1,2,\ldots$, and the balancing condition is satisfied, namely

$$q^2a^3 = bcdefgh.$$
(D.57)

In particular, by setting $h = q^{-n}$, we have

$$_{10}W_9(a;b,c,d,e,f,g,q^{-n};q,q)$$
(D.58)

$$= \frac{(aq, aq/ef, aq/eg, aq/fg; q)_n}{(aq/e, aq/f, aq/g, aq/efg; q)_n} {}_{10}W_9\big(qa^2/bcd; aq/bc, aq/bd, aq/cd, e, f, g, q^{-n}; q, q\big).$$

Proposition D.23. *Let $n, i, j \in \mathbb{Z}_{\geq 0}$ and $i, j \leq n$. Applying Bailey's transformation twice, we have*

$$
{}_{10}W_9\big(q^{-n}z/\alpha; q^{-i}, q^{-j}, q^{-n+i}w/\alpha, q^{-n+j}\beta/w, z/\alpha\beta, qz/\alpha, q^{-n+1}z; q, q\big)
$$
$$
= \frac{(1/\alpha, 1/\beta, w/z, \alpha\beta/zw; q)_n}{(\alpha/z, \beta/z, 1/w, w/\alpha\beta; q)_n} \frac{(\beta/z, w/\alpha\beta, q^{-n+1}z/\alpha, q^{-n+1}w; q)_i}{(1/\alpha, w/z, q^{-n+1}\beta, q^{-n+1}zw/\alpha\beta; q)_i}
$$
$$
\times \frac{(\beta/z, 1/w, q^{-n+1}z/\alpha, q^{-n+1}\alpha\beta/w; q)_j}{(1/\alpha, \alpha\beta/zw, q^{-n+1}\beta, q^{-n+1}z/w; q)_j}
$$
$$
\times {}_{10}W_9\big(q^{-n}z/\beta; q^{-n+i}, q^{-n+j}, q^{-i}\alpha/w, q^{-j}w/\beta, z/\alpha\beta, qz/\beta, q^{-n+1}z; q, q\big). \tag{D.59}
$$

Proof. Observe that the balancing condition as in (D.57) is satisfied

$$
q^2\left(\frac{q^{-n}z}{\alpha}\right)^3 = q^{-i}q^{-j}\frac{q^{-n+i}w}{\alpha}\frac{q^{-n+j}\beta}{w}\frac{z}{\alpha\beta}\frac{qz}{\alpha}q^{-n+1}z.
$$

Hence, Bailey's transformation formula (D.58) applies and we have

$$
{}_{10}W_9\big(q^{-n}z/\alpha; q^{-i}, q^{-j}, q^{-n+i}w/\alpha, q^{-n+j}\beta/w, z/\alpha\beta, qz/\alpha, q^{-n+1}z; q, q\big)
$$
$$
= {}_{10}W_9\big(q^{-n}z/\alpha; q^{-j}, q^{-n+j}\beta/w, qz/\alpha, q^{-n+i}w/\alpha, z/\alpha\beta, q^{-n+1}z, q^{-i}; q, q\big)
$$
$$
= \frac{(q^{-n+1}z/\alpha, q^{-i+1}\alpha\beta/w, q^{-i+n}/w, \beta/z; q)_i}{(q^{-i+1}z/w, q^{-n+1}\beta, 1/\alpha, q^{-i+n}\alpha\beta/zw; q)_i}
$$
$$
\times {}_{10}W_9\big(q^{-n}zw/\alpha\beta; qzw/\alpha\beta, q^{-n+j}, q^{-j}w/\beta, q^{-n+i}w/\alpha, z/\alpha\beta, q^{-n+1}z, q^{-i}; q, q\big).
$$

Applying Bailey's transformation once again, we have

$$
{}_{10}W_9\big(q^{-n}zw/\alpha\beta; qzw/\alpha\beta, q^{-n+j}, q^{-j}w/\beta, q^{-n+i}w/\alpha, z/\alpha\beta, q^{-n+1}z, q^{-i}; q, q\big)
$$
$$
= {}_{10}W_9\big(q^{-n}zw/\alpha\beta; q^{-i}, q^{-n+i}w/\alpha, qzw/\alpha\beta, q^{-j}w/\beta, z/\alpha\beta, q^{-n+1}z, q^{-n+j}; q, q\big)
$$
$$
= \frac{(q^{-n+1}zw/\alpha\beta, q^{-n+j+1}\beta, q^j/\alpha, w/z; q)_{n-j}}{(q^{-n+j+1}z/\alpha, q^{-n+1}w, w/\alpha\beta, q^j\beta/z; q)_{n-j}}
$$
$$
\times {}_{10}W_9\big(q^{-n}z/\beta; qz/\beta, q^{-n+i}, q^{-i}\alpha/w, q^{-j}w/\beta, z/\alpha\beta, q^{-n+1}z, q^{-n+j}; q, q\big).
$$

Simplifying the prefactor we have (D.59). $\blacksquare$

Now we are ready to state our proof of Theorem D.3, therewith its reformulation Theorem D.13.

Proof of Theorem D.3. In view of (D.44)–(D.46) in Proposition D.19, and (D.59) in Proposition D.23, we have the equality of matrices

$$
\frac{1}{\mathsf{L}_{i0}^1 \cdot d_0^R \cdot \mathsf{U}_{0j}^1}\mathsf{L}^1 \cdot D_R \cdot \mathsf{U}^1 = \frac{1}{\mathsf{L}_{i0}^1 \cdot d_0^R \cdot \mathsf{U}_{0j}^1}D_A \cdot \mathsf{U}^2 \cdot D_R' \cdot \mathsf{L}^2 \cdot D_A'^{-1},
$$

which is equivalent to the commutativity $RD_2A = ARD_2$. $\blacksquare$

E Truncation by tuning mass parameters

In [10], we employed the following orbifolded Nekrasov factor for the affine Laumon space:

$$
\mathsf{N}_{\lambda,\mu}^{(k|N)}(u|q,\kappa) = \mathsf{N}_{\lambda,\mu}^{(k)}(u|q,\kappa) \tag{E.1}
$$
$$
= \prod_{\substack{j\geq i\geq 1 \\ j-i\equiv k \pmod N}} \left[uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q\right]_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\substack{\beta\geq\alpha\geq 1 \\ \beta-\alpha\equiv -k-1 \pmod N}} \left[uq^{\lambda_\alpha-\mu_\beta}\kappa^{\alpha-\beta-1};q\right]_{\mu_\beta-\mu_{\beta+1}},
$$

with $[u;q]_n := u^{-n/2}q^{-n(n-1)/4}(u;q)_n$. To work out the tuning condition of mass parameters required for the truncation of the Young diagrams, it is convenient to recast (E.1) to the "Nakajima" form, which is expressed as a product over the boxes of the Young diagrams (λ,μ). In the following, we temporally replace $[u;q]_n$ by $(u;q)_n$, since we can neglect the monomial factors for our purpose.

