Rogers–Ramanujan Type Identities Involving Double Sums

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Abstract. We prove four new Rogers–Ramanujan-type identities for double series. They follow from the classical Rogers–Ramanujan identities using the constant term method and properties of Rogers–Szegő polynomials.

Key words: Rogers–Ramanujan type identities; sum-product identities; constant term method

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1 Introduction

In 1894, L.J. Rogers [10] discovered numerous sum-product q-series identities. Among his findings, he proved the following identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{\left(q,q^4;q^5\right)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{\left(q^2,q^3;q^5\right)_\infty}, \tag{1.2}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left(q^4;q^4\right)_n}=\frac{1}{\left(-q^2;q^2\right)_\infty\left(q,q^4;q^5\right)_\infty}, \tag{1.3}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{\left(q^4;q^4\right)_n}=\frac{1}{\left(-q^2;q^2\right)_\infty\left(q^2,q^3;q^5\right)_\infty}. \tag{1.4}$$

We briefly introduce the notations used in this paper. We always assume $|q|<1$ for convergence. The standard q-series notations are as follows [6]:

$$(a;q)_0:=1,\qquad (a;q)_n:=\prod_{k=0}^{n-1}\left(1-aq^k\right),\qquad (a;q)_\infty:=\prod_{k=0}^{\infty}\left(1-aq^k\right),$$
$$(a_1,\dots,a_m;q)_n=(a_1;q)_n\cdots(a_m;q)_n,\qquad n\in\mathbb{N}\cup\{\infty\}.$$

The identities referred to as (1.1) and (1.2) are recognized as the Rogers–Ramanujan identities, having been rediscovered by Ramanujan prior to the year 1913. They have inspired a lot of

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work on finding identities of similar forms. Rogers–Ramanujan type identities function as one of the witnesses for deep connections between the theory of q-series and modular forms. After multiplying with suitable powers of q, the right-hand side of (1.1) and (1.2) become modular forms which is not clear from the sum sides. An important question in the theory of q-series and modular forms is to judge what kind of basic hypergeometric series qualify as modular forms. This question remains an open challenge in the field. In a series of works, W. Nahm [7, 8, 9] considered the series

$$f_{A,B,C}(q) := \sum_{n=(n_1,\dots,n_r)^T\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^TAn+n^TB+C}}{(q;q)_{n_1}\cdots(q;q)_{n_r}},$$

where $r \geq 1$ is a positive integer, A is a real positive definite symmetric $r\times r$ matrix, B is a vector of length r, and C is a scalar.

Nahm [9] proposed a conjecture that provides sufficient and necessary conditions on the matrix part of a modular triple. The conjecture is formulated in terms of the Bloch group and a system of polynomial equations induced by the matrix part. D. Zagier [13] gave a precise statement of this conjecture. When the rank $r = 1$, the identities (1.1)–(1.4) showed that

$$(A,B,C) = (2,0,-1/60), \qquad (2,1,11/60), \qquad (1/2,0,-1/40), \qquad (1/2,1/2,1/40)$$

are all modular triples. Zagier [13] studied Nahm's problem and identified many possible modular triples. In particular, for rank $r = 1$, Zagier substantiated Nahm's conjecture and proved that there exactly seven modular triples. Besides the four aforementioned triples, the remaining triples are

$$(1,0,-1/48), \qquad (1,1/2,1/24), \qquad (1,-1/2,1/24),$$

which is easily justified by Euler's identities.

Following the notion in [11], some Rogers–Ramanujan type identities are characterized by a distinct structural pattern. For a given integer k, an identity of the following shape is defined as finite sum of

$$\sum_{(i_1,\dots,i_k)\in S} \frac{(-1)^{t(i_1,\dots,i_k)}q^{Q(i_1,\dots,i_k)}}{(q^{n_1};q^{n_1})_{i_1}\cdots(q^{n_k};q^{n_k})_{i_k}} = \prod_{(a,n)\in P}(q^a;q^n)_\infty^{r(a,n)}$$

as Rogers–Ramanujan type identities of index$(n_1,n_2,\dots,n_k)$. Here $t(i_1,\dots,i_k)$ is an integer-valued function, $Q(i_1,\dots,i_k)$ is a rational polynomials in variables $i_1,\dots,i_k,n_1,\dots,n_k$ are positive integers with $\gcd(n_1,n_2,\dots,n_k) = 1$, S is a subset of $\mathbb{Z}^k$, P is a finite subset of $\mathbb{Q}^2$ and $r(a,n)$ are integer-valued functions.

