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On Relative Tractor Bundles

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Abstract. This article contributes to the relative BGG-machinery for parabolic geometries. Starting from a relative tractor bundle, this machinery constructs a sequence of differential operators that are naturally associated to the geometry in question. In many situations of interest, it is known that this sequence provides a resolution of a sheaf that can locally be realized as a pullback from a local leaf space of a foliation that is naturally available in this situation. An explicit description of the latter sheaf was only available under much more restrictive assumptions. For any geometry which admits relative tractor bundles, we construct a large family of such bundles for which we obtain a simple, explicit description of the resolved sheaves under weak assumptions on the torsion of the geometry. In particular, we discuss the cases of Legendrean contact structures and of generalized path geometries, which are among the most important examples for which the relative BGG machinery is available. In both cases, we show that essentially all relative tractor bundles are obtained by our construction and our description of the resolved sheaves applies whenever the BGG sequence is a resolution.

Key words: relative BGG-machinery; relative BGG resolution; relative tractor bundle; parabolic geometries; Legendrean contact structure; generalized path geometry

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1 Introduction

Tractor bundles and relative tractor bundles are a special class of vector bundles that are naturally associated to manifolds endowed with a geometric structure from the class of parabolic geometries. A major reason for their importance lies in the role in the construction of Bernstein–Gelfand–Gelfand sequences (or BGG sequences) and in the relative version of this construction, respectively. To explain this, we have to review some background: The motivation for these constructions comes from algebraic results in the realm of semisimple Lie algebras. The starting point was [3], in which of I.N. Bernšteĭn, I.M. Gel'fand and S.I. Gel'fand constructed a resolution of any finite-dimensional representation of a complex semisimple Lie algebra $\mathfrak{g}$ by homomorphisms of Verma modules. This was generalized by J. Lepowsky to resolutions by homomorphisms of generalized Verma modules for a given parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, see [17].

The connection to geometry comes from a duality connecting homomorphisms between generalized Verma modules for $(\mathfrak{g}, \mathfrak{p})$ to invariant differential operators acting on sections of homogeneous vector bundles over the generalized flag variety G/P (for an appropriate Lie group G with Lie algebra $\mathfrak{g}$ and subgroup $P \subset G$ corresponding to $\mathfrak{p}$) coming from irreducible representations of P. The space G/P in turn is the homogeneous model for parabolic geometries of type (G, P), a concept that extends to a general (real or complex) semisimple Lie group G and parabolic subgroup $P \subset G$. The parabolic geometries are defined (uniformly) as Cartan geometries of type (G, P) but there are general results showing that they are equivalent descriptions of simpler underlying structures, whose explicit descriptions are very diverse, see [7]. Among these underlying structures there are well-known examples like conformal structures and Levi-non-degenerate CR structures of hypersurface type that are intensively studied using a variety of other methods. In particular, understanding differential operators intrinsic to such geometries has received a lot of interest.

It turned out that things can not only be moved to the geometric side, but the geometric version of the construction then applies to arbitrary curved geometries of type (G, P). Building on [1], the first general geometric construction was given in [8], with improvements in [4] and later in [10]. The basic input for the construction is a so-called *tractor bundle* that is induced by a representation $\mathbb{V}$ of the group G (which is restricted to P). On the homogeneous model G/P, any such bundle is canonically trivialized and hence endowed with a canonical linear connection. In curved cases, tractor bundles are non-trivial in general, but they still inherit canonical linear connections, whose curvature equivalently encodes the curvature of the Cartan connection. Coupling these *tractor connections* to the exterior derivative, one obtains a twisted de Rham sequence.

The BGG construction then compresses the operators in this sequence to a sequence of higher order operators acting on sections of bundles associated to completely reducible representations of P, which form the BGG sequence. The latter bundles are induced by Lie algebra homology spaces and Kostant's version of the Bott–Borel–Weyl theorem [16] implies that they correspond to the Verma modules showing up in Lepowsky's construction. On locally flat geometries, the de Rham sequence is a fine resolution of the sheaf of local parallel sections of the tractor bundle, and it turns out that the same holds for the induced BGG sequence, so in particular both sequences compute the same cohomology. In the curved case, there still is a close relation between the operators in the BGG sequences and the covariant exterior derivative, which has been exploited very successfully in many applications.