By taking the product over $\mathsf{N}^{(0)}$ to $\mathsf{N}^{(N-1)}$, we can remove the selection rule,

$$
\mathsf{N}_{\lambda,\mu}(u|q,\kappa) := \prod_{k=0}^{N-1} \mathsf{N}_{\lambda,\mu}^{(k)}(u|q,\kappa)
$$
$$
= \prod_{j\geq i\geq 1} \left(uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q\right)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\beta\geq\alpha\geq 1} \left(uq^{\lambda_\alpha-\mu_\beta}\kappa^{\alpha-\beta-1};q\right)_{\mu_\beta-\mu_{\beta+1}}. \tag{E.2}
$$

Namely, we can obtain $\mathsf{N}_{\lambda,\mu}^{(k|N)}(u|q,\kappa)$ by factorizing the total Nekrasov factor $\mathsf{N}_{\lambda,\mu}(u|q,\kappa)$ according to the power of κ. Now we can transform the total Nekrasov factor as follows:

$$
\mathsf{N}_{\lambda,\mu}(u|q,\kappa) = \prod_{j\geq i\geq 1} \frac{\left(uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q\right)_\infty}{\left(uq^{-\mu_i+\lambda_j}\kappa^{-i+j};q\right)_\infty} \cdot \prod_{i\geq j\geq 1} \frac{\left(uq^{\lambda_j-\mu_i}\kappa^{j-i-1};q\right)_\infty}{\left(uq^{\lambda_j-\mu_{i+1}}\kappa^{j-i-1};q\right)_\infty}
$$
$$
= \prod_{i,j=1}^{\infty} \frac{\left(uq^{\lambda_j-\mu_i}\kappa^{j-i-1};q\right)_\infty}{\left(uq^{\lambda_j-\mu_i}\kappa^{j-i};q\right)_\infty} \cdot \frac{\left(u\kappa^{j-i};q\right)_\infty}{\left(u\kappa^{j-i-1};q\right)_\infty}
$$
$$
= \prod_{(i,j)\in\lambda} \left(1-uq^{\lambda_i-j}\kappa^{-\mu_j^\vee+i-1}\right) \cdot \prod_{(i,j)\in\mu} \left(1-uq^{-\mu_i+j-1}\kappa^{\lambda_j^\vee-i}\right). \tag{E.3}
$$

where we have used the combinatorial identity (cf. proposition in [11], (2.12) = (2.9) with $\kappa=t^{-1}$) for the last equality. When one of the partitions is empty as in the case of fundamental matter contributions, the Nekrasov factor is simplified considerably,

$$
\mathsf{N}_{\lambda,\varnothing}(u|q,\kappa) = \prod_{(i,j)\in\lambda} \left(1-uq^{\lambda_i-j}\kappa^{i-1}\right),
$$
$$
\mathsf{N}_{\varnothing,\mu}(u|q,\kappa) = \prod_{(i,j)\in\mu} \left(1-uq^{-\mu_i+j-1}\kappa^{-i}\right). \tag{E.4}
$$

When $N=2$, the factors from the first rows of λ and μ are contained in $\mathsf{N}_{\lambda,\varnothing}^{(0)}(u|q,\kappa)$ and $\mathsf{N}_{\varnothing,\mu}^{(1)}(u|q,\kappa)$, respectively, which are explicitly

$$
\prod_{k=0}^{\lambda_1-1}\left(1-uq^k\right), \qquad \prod_{k=1}^{\mu_1}\left(1-uq^{-k}\kappa^{-1}\right). \tag{E.5}
$$

Hence, we can see that if $u=q^{-m}$, $\mathsf{N}_{\lambda,\varnothing}^{(0)}(u|q,\kappa)$ vanishes for $m\leq\lambda_1-1$. Similarly, if $u\kappa^{-1}=q^{n+1}$, $\mathsf{N}_{\varnothing,\mu}^{(1)}(u|q,\kappa)$ vanishes for $n+1\leq\mu_1$.

In the localization formula of the partition function, the numerator of the contribution from a fixed point $(\lambda^{(1)}, \lambda^{(2)})$, which is identified with the matter contribution is explicitly given by

$$\prod_{i,j=1}^{2} \mathsf{N}_{\varnothing,\lambda^{(j)}}^{(j-i|2)}(u_i/v_j|q,\kappa)\mathsf{N}_{\lambda^{(i)},\varnothing}^{(j-i|2)}(v_i/w_j|q,\kappa)$$

$$= \mathsf{N}_{\varnothing,\lambda^{(1)}}^{(0)}(u_1/v_1|q,\kappa)\mathsf{N}_{\varnothing,\lambda^{(2)}}^{(1)}(u_1/v_2|q,\kappa)\mathsf{N}_{\varnothing,\lambda^{(1)}}^{(1)}(u_2/v_1|q,\kappa)\mathsf{N}_{\varnothing,\lambda^{(2)}}^{(0)}(u_2/v_2|q,\kappa)$$

$$\times \mathsf{N}_{\lambda^{(1)},\varnothing}^{(0)}(v_1/w_1|q,\kappa)\mathsf{N}_{\lambda^{(1)},\varnothing}^{(1)}(v_1/w_2|q,\kappa)\mathsf{N}_{\lambda^{(2)},\varnothing}^{(1)}(v_2/w_1|q,\kappa)\mathsf{N}_{\lambda^{(2)},\varnothing}^{(0)}(v_2/w_2|q,\kappa). \quad \text{(E.6)}$$

Let us look at the mass parameter tuning conditions such that the columns of the Young diagrams of $\lambda^{(1)}$ and $\lambda^{(2)}$ are at most m and n, respectively. We need a zero coming from $\mathsf{N}_{\lambda^{(1)},\varnothing}^{(0)}(v_1/w_1|q,\kappa)$ or $\mathsf{N}_{\varnothing,\lambda^{(1)}}^{(1)}(u_2/v_1|q,\kappa)$, namely,

$$(v_1/w_1) = q^{-m}, \qquad (u_2/v_1)\kappa^{-1} = q^{m+1}. \quad \text{(E.7)}$$

Similarly, we need a zero coming from $\mathsf{N}_{\lambda^{(2)},\varnothing}^{(0)}(v_2/w_2|q,\kappa)$ or $\mathsf{N}_{\varnothing,\lambda^{(2)}}^{(1)}(u_1/v_2|q,\kappa)$, namely,

$$(v_2/w_2) = q^{-n}, \qquad (u_1/v_2)\kappa^{-1} = q^{n+1}. \quad \text{(E.8)}$$

Recall that we use the following specialization (see (4.7)) in this paper:

$$u_1 = \frac{qQ}{d_3}, \quad u_2 = \frac{\kappa q}{d_1}, \qquad v_1 = 1, \quad v_2 = \frac{Q}{\kappa}, \qquad w_1 = \frac{1}{d_2}, \quad w_2 = \frac{Q}{d_4\kappa}. \quad \text{(E.9)}$$

Hence, the above conditions (E.7) and (E.8) are translated to

$$d_2 = q^{-m}, \qquad d_1 = q^{-m}, \quad \text{(E.10)}$$

and

$$d_4 = q^{-n}, \qquad d_3 = q^{-n}. \quad \text{(E.11)}$$

The conditions for d_1 and d_2 are the same and those for d_3 and d_4 are also the same. Thus, there are four equivalent choices for the truncation of the Young diagrams. In the main text of the paper, we choose $d_2 = q^{-m}$ and $d_3 = q^{-n}$.