In 2021, Andrews and Uncu [2] proved an identity of index $(1,3)$ and further conjectured that [2, Conjecture 1.2]

$$\sum_{i,j\geq 0} \frac{(-1)^jq^{3j(3j+1)/2+i^2+3ij+i+j}}{(q;q)_i\left(q^3;q^3\right)_j} = \frac{1}{\left(q^2,q^3;q^6\right)_\infty}.$$

This was first proved by Chern [5] and then by Wang [11]. Besides, Cao and Wang [4] established some Rogers–Ramanujan type identities of indexes

$$(1,1), \qquad (1,2), \qquad (1,1,1), \qquad (1,1,3), \qquad (1,2,2), \qquad (1,2,3), \qquad (1,2,4).$$

For instance, they proved that for any $u \in \mathbb{C}$

$$\sum_{i,j,k\geq 0} \frac{(-1)^{i+j}u^{i+3k}q^{(i^2-i)/2+(i-2j+3k)^2/4}}{(q;q)_i\left(q^2;q^2\right)_j\left(q^3;q^3\right)_k} = \frac{\left(u^2;q\right)_\infty\left(q,-u^2;q^2\right)_\infty}{\left(-u^6;q^6\right)_\infty},$$

$$\sum_{i,j,k\geq 0} \frac{(-1)^{(i-2j+3k)/2}u^{i+k}q^{(i^2-i)/2+(i-2j+3k)^2/4}}{(q;q)_i\left(q^2;q^2\right)_j\left(q^3;q^3\right)_k} = \frac{\left(q;q^2\right)_\infty\left(-u^2;q^3\right)_\infty}{\left(u^2;q^6\right)_\infty}.$$

Furthermore, Wang proved Zagier's rank three examples for Nahm's problem one by one in [12].

Motivated by the constant term method [1] and the identities (1.1)–(1.4), we present the following theorem.

Theorem 1. *We have*

$$\sum_{n,m\geq 0} \frac{(-1)^{\binom{n-m}{2}}q^{\frac{3m^2}{4}+\frac{mn}{2}+\frac{3n^2}{4}}}{(q;q)_m(q;q)_n} = \frac{1}{\left(q^2,q^8;q^{10}\right)_\infty}, \tag{1.5}$$

$$\sum_{n,m\geq 0} \frac{(-1)^{\binom{n-m}{2}}q^{\frac{3m^2}{4}+\frac{mn}{2}+\frac{3n^2}{4}+m+n}}{(q;q)_m(q;q)_n} = \frac{1}{\left(q^4,q^6;q^{10}\right)_\infty}, \tag{1.6}$$

$$\sum_{n,m\geq 0} \frac{(-1)^{m}q^{\frac{m^2}{4}+\frac{mn}{2}+\frac{n^2}{4}}}{\left(q^2;q^2\right)_m\left(q^2;q^2\right)_n} = \frac{1}{\left(-q^2;q^2\right)_\infty\left(q,q^4;q^5\right)_\infty}, \tag{1.7}$$

$$\sum_{n,m\geq 0} \frac{(-1)^{m}q^{\frac{m^2}{4}+\frac{mn}{2}+\frac{n^2}{4}+m+n}}{\left(q^2;q^2\right)_m\left(q^2;q^2\right)_n} = \frac{1}{\left(-q^2;q^2\right)_\infty\left(q^2,q^3;q^5\right)_\infty}. \tag{1.8}$$

The paper is organized as follows. In Section 2, we introduce some basic identities and the constant term method. In Section 3, we demonstrate the proof of Theorem 1 using the constant term method.

2 Preliminaries

In this section, we first collect some useful identities on basic hypergeometric series. The q-binomial theorem [6, p. 8] is defined as

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}z^n, \qquad |z|<1.$$

As corollaries, Euler's q-exponential identities assert [6]:

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |z|<1,$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}z^n}{(q;q)_n} = (-z;q)_\infty.$$

The Jacobi's triple product identity is given by [6, p. 15]

$$\left(q,zq^{\frac{1}{2}},q^{\frac{1}{2}}/z;q\right)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}z^n, \qquad z\neq 0.$$

Recall the new representation of Rogers–Szegő polynomials given by Berkovich and Warnaar [3, Theorem 8.1]

$$H_n(t;q)=\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} t^{2r}\left(-q/t;q^2\right)_r\left(-t;q^2\right)_{\lfloor\frac{n+1}{2}\rfloor-r}\begin{bmatrix}\lfloor n/2\rfloor\\ r\end{bmatrix}_{q^2}, \tag{2.1}$$

where $H_n(t;q)$ was originally defined as

$$H_n(t;q)=\sum_{j=0}^{n} t^j\begin{bmatrix}n\\ j\end{bmatrix}_q.$$

By (2.1), we directly deduce that $H_{2n}(-1;q)=\left(q;q^2\right)_n$ and $H_{2n+1}(-1;q)=0$, which are also deduced from the generating function for these polynomials, that was known to Rogers [6].

Besides, for a series $f(z)=\sum_{n=-\infty}^{\infty}a(n)z^n$, we denote the coefficient of z^n by $[z^n]f(z)=a(n)$. Specifically, we use $CT_z f(z)$ to denote the constant term $\left[z^0\right]f(z)$. It is a well-established fact that

$$\oint_K f(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}z}=CT_z f(z)=\left[z^0\right]f(z),$$

where K is a positively oriented, simple closed contour around the origin. Consequently, we often calculate the constant term or integral. By doing so we can transform the original series into new series, which can then be assessed by some well-known identities. For simplicity, when integral calculus is not necessary, we prefer to utilize the constant term method in proofs.