In the setting of the homogeneous models, a relative version of the BGG construction was already used in the book [2] on the Penrose transform. Here one considers two nested parabolic subgroups $Q \subset P \subset G$, which immediately leads to a projection $G/Q \to G/P$. It turns out that the fibers P/Q of this projection again are generalized flag varieties (of a smaller group) and one then uses a "BGG resolution along the fibers". A geometric version of this relative theory was obtained in [10] building on algebraic background from [9]. Starting from a geometry of type (G, Q) on a manifold M, the additional parabolic subgroup P gives rise to a natural distribution $T_\rho M \subset TM$, called the *relative tangent bundle*. The second ingredient one needs is a *relative tractor bundle* which is induced by a completely reducible representation of the group P and the theory of such representations is well understood. It turns out that any such bundle inherits a canonical partial connection $\big($which allows for differentiation in directions in $T_\rho M$ only$\big)$, called the *relative tractor connection*. With some technical subtleties $\big($in particular in the case that $T_\rho M$ is not involutive$\big)$ an analog of the geometric BGG construction discussed above then produces operators between bundles associated to completely reducible representations of Q.

For appropriate initial representations, the relative BGG construction produces operators that cannot be obtained by the original BGG construction, in particular in cases of so-called singular infinitesimal character. In other cases, it provides alternative constructions of BGG operators. A very important feature is that the relative construction provides complexes and resolutions in many non-flat cases. On the one hand, this needs involutivity of the distribution $T_\rho M$, which can be easily characterized in terms of the curvature and the harmonic curvature of the original geometry of type (G, Q). Under slightly stronger curvature conditions one obtains a fine resolution of a sheaf that locally is a pull-back of a sheaf on a local leaf space for the distribution $T_\rho M$. An issue that remained open in [10] is how to concretely identify this sheaf, even in the case that M is globally the total space of a bundle over some base space N with vertical subbundle $T_\rho M$.

It is exactly this point that we address in the current paper. Given $Q \subset P \subset G$ as above, a geometry of type (G, Q) on M canonically determines natural subbundles $T_P^{i'} M \subset TM$, see Section 2.3. The conditions on the torsion of the geometry referred to below are on its behavior with respect to these subbundles. We show in each such situation, there is a large class of relative tractor bundles which, assuming involutivity of $T_\rho M$ and a weak condition on the torsion, can locally be explicitly identified with pullbacks of vector bundles on appropriate local leaf spaces for $T_\rho M$. This is proved in part (1) of Theorem 3.4. Part (2) of this theorem shows that under a slightly stronger condition on the torsion, the sections of these relative tractor bundles, which are parallel for the relative tractor connection, locally are exactly the pullbacks of sections of the bundles on local leaf spaces. These are exactly the sheaves that, assuming appropriate conditions on the curvature, are resolved by the corresponding relative BGG sequence. Moreover, we can explicitly relate the relative tractor connections on this bundle to well-known operations, like the adjoint tractor connection and (in the case that $T_\rho M$ is involutive) to the Bott connection associated to the induced foliation, see Theorem 3.3. These results originally arose during the work on the PhD theses of the second and third author under the direction of the first author for two specific examples of structures.

It should be remarked at this point that the relative theory makes sense only if Q does not correspond to a maximal parabolic subalgebra of $\mathfrak{g}$, since otherwise there is no intermediate parabolic subgroup P. Consequently, the theory does not apply to examples like conformal structures or CR structures of hypersurface type. Still, there are at least two important examples of parabolic geometries for which relative BGG sequences are available, namely Legendrean (or Lagrangean) contact structures and generalized path geometries and these are the two examples referred to above. Both these structures can exist only on manifolds of odd dimension. The former consists of a contact structure together with a chosen decomposition of the contact subbundle into the direct sum of two Legendrean subbundles. These structures are closely similar to CR structures and have been studied intensively during the last years and successfully applied to classification problems in CR geometry, see, e.g., [11, 12] and [18]. For this geometry, our construction essentially provides all relative tractor bundles. Generalized path geometries are also defined by a configuration of distributions, see Section 3.5 below. Their importance comes from the fact that they provide an equivalent encoding of systems of second order ODEs, see [14]. For generalized path geometries, we obtain essentially all relative tractor bundles for one of the possible intermediate parabolic subgroups and a large subclass for the other.