F   Affine Laumon space and orbifolded Nekrasov factor

The affine Laumon space is the moduli space of parabolic torsion free sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Originally in [35, 36], a partial compactification of the space of quasi-maps from $\mathbb{P}^1$ to the flag variety of GL_N was defined. The geometric representation theory of the compactification is controlled by $\mathfrak{gl}_N$. There is an action of the $\mathfrak{gl}_N$-Yangian on the equivariant cohomology of the Laumon space [25]. The affine Laumon space is an affine analogue, where the base manifold is replaced by $\mathbb{P}^1 \times \mathbb{P}^1$ and we have an action of the affine Yangian of $\mathfrak{gl}_N$. In [44], it was shown that $U_{\mathsf{v}}(\widehat{\mathfrak{gl}}_n)$ acts on the equivariant K-theory of the affine Laumon space, which cannot be irrelevant to the present work. In algebraic geometry such "affine" quasi-maps are described by torsion free sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with a canonical framing at $\{\infty\} \times \{\infty\}$ and a parabolic structure on $\mathbb{P}^1 \times \{0\}$.

In the context of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories the affine Laumon space was first featured in [15, 16], where the conjecture on the relation of the Nekrasov partition function and the Seiberg–Witten prepotential was proved. Then as a generalization of the AGT correspondence for Virasoro and $\mathcal{W}_N$ conformal blocks, Alday–Tachikawa [6] pointed out that

we could employ the affine Laumon space to describe the theory with a surface defect whose classification by the embedding of $\mathfrak{sl}_2$ subalgebra into $\mathfrak{sl}_N$ is exactly the same as that of the parabolic structure on $\mathbb{P}^1 \times \{0\}$. In [6], it is conjectured when the defect is of the full type the instanton partition function gives a conformal block for the $\mathfrak{sl}_N$ current algebra.[23]

It is known that the parabolic sheaves are equally described by the sheaves with the $\mathbb{Z}_M$ orbifold action $\mathbb{P}^1 \times \mathbb{P}^1 \ni (z, w) \to (z, \omega w)$ with $\omega^M = 1$. Moreover a natural toric action on $\mathbb{P}^1 \times \mathbb{P}^1$ induces that on the affine Laumon space. The fixed points of the toric action are labelled by N-tuples of Young diagrams, which allows us to use the technique of the equivariant localization. In practice the relevant quiver is called the chain-saw quiver [26]. In particular, the equivariant character of the Yangian action on the equivariant cohomology of the affine Laumon space was computed in [25]. It is straightforward to up-lift the formula in [25] to the K theory version, which we use in the present paper. On the method of orbifolding for the computation of the instanton partition function with a surface defect, see, for example, [19, 33, 45, 46, 47].

In [17], the relation of the asymptotically free Macdonald polynomials and the geometric representation theory of the Laumon space was clarified. In an attempt at generalizing this result to an affine version, a formula of the non-stationary Ruijsenaars function is obtained in [52] based on the affine screening operators. The non-stationary Ruijsenaars function proposed in [52] agrees with the instanton partition function obtained from the equivariant Euler character of the affine Laumon space.[24] For the relation of the formula in [52], which we employ in the present paper, and the formula of the equivariant character in [25], see, for example, [12, Appendix B]. See also [43] on the relations of the geometry of affine Laumon space, intertwines of the quantum toroidal algebra and associated integrable systems.

As we have seen in Appendix E, the total Nekrasov factor

$$
\begin{aligned}
\mathsf{N}_{\lambda,\mu}(u|q,\kappa) &= \prod_{(i,j)\in\lambda} \left(1 - uq^{\lambda_i - j}\kappa^{-\mu_j^\vee + i - 1}\right) \cdot \prod_{(i,j)\in\mu} \left(1 - uq^{-\mu_i + j - 1}\kappa^{\lambda_j^\vee - i}\right), \\
&= \prod_{i,j=1}^{\infty} \frac{\left(uq^{\lambda_j - \mu_i}\kappa^{j-i-1}; q\right)_\infty}{\left(uq^{\lambda_j - \mu_i}\kappa^{j-i}; q\right)_\infty} \cdot \frac{\left(u\kappa^{j-i}; q\right)_\infty}{\left(u\kappa^{j-i-1}; q\right)_\infty}
\end{aligned}
\tag{F.1}
$$

can be factorized into $\mathsf{N}_{\lambda,\mu}^{(k|n)}(u|q,\kappa)$, $0 \le k \le n-1$, according to the power of κ. The combinatorial identities in [11, (2.8) and (2.13)] also show the following "transposed" formula for the total Nekrasov factor:

$$
\begin{aligned}
\mathsf{N}_{\lambda,\mu}(u|q, t=\kappa^{-1}) &= \prod_{(i,j)\in\mu} \left(1 - uq^{\lambda_i - j}t^{\mu_j^\vee - i + 1}\right) \cdot \prod_{(i,j)\in\lambda} \left(1 - uq^{-\mu_i + j - 1}t^{-\lambda_j^\vee + i}\right) \\
&= \prod_{i,j=1}^{\infty} \frac{\left(ut^{\mu_i^\vee - \lambda_j^\vee + 1}q^{j-i}; t\right)_\infty}{\left(ut^{\mu_i^\vee - \lambda_j^\vee + 1}q^{j-i-1}; t\right)_\infty} \cdot \frac{\left(utq^{j-i-1}; t\right)_\infty}{\left(utq^{j-i}; t\right)_\infty}.
\end{aligned}
\tag{F.2}
$$

Compared with (F.1), the partitions λ and μ in the formula (F.2) are exchanged in the product over the boxes of Young diagram, while the form of each factor is kept intact. Factorization of (F.2) according to the power of κ gives the following formula of the orbifolded Nekrasov factor:

$$
\mathsf{N}_{\lambda\mu}^{(k|n)}(u|q,\kappa) = \prod_{i\le j} \prod_{\substack{\ell=0 \\ \lambda_{j+1}^\vee - \mu_i^\vee + \ell \equiv k}}^{\lambda_j^\vee - \lambda_{j+1}^\vee - 1} \left[uq^{j-i}\kappa^{\lambda_{j+1}^\vee - \mu_i^\vee + \ell}\right]
$$

---

[23]For the codimension two surface defect of general type the conjecture is that the corresponding chiral algebra is the $\mathcal{W}$ algebra obtained from Drinfeld–Sokolov reduction of $\widehat{\mathfrak{sl}}_N$ current algebra, see, for example, [33].