3 The proof of Theorem 1

In this section, we present the proof of Theorem 1 by the constant term method and famous q-series identities.

Proof. For (1.5), we obtain that

$$\begin{aligned}
\sum_{n,m\geq 0}\frac{(-1)^{\binom{n-m}{2}}q^{\frac{3}{4}m^2+\frac{1}{2}mn+\frac{3}{4}n^2}}{(q;q)_m(q;q)_n}&=\sum_{n,m\geq 0}\frac{\mathrm{i}^{n-m}q^{\frac{3}{4}m^2+\frac{1}{2}mn+\frac{3}{4}n^2}}{(q;q)_m(q;q)_n}\\
&=\sum_{n,m\geq 0}\frac{\mathrm{i}^{n-m}q^{\frac{1}{2}\binom{m+n}{2}+\binom{m}{2}+\frac{3m}{4}+\binom{n}{2}+\frac{3n}{4}}}{(q;q)_m(q;q)_n}\\
&=CT_z\sum_{m\geq 0}\frac{\mathrm{i}^m z^m q^{\binom{m}{2}+\frac{3m}{4}}}{(q;q)_m}\sum_{n\geq 0}\frac{(-\mathrm{i})^n z^n q^{\binom{n}{2}+\frac{3n}{4}}}{(q;q)_n}\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{1}{2}\binom{k}{2}}z^{-k}\\
&=CT_z\left(-z^2q^{\frac{3}{2}};q^2\right)_\infty\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{1}{2}\binom{k}{2}}z^{-k}\\
&=CT_z\sum_{n\geq 0}\frac{q^{2\binom{n}{2}}z^{2n}q^{\frac{3n}{2}}}{\left(q^2;q^2\right)_n}\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{1}{2}\binom{k}{2}}z^{-k}\\
&=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{\left(q^2;q^2\right)_n}.
\end{aligned}$$

Using (1.1), we complete the proof of (1.5).

For (1.6), we similarly write the item $q^{\frac{1}{2}\binom{m+n}{2}}$ into $q^{\frac{1}{2}\binom{k}{2}}$. By the constant item method and Euler's q-exponential identities, we can easily prove it.

After exchanging the summation, we prove (1.7) using the special case of (2.1),

$$\begin{aligned}
\sum_{n,m\geq 0}\frac{(-1)^m q^{\frac{m^2}{4}+\frac{mn}{2}+\frac{n^2}{4}}}{\left(q^2;q^2\right)_m\left(q^2;q^2\right)_n} &= \sum_{n\geq 0}\sum_{m=0}^{n}\frac{(-1)^n q^{\frac{n^2}{4}}}{\left(q^2;q^2\right)_n}\cdot(-1)^m\begin{bmatrix} n\\ m\end{bmatrix}_{q^2}\\
&= \sum_{n\geq 0}\frac{(-1)^n q^{\frac{n^2}{4}}}{\left(q^2;q^2\right)_n}\cdot H_n\left(-1;q^2\right)\\
&= \sum_{n\geq 0}\frac{(-1)^{2n} q^{n^2}}{\left(q^2;q^2\right)_{2n}}\cdot\left(q^2;q^4\right)_n = \sum_{n\geq 0}\frac{q^{n^2}}{\left(q^4;q^4\right)_n}.
\end{aligned}$$

With the help of (1.3) and (1.7) is proved.

Similar to the method used in (1.7) and (1.8) can also be proved by the constant term method,

$$\begin{aligned}
\sum_{n,m\geq 0}\frac{(-1)^m q^{\frac{m^2}{4}+\frac{mn}{2}+\frac{n^2}{4}+m+n}}{\left(q^2;q^2\right)_m\left(q^2;q^2\right)_n} &= \sum_{n,m\geq 0}\frac{(-\mathrm{i})^{n-m} q^{\frac{3m+3n}{2}}}{\left(q^2;q^2\right)_m\left(q^2;q^2\right)_n}\cdot q^{\frac{(m+n)(m+n-2)}{4}}\mathrm{i}^{n+m}\\
&= CT_z\sum_{m\geq 0}\frac{q^{\frac{3m}{2}}z^m(i)^m}{\left(q^2;q^2\right)_m}\sum_{n\geq 0}\frac{q^{\frac{3n}{2}}z^n(-\mathrm{i})^n}{\left(q^2;q^2\right)_n}\sum_{k=-\infty}^{\infty}q^{\frac{k(k-2)}{4}}z^{-k}\mathrm{i}^k\\
&= CT_z\frac{1}{\left(-z^2q^3;q^4\right)_\infty}\sum_{k=-\infty}^{\infty}q^{\frac{k(k-2)}{4}}z^{-k}\mathrm{i}^k\\
&= \sum_{n=0}^{\infty}\frac{\left(-z^2q^3\right)^n}{\left(q^4;q^4\right)_n}(-1)^n z^{-2n}q^{2\binom{n}{2}}\\
&= \sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{\left(q^4;q^4\right)_n}.
\end{aligned}$$

Combining with (1.4), we find that (1.8) holds. ■

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