Let us briefly outline the contents of the paper. In Section 2, we first collect the necessary setup for nested parabolics. As a new ingredient compared to the discussion in [9, 10], we focus on the natural bigrading of the Lie algebra $\mathfrak{g}$ induced by the nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$, which is very convenient for our purposes. Considering an appropriate filtration induced by this bigrading, one quickly arrives at a class of representations that induce basic relative tractor bundles, which can be obtained as subquotients of the adjoint tractor bundle of the geometry.

In Section 3, we start by showing that the relative tractor connections on the bundles in our class are induced by the adjoint tractor connection. Next, we derive an explicit formula for the relative tractor connections in terms of the Lie bracket of vector fields and the torsion of the geometry. The relation to the Lie bracket is crucial for the proof of Theorem 3.4, which is the main technical result of our article. Under appropriate conditions on the torsion, we obtain a filtration of the tangent bundle of local leaf spaces and relate the relative tractor bundles we have constructed to the components of the associated graded vector bundle to this filtration. Under slightly stronger conditions, it is shown that sections, which are parallel for the relative tractor connection, coincide with pullbacks of sections on the local leaf space. This quickly leads to a large class of relative tractor bundles for which similar results are available, see Section 3.3.

In the last part of the article, we discuss the two examples which gave rise to the developments of this article, namely Legendrean contact structures in Section 3.4 and (generalized) path geometries in Section 3.5. In both cases, the situation is relatively simple in the sense that the filtrations of the tangent bundle of local leaf spaces are trivial, so the special relative tractor bundles we construct are all directly related to tensor bundles on local leaf spaces. For both examples of structures, there are two possible intermediate parabolic subgroups between Q and G. We in particular prove that, up to isomorphism, our construction produces essentially all relative tractor bundles for both intermediate groups in the Legendrean contact case and for one of the intermediate groups in the case of generalized path geometries.

2 A basic class of relative tractor bundles

We start by briefly recalling the setup of two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a semisimple Lie algebra $\mathfrak{g}$ with a compatible choice $Q \subset P \subset G$ of groups as discussed in [9]. The focus here is on a natural bigrading induced by the pair $(\mathfrak{q}, \mathfrak{p})$.

2.1 The bigrading determined by a pair of parabolics

There is a well-known correspondence between standard parabolic subalgebras of a real or complex semisimple Lie algebra $\mathfrak{g}$, gradings of $\mathfrak{g}$, and (after an appropriate choice of Cartan subalgebra) subsets of the set of simple roots respectively simple restricted roots. By definition, any (restricted) root can be uniquely written as a linear combination of simple (restricted) roots with integer coefficients that are either all ≥ 0 or all ≤ 0. Given a subset Σ of simple (restricted) roots, one defines the Σ-height of a root α to be the sum of the coefficients of all elements of Σ in this expansion of α. The grading corresponding to Σ is then defined by putting the Cartan subalgebra into degree zero and giving (restricted) root spaces degree equal to the Σ-height of the root. The compatibility of the (restricted) root decomposition with the Lie bracket of $\mathfrak{g}$, readily implies compatibility of this grading with the Lie bracket. In particular, the sum of all grading components of non-positive degree is a subalgebra of $\mathfrak{g}$ and this is the parabolic subalgebra determined by Σ. See [7, Sections 3.2.1 and 3.2.9] for details.

In this language, two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ correspond to subsets $\Sigma_{\mathfrak{q}}$ and $\Sigma_{\mathfrak{p}}$ such that $\Sigma_{\mathfrak{p}} \subset \Sigma_{\mathfrak{q}}$. This readily implies that for each positive (restricted) root α the $\Sigma_{\mathfrak{p}}$-height of α is $\leq$ its $\Sigma_{\mathfrak{q}}$-height. Using this observation, we define the bigrading determined by $(\mathfrak{q}, \mathfrak{p})$.

Definition 2.1. Consider a real or complex semisimple Lie algebra $\mathfrak{