[24]Physically this corresponds to the gauge theory with adjoint hypermultiplet, sometimes called $\mathcal{N} = 2^*$ theory.

$$\times \prod_{\substack{i \le j}} \prod_{\substack{\ell=0 \\ \lambda_i^\vee - \mu_j^\vee + \ell \equiv k}}^{\mu_j^\vee - \mu_{j+1}^\vee - 1} \left[u q^{i-j-1} \kappa^{\lambda_i^\vee - \mu_j^\vee + \ell} \right], \tag{F.3}$$

where $[u] = u^{-1/2} - u^{1/2}$. Remark that usually the Nekrasov factor (such as $\mathsf{N}_{\lambda,\mu}(u|q,\kappa)$ as above) is expressed in terms of $(u;q)_n$. But we find in the case of the affine Laumon space it is more appropriate to use

$$[u;q]_n = u^{-n/2} q^{-n(n-1)/4} (u;q)_n = [u][qu] \cdots \left[q^{n-1} u \right]. \tag{F.4}$$

When $n \to \infty$, we have to regularize the monomial factor which connects $[u;q]_n$ and $(u;q)_n$. One of the ways of the regularization is to define $[u;q]_\infty$ by

$$[u;q]_\infty := \frac{(u;q)_\infty}{\vartheta_{q^{1/2}}(-u^{1/2})} = \frac{(u^{1/2};q^{1/2})_\infty}{(-q^{1/2}u^{-1/2};q^{1/2})_\infty}, \tag{F.5}$$

where $\vartheta_p(z) := (z;p)_\infty (pz^{-1};p)_\infty$. One can check

$$\frac{[u;q]_\infty}{[q^n u;q]_\infty} = \frac{(u^{1/2};q^{1/2})_\infty}{(-q^{1/2}u^{-1/2};q^{1/2})_\infty} \frac{(-q^{(1-n)/2}u^{-1/2};q^{1/2})_\infty}{(q^{n/2}u^{1/2};q^{1/2})_\infty}$$
$$= (u^{1/2};q^{1/2})_n \left(-q^{(1-n)/2}u^{-1/2};q^{1/2}\right)_n = [u;q]_n. \tag{F.6}$$

In particular, we have $[u;q]_\infty = (u^{-1/2} - u^{1/2})[qu;q]_\infty$. We also find

$$\frac{[u;q]_\infty}{[q/u;q]_\infty} = \frac{(u^{1/2};q^{1/2})_\infty}{(-q^{1/2}u^{-1/2};q^{1/2})_\infty} \frac{(-u^{1/2};q^{1/2})_\infty}{(q^{1/2}u^{-1/2};q^{1/2})_\infty} = \frac{(u;q)_\infty}{(q/u;q)_\infty}. \tag{F.7}$$

It is possible to express the result of the selection rule in the orbifolded Nekrasov factor (F.3) in terms of the floor function $\lfloor \bullet \rfloor$. In the following manipulations, we use the formulas for the floor function,

$$\left\lfloor \frac{\ell}{n} \right\rfloor + 1 = -\left\lfloor \frac{-\ell-1}{n} \right\rfloor, \qquad \left\lfloor \frac{\ell+m}{n} \right\rfloor = \left\lfloor \frac{\ell}{n} \right\rfloor + \left\lfloor \frac{(\ell)+m}{n} \right\rfloor, \qquad \ell, m \in \mathbb{Z}, \tag{F.8}$$

where (ℓ) denotes the residue of an integer ℓ modulo n.

Proposition F.1. *The orbifolded Nekrasov factor* (F.3) *is given by the following formula:*

$$\mathsf{N}_{\lambda\mu}^{(k|n)} = \prod_{i \le j} \left[q^{j-i} \kappa^{k+n\lfloor \frac{\lambda_{j+1}^\vee + n-1-k-(\mu_i^\vee)}{n} \rfloor - n\lfloor \frac{\mu_i^\vee}{n} \rfloor}; \kappa^n \right]_{\lfloor \frac{\lambda_j^\vee + n-1-k-(\mu_i^\vee)}{n} \rfloor - \lfloor \frac{\lambda_{j+1}^\vee + n-1-k-(\mu_i^\vee)}{n} \rfloor}$$
$$\times \prod_{i \le j} \left[q^{i-j-1} \kappa^{k+n\lfloor \frac{\lambda_i^\vee + n-1-k-(\mu_j^\vee)}{n} \rfloor - n\lfloor \frac{\mu_j^\vee}{n} \rfloor}; \kappa^n \right]_{\lfloor \frac{\mu_j^\vee + k + (-\lambda_i^\vee)}{n} \rfloor - \lfloor \frac{\mu_{j+1}^\vee + k + (-\lambda_i^\vee)}{n} \rfloor}$$
$$= \prod_{i \le j} \left[q^{j-i} \kappa^{k+n\lfloor \frac{\lambda_{j+1}^\vee + n-1-k-(\mu_i^\vee)}{n} \rfloor - n\lfloor \frac{\mu_i^\vee}{n} \rfloor}; \kappa^n \right]_{\lfloor \frac{\lambda_j^\vee - 1-k-(\mu_i^\vee)}{n} \rfloor - \lfloor \frac{\lambda_{j+1}^\vee - 1-k-(\mu_i^\vee)}{n} \rfloor}$$
$$\times \prod_{i \le j} \left[q^{i-j-1} \kappa^{k+n\lfloor \frac{\lambda_i^\vee + n-1-k-(\mu_j^\vee)}{n} \rfloor - n\lfloor \frac{\mu_j^\vee}{n} \rfloor}; \kappa^n \right]_{\lfloor \frac{\mu_j^\vee + k + (-\lambda_i^\vee)}{n} \rfloor - \lfloor \frac{\mu_{j+1}^\vee + k + (-\lambda_i^\vee)}{n} \rfloor}. \tag{F.9}$$

Proof. Since the selection rule on ℓ is imposed by $\equiv$ (the equality mod n), $\mathsf{N}_{\lambda\mu}^{(k|n)}$ can be rewritten in terms of κ^n-shifted factorial $(z_0; \kappa^n)_m$. Let us check that the initial value z_0 and the number m of factors for each shifted factorial agree with those in the formula (F.9). To compute them for the first shifted factorial, let us assume

$$\mu_i^\vee = q_1 n + r_1, \qquad \lambda_{j+1}^\vee = q_2 n + r_2, \qquad \lambda_i^\vee = q_3 n + r_3, \qquad 0 \leq r_i \leq n - 1. \tag{F.10}$$

Namely, we may write, for example, $q_1 = \lfloor \frac{\mu_i^\vee}{n} \rfloor$ and $r_1 = (\mu_i^\vee)$. But in the following we will use q_i and r_i for simplicity of notations. The selection rule in (F.3) tells

$$\ell \equiv k + r_1 - r_2. \tag{F.11}$$

Since $0 \leq k, r_1, r_2 \leq n - 1$, we have $1 - n \leq k + r_1 - r_2 \leq 2n - 2$. Accordingly, we can write the initial value of ℓ uniformly as

$$\ell_0 = k + r_1 - r_2 - n \left\lfloor \frac{k + r_1 - r_2}{n} \right\rfloor. \tag{F.12}$$

Hence, we see

$$z_0 = q^{j-i} \kappa^{\lambda_{j+1}^\vee - \mu_i^\vee + \ell_0} = q^{j-i} \kappa^{k + n \lfloor \frac{\lambda_{j+1}^\vee - \mu_i^\vee}{n} \rfloor - n \lfloor \frac{k + r_1 - r_2}{n} \rfloor}. \tag{F.13}$$

Now, let us count the number of factors in the first shifted factorial. Since the initial value of ℓ is given by (F.12) and the upper bound is $\lambda_j^\vee - \lambda_{j+1}^\vee - 1$, we see

$$
\begin{aligned}
m &= \left\lfloor \frac{\lambda_j^\vee - \lambda_{j+1}^\vee - 1 - \ell_0}{n} \right\rfloor + 1 \\
&= (q_3 - q_2) + 1 + \left\lfloor \frac{r_3 - r_1 - k - 1}{n} \right\rfloor + \left\lfloor \frac{k + r_1 - r_2}{n} \right\rfloor \\
&= \left\lfloor \frac{\lambda_j^\vee - r_1 - k - 1}{n} \right\rfloor + \left\lfloor \frac{k + r_1 - \lambda_{j+1}^\vee}{n} \right\rfloor + 1
\end{aligned} \tag{F.14}
$$

Since

$$\left\lfloor \frac{\mu_i^\vee}{n} \right\rfloor = q_1, \tag{F.15}$$

and

$$\left\lfloor \frac{\lambda_{j+1}^\vee + n - 1 - k - (\mu_i^\vee)}{n} \right\rfloor = \begin{cases} q_2 + 1, & 0 \leq r_2 - r_1 - 1 - k, \\ q_2, & -n \leq r_2 - r_1 - 1 - k < 0, \\ q_2 - 1, & r_2 - r_1 - 1 - k < -n, \end{cases} \tag{F.16}$$

the power of κ is

$$\begin{cases} \lambda_{j+1}^\vee - \mu_i^\vee + k + r_1 - r_2 + n, & 1 \leq r_2 - r_1 - k, \\ \lambda_{j+1}^\vee - \mu_i^\vee + k + r_1 - r_2, & 1 - n \leq r_2 - r_1 - k < 1, \\ \lambda_{j+1}^\vee - \mu_i^\vee + k + r_1 - r_2 - n, & r_2 - r_1 - k < 1 - n. \end{cases} \tag{F.17}$$

Comparing this with (F.12), we can confirm an agreement. For the number of factors, what we have to check is

$$\left\lfloor \frac{k + r_1 - \lambda_{j+1}^\vee}{n} \right\rfloor + 1 = - \left\lfloor \frac{\lambda_{j+1}^\vee - 1 - k - (\mu_i^\vee)}{n} \right\rfloor. \tag{F.18}$$

But this is a special case of the first formula of (F.8). We have checked a complete agreement of the first factorial.

Let us proceed to the second factorial. Concerning the initial condition, we can see the formula for the second is obtained from the first simply by the replacement,

$$\left(\mu_i^\vee, \lambda_{j+1}^\vee\right) \longrightarrow \left(\mu_j^\vee, \lambda_i^\vee\right). \tag{F.19}$$

Thus an agreement is proved by appropriate changes of variables. However, counting the number of factors is more involved. Let us assume

$$\lambda_i^\vee = q_4 n + r_4, \qquad \mu_j^\vee = q_5 n + r_5, \qquad \mu_{j+1}^\vee = q_6 n + r_6, \qquad 0 \le r_i \le n-1. \tag{F.20}$$

Then a similar computation as before gives the number of factors in the formula (F.3)

$$m' = \left\lfloor \frac{\mu_j^\vee - \mu_{j+1}^\vee - 1 - \ell_0'}{n} \right\rfloor + 1, \tag{F.21}$$

where

$$\ell_0' = k + r_5 - r_4 - n \left\lfloor \frac{k + r_5 - r_4}{n} \right\rfloor. \tag{F.22}$$

Hence,

$$m' = (q_5 - q_6) + 1 + \left\lfloor \frac{r_4 - r_6 - k - 1}{n} \right\rfloor + \left\lfloor \frac{k + r_5 - r_4}{n} \right\rfloor$$
$$= \left\lfloor \frac{r_4 - \mu_{j+1}^\vee - k - 1}{n} \right\rfloor + \left\lfloor \frac{\mu_j^\vee + k - r_4}{n} \right\rfloor + 1. \tag{F.23}$$

We note $\left(-\lambda_i^\vee\right) = n - r_4$ for $r_4 \neq 0$, but $\left(-\lambda_i^\vee\right) = -r_4 = 0$ for $r_4 \neq 0$. But such a difference does not matter, since we take a difference of the floor function. Hence, we can safely use $\left(-\lambda_i^\vee\right) = -r_4$ and what we have to check is

$$\left\lfloor \frac{r_4 - \mu_{j+1}^\vee - k - 1}{n} \right\rfloor + 1 = -\left\lfloor \frac{\mu_{j+1}^\vee + k + (-\lambda_i^\vee)}{n} \right\rfloor, \tag{F.24}$$

which follows from the first formula of (F.8). ∎

We can also write the formula of Proposition F.1 in terms of the ratio of the regularized infinite products $[u;q]_\infty$ defined by (F.5). Using the formulas (F.6) and (F.8), we find

$$N_{\lambda,\mu}^{(k|n)}(u|q,\kappa) = \frac{\displaystyle\prod_{j>i\ge1} \left[uq^{j-i-1}\kappa^{k-n\lfloor \frac{\mu_i^\vee}{n}\rfloor + n\lfloor \frac{\lambda_j^\vee}{n}\rfloor - n\lfloor \frac{-(\lambda_j^\vee)+k+(\mu_i^\vee)}{n}\rfloor};\kappa^n\right]_\infty}{\displaystyle\prod_{j\ge i\ge1} \left[uq^{j-i}\kappa^{k-n\lfloor \frac{\mu_i^\vee}{n}\rfloor + n\lfloor \frac{\lambda_j^\vee}{n}\rfloor - n\lfloor \frac{-(\lambda_j^\vee)+k+(\mu_i^\vee)}{n}\rfloor};\kappa^n\right]_\infty}$$
$$\times \frac{\displaystyle\prod_{i\ge j\ge1} \left[uq^{j-i-1}\kappa^{k+n\lfloor \frac{\lambda_j^\vee}{n}\rfloor - n\lfloor \frac{\mu_i^\vee}{n}\rfloor - n\lfloor \frac{(\mu_i^\vee)+k-(\lambda_j^\vee)}{n}\rfloor};\kappa^n\right]_\infty}{\displaystyle\prod_{i>j\ge1} \left[uq^{j-i}\kappa^{k+n\lfloor \frac{\lambda_j^\vee}{n}\rfloor - n\lfloor \frac{\mu_i^\vee}{n}\rfloor - n\lfloor \frac{(\mu_i^\vee)+k-(\lambda_j^\vee)}{n}\rfloor};\kappa^n\right]_\infty}$$
$$= \prod_{i,j=1}^\infty \frac{\left[uq^{j-i-1}\kappa^{k-n\lfloor \frac{\mu_i^\vee+k-\lambda_j^\vee}{n}\rfloor};\kappa^n\right]_\infty}{\left[uq^{j-i}\kappa^{k-n\lfloor \frac{\mu_i^\vee+k-\lambda_j^\vee}{n}\rfloor};\kappa^n\right]_\infty} \frac{\left[uq^{j-i}\kappa^k;\kappa^n\right]_\infty}{\left[uq^{j-i-1}\kappa^k;\kappa^n\right]_\infty}. \tag{F.25}$$

For later convenience, let us express the orbifolded Nekrasov factor in terms of $t = \kappa^{-n}$-shifted factorials. We have

$$
\begin{aligned}
&\mathsf{N}^{(k|n)}_{\lambda,\mu}(u|q,t) \\
&= \prod_{j \geq i \geq 1} \left[uq^{j-i}t^{-\frac{k}{n}+\lfloor\frac{\mu_i^\vee}{n}\rfloor-\lfloor\frac{\lambda_{j+1}^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor}; t^{-1} \right]_{\lfloor\frac{\lambda_j^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor-\lfloor\frac{\lambda_{j+1}^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor} \\
&\quad \times \prod_{j \geq i \geq 1} \left[uq^{i-j-1}t^{-\frac{k}{n}-\lfloor\frac{\lambda_i^\vee}{n}\rfloor+\lfloor\frac{\mu_j^\vee+k-(\lambda_i^\vee)}{n}\rfloor}; t^{-1} \right]_{\lfloor\frac{\mu_j^\vee+k-(\lambda_i^\vee)}{n}\rfloor-\lfloor\frac{\mu_{j+1}^\vee+k-(\lambda_i^\vee)}{n}\rfloor} \\
&= \prod_{j \geq i \geq 1} \left[uq^{j-i}t^{1-\frac{k}{n}+\lfloor\frac{\mu_i^\vee}{n}\rfloor-\lfloor\frac{\lambda_j^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor}; t \right]_{\lfloor\frac{\lambda_j^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor-\lfloor\frac{\lambda_{j+1}^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor} \\
&\quad \times \prod_{j \geq i \geq 1} \left[uq^{i-j-1}t^{1-\frac{k}{n}-\lfloor\frac{\lambda_i^\vee}{n}\rfloor+\lfloor\frac{\mu_{j+1}^\vee+k-(\lambda_i^\vee)}{n}\rfloor}; t \right]_{\lfloor\frac{\mu_j^\vee+k-(\lambda_i^\vee)}{n}\rfloor-\lfloor\frac{\mu_{j+1}^\vee+k-(\lambda_i^\vee)}{n}\rfloor} \\
&= \prod_{j \geq i \geq 1} \frac{\left[uq^{j-i}t^{1-\frac{k}{n}+\lfloor\frac{\mu_i^\vee}{n}\rfloor-\lfloor\frac{\lambda_j^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor}; t \right]_\infty}{\left[uq^{j-i}t^{1-\frac{k}{n}+\lfloor\frac{\mu_i^\vee}{n}\rfloor-\lfloor\frac{\lambda_{j+1}^\vee+n-1-k-(\mu_i^\vee)}{n}\rfloor}; t \right]_\infty} \\
&\quad \times \prod_{j \geq i \geq 1} \frac{\left[uq^{i-j-1}t^{1-\frac{k}{n}-\lfloor\frac{\lambda_i^\vee}{n}\rfloor+\lfloor\frac{\mu_{j+1}^\vee+k-(\lambda_i^\vee)}{n}\rfloor}; t \right]_\infty}{\left[uq^{i-j-1}t^{1-\frac{k}{n}-\lfloor\frac{\lambda_i^\vee}{n}\rfloor+\lfloor\frac{\mu_j^\vee+k-(\lambda_i^\vee)}{n}\rfloor}; t \right]_\infty},
\end{aligned}
\tag{F.26}
$$

where we have used

$$
\left[u; t^{-1}\right]_n = \left[ut^{-n+1}; t\right]_n, \qquad n > 0.
\tag{F.27}
$$

Finally, the same manipulation as the last equality of (F.25) gives

$$
\mathsf{N}^{(k|n)}_{\lambda,\mu}(u|q,t) = \prod_{i,j=1}^{\infty} \frac{\left[uq^{j-i}t^{1-\frac{k}{n}+\lfloor\frac{\mu_i^\vee+k-\lambda_j^\vee}{n}\rfloor}; t \right]_\infty}{\left[uq^{j-i-1}t^{1-\frac{k}{n}+\lfloor\frac{\mu_i^\vee+k-\lambda_j^\vee}{n}\rfloor}; t \right]_\infty} \frac{\left[uq^{j-i-1}t^{1-\frac{k}{n}}; t \right]_\infty}{\left[uq^{j-i}t^{1-\frac{k}{n}}; t \right]_\infty}.
\tag{F.28}
$$

When one of the partitions is empty, $\mathsf{N}^{(k|n)}_{\lambda,\mu}$ simplifies as follows:

$$
\begin{aligned}
\mathsf{N}^{(k|n)}_{\lambda,\varnothing}(u|q,\kappa) &= \frac{\prod_{j \geq i \geq 1} \left[uq^{j-i}\kappa^k; \kappa^n \right]_{\lfloor\frac{\lambda_j^\vee+n-1-k}{n}\rfloor}}{\prod_{j \geq i \geq 1} \left[uq^{(j+1)-(i+1)}\kappa^k; \kappa^n \right]_{\lfloor\frac{\lambda_{j+1}^\vee+n-1-k}{n}\rfloor}} \\
&= \prod_{i \geq 1} \left[uq^{i-1}\kappa^k; \kappa^n \right]_{\lfloor\frac{\lambda_i^\vee+n-1-k}{n}\rfloor},
\end{aligned}
\tag{F.29}
$$

$$
\begin{aligned}
\mathsf{N}^{(k|n)}_{\varnothing,\mu}(u|q,\kappa) &= \frac{\prod_{j \geq i \geq 1} \left[uq^{i-j-1}\kappa^{k-n\lfloor\frac{\mu_j^\vee+k}{n}\rfloor}; \kappa^n \right]_{\lfloor\frac{\mu_j^\vee+k}{n}\rfloor}}{\prod_{j \geq i \geq 1} \left[uq^{i-(j+1)}\kappa^{k-n\lfloor\frac{\mu_{j+1}^\vee+k}{n}\rfloor}; \kappa^n \right]_{\lfloor\frac{\mu_{j+1}^\vee+k}{n}\rfloor}} \\
&= \prod_{i \geq 1} \left[uq^{-i}\kappa^{k-n\lfloor\frac{\mu_i^\vee+k}{n}\rfloor}; \kappa^n \right]_{\lfloor\frac{\mu_i^\vee+k}{n}\rfloor}.
\end{aligned}
\tag{F.30}
$$

G   (Anti-)symmetrization in a factorized form

A basis of the space of cohomology classes for the N-tuple Jackson integral is defined by using the anti-symmetrization of the variables $z = (z_1, \ldots, z_N)$ with the weight function $\Delta(q, z)$ (see (3.2)). In general, the result of (anti-)symmetrization is expanded in terms of symmetric polynomials (divided by the Vandermonde determinant for anti-symmetrization). However, from the viewpoint of the correspondence of the Jackson integral and the Bethe vector for quantum integrable system [48], what we need is the (anti-)symmetrization that keeps the factorization. One can rephrase it as a summation over partitions of the index set $[N] := \{1, 2, \ldots, N\}$. We give a proof for the following well-known formulas for completeness.

Proposition G.1. *Let $I_1 \sqcup \cdots \sqcup I_s$ be a partition of $\{1, 2, \ldots, N\}$ with $|I_a| = k_a \geq 1$ and $\mathcal{A}$ be the anti-symmetrization operator of $z = (z_1, \ldots, z_N)$. For any functions $f_a(x)$, $1 \leq a \leq s$, we have*

$$\frac{1}{\Delta(1, z)} \mathcal{A} \left(\prod_{i_1=1}^{k_1} f_1(z_{i_1}) \prod_{i_2=1}^{k_2} f_2(z_{k_1+i_2}) \cdots \prod_{i_s=1}^{k_s} f_s(z_{k_1+\cdots+k_{s-1}+i_s}) \Delta(q, z) \right)$$

$$= \prod_{a=1}^{s} [k_a]_{q^{-1}}! \sum_{I_1 \sqcup \cdots \sqcup I_s} \left\{ \prod_{a=1}^{s} \prod_{i_a \in I_a} f_a(z_{i_a}) \prod_{1 \leq a < b \leq s} \prod_{i \in I_a} \prod_{j \in I_b} \frac{z_i - q^{-1} z_j}{z_i - z_j} \right\}, \tag{G.1}$$

where the sum $\sum_{I_1 \sqcup \cdots \sqcup I_s}$ on the right-hand side is taken over the partitions $I_1 \sqcup \cdots \sqcup I_s$.

Proof. For a permutation $\sigma \in S_N$, we define $I_a = \{\sigma(k_1 + \cdots + k_{a-1} + i) \mid 1 \leq i \leq k_a\}$. Since

$$\frac{1}{\Delta(1, z)} \mathcal{A}(F(z) \Delta(q, z)) = \sum_{\sigma \in S_N} \sigma \left(F(z) \frac{\Delta(q, z)}{\Delta(1, z)} \right),$$

for any $F(z) = F(z_1, \ldots, z_N)$, we can recast the left-hand side of (G.1) to

$$\sum_{\sigma \in S_N} \left(\prod_{a=1}^{s} \prod_{i \in I_a} f_a(z_i) \frac{\Delta(q, z_{I_a})}{\Delta(z_{I_a})} \prod_{1 \leq a < b \leq s} \prod_{i \in I_a} \prod_{j \in I_b} \frac{z_i - q^{-1} z_j}{z_i - z_j} \right).$$

Let S_I be the symmetric group of a set I. Then according to the coset decomposition $S_N = \sqcup_{I_1 \sqcup \cdots \sqcup I_s} S_{I_1} \times \cdots \times S_{I_s}$, we take the sum in the following manner:

$$\sum_{\sigma \in S_N} = \sum_{I_1 \sqcup \cdots \sqcup I_s} \sum_{w_1 \in S_{I_1}} \cdots \sum_{w_s \in S_{I_s}}.$$

By applying the formula [37, Section III, equation (1.4)]

$$\sum_{w_a \in S_{I_a}} w_a \left(\frac{\Delta(q, z_{I_a})}{\Delta(1, z_{I_a})} \right) = [k_a]_{q^{-1}}!$$

for each sum over S_{I_a}, we obtain the desired result (G.1). ∎

H   Shakirov's equation as a coupled system

Consider Shakirov's equation in the form

$$\psi = \mathcal{H}_S T_{t,\Lambda}^{-1} T_{qtQ,x}^{-1} \psi, \tag{H.1}$$

$$\mathcal{H}_{\mathrm{S}} = \frac{1}{\varphi(qx)\varphi(\frac{\Lambda}{x})}\mathcal{B}\frac{\varphi(\Lambda)\varphi\big(q^{-1}d_1d_2d_3d_4\Lambda\big)}{\varphi(-d_1x)\varphi(-d_2x)\varphi\big(-d_3\frac{\Lambda}{x}\big)\varphi\big(-d_4\frac{\Lambda}{x}\big)}\mathcal{B}\frac{1}{\varphi\big(\frac{d_1d_2}{q}x\big)\varphi\big(d_3d_4\frac{\Lambda}{x}\big)}.$$

It has a unique formal series solution normalized as $\psi = 1 + \mathcal{O}\big(x, \frac{\Lambda}{x}\big)$. In terms of the affine Laumon partition function $\mathcal{Z}_{\mathrm{AL}}$, the solution is given by

$$\psi(\Lambda, x) = \mathcal{Z}_{\mathrm{AL}}\left(\begin{array}{cc} u_1, u_2 \\ v_1, v_2 \\ w_1, w_2 \end{array}\middle| x_1, \frac{\Lambda_1}{x_1}\middle| q, t\right)$$

$$= \mathcal{Z}_{\mathrm{AL}}\left(\begin{array}{cc} \frac{qQ}{d_3}, \frac{q\kappa}{d_1} \\ 1, \frac{Q}{\kappa} \\ \frac{1}{d_2}, \frac{\kappa}{\kappa d_4} \end{array}\middle| -\frac{\sqrt{Qd_1d_2}}{\kappa}x, -\sqrt{\frac{d_3d_4}{q^2Q}}\frac{\Lambda}{x}\middle| q, \kappa^{-2}\right). \tag{H.2}$$

We will give a coupled form of Shakirov's equation and its solution. To do this, we define operators T and K by[25]

$$T = \left\{d_2 \mapsto \frac{q}{tQd_2},\ d_4 \mapsto \frac{qQ}{d_4},\ \Lambda \mapsto \frac{d_2d_4\Lambda}{q},\ x \mapsto -\frac{d_2x}{q}\right\},$$

$$K = \frac{1}{\varphi(qx)\varphi\big(\frac{\Lambda}{x}\big)}\mathcal{B}\frac{1}{\varphi(-d_1x)\varphi\big(-d_3\frac{\Lambda}{x}\big)}. \tag{H.3}$$

On the parameters u_i, v_i, w_i, x_1, Λ_1, the operator T acts as

$$T = \left\{w_i \mapsto \frac{tQ}{qw_i},\ x_1 \mapsto -\frac{x_1}{\sqrt{qtQ}},\ \Lambda_1 \mapsto \frac{\Lambda_1}{\sqrt{t}}\right\}. \tag{H.4}$$

In our previous paper [10], we used the renormalized q-Borel transformation $\widetilde{\mathcal{B}} := T^{-1}_{(qt^{1/2}Q)^{1/2},x}\mathcal{B}$ in writing down the coupled system in order to make the correspondence with the qq-Painlevé VI equation clear. If this is taken into account, the action of T on the expansion parameters becomes $x_1 = x \mapsto -t^{-1/4}x_1$ and hence $x_2 = \Lambda_1/x_1 \mapsto -t^{-1/4}x_2$. The T action is the same for x_1 and x_2 and coincides with the action of the shift operator $\widetilde{T}_{\mathsf{p},b}$ with $\mathsf{p} = t^{1/4}$, which is regarded as a square root of the time evolution of the parameters b_i in the qq-Painlevé VI equation (see [10, equation (6.3)]).

Proposition H.1. *The normalized solution ψ of (H.1) and its parameter transform $\chi := T\psi$ satisfy the following coupled system of equations:*

$$\psi = gK\chi, \qquad \chi = TgKT\psi, \tag{H.5}$$

where

$$g = \frac{\big(\frac{td_2d_4}{q}\Lambda, d_1d_3\Lambda; q, t\big)_\infty}{\big(t\Lambda, \frac{td_1d_2d_3d_4}{q}\Lambda; q, t\big)_\infty}. \tag{H.6}$$

Conversely the equation (H.1) follows from the coupled system (H.5).

Proof. From (H.1) and (H.3), one can verify the following relations:

$$T^2 = T^{-1}_{t,\Lambda}T^{-1}_{qQt,x}, \tag{H.7}$$

$$TK = \frac{1}{\varphi(-d_2x)\varphi\big(-d_4\frac{\Lambda}{x}\big)}\mathcal{B}\frac{1}{\varphi\big(\frac{d_1d_2}{q}x\big)\varphi\big(d_3d_4\frac{\Lambda}{x}\big)}T, \tag{H.8}$$

---

[25]One may use d_1 (or d_3) instead of d_2 (or d_4).

$$KTKT = K \frac{1}{\varphi(-d_2 x)\varphi(-d_4 \frac{\Lambda}{x})} \mathcal{B} \frac{1}{\varphi(\frac{d_1 d_2}{q}x)\varphi(d_3 d_4 \frac{\Lambda}{x})} T^2 = \frac{1}{\varphi(\Lambda)\varphi(\frac{d_1 d_2 d_3 d_4}{q}\Lambda)} \mathcal{H}_{\mathrm{S}} T^2, \quad \text{(H.9)}$$

$$KT\mathcal{H}_{\mathrm{S}} T^2 = \varphi\left(\frac{d_2 d_4}{q}\Lambda\right)\varphi\left(\frac{d_1 d_3}{t}\Lambda\right)(KT)^3 = \frac{\varphi(\frac{d_2 d_4}{q}\Lambda)\varphi(\frac{d_1 d_3}{t}\Lambda)}{\varphi(\Lambda)\varphi(\frac{d_1 d_2 d_3 d_4}{q}\Lambda)} \mathcal{H}_{\mathrm{S}} T^2 KT. \quad \text{(H.10)}$$

From (H.7), (H.10) and (H.1), we have

$$KT\psi = KT\mathcal{H}_{\mathrm{S}} T^2 \psi = \frac{\varphi(\frac{d_2 d_4}{q}\Lambda)\varphi(\frac{d_1 d_3}{t}\Lambda)}{\varphi(\Lambda)\varphi(\frac{d_1 d_2 d_3 d_4}{q}\Lambda)} \mathcal{H}_{\mathrm{S}} T^2 KT\psi. \quad \text{(H.11)}$$

Noting that

$$T_{t,\Lambda}^{-1}(g) = \frac{\varphi(\frac{d_2 d_4}{q}\Lambda)\varphi(\frac{d_1 d_3}{t}\Lambda)}{\varphi(\Lambda)\varphi(\frac{d_1 d_2 d_3 d_4}{q}\Lambda)} g,$$

we see the function $gKT\psi$ satisfies the same equation as (H.1) with the same initial condition as ψ. By the uniqueness of the solution, we obtain the first relation $\psi = gKT\psi = gK\chi$. The second relation is the T-transform of the first one. To check the converse, we note that

$$g \cdot T \cdot g = \frac{\left(\Lambda, \frac{d_1 d_2 d_3 d_4}{q}\Lambda; q, t\right)_\infty}{\left(t\Lambda, \frac{td_1 d_2 d_3 d_4}{q}\Lambda; q, t\right)_\infty} \cdot T = \varphi(\Lambda)\varphi\left(\frac{d_1 d_2 d_3 d_4}{q}\Lambda\right) \cdot T. \quad \text{(H.12)}$$

Hence,

$$\psi = g \cdot KT \cdot g \cdot KT\psi = \varphi(\Lambda)\varphi\left(\frac{d_1 d_2 d_3 d_4}{q}\Lambda\right) \cdot KTKT\psi = \mathcal{H}_{\mathrm{S}} T^2 \psi. \quad \text{(H.13)}$$

This completes the proof. ∎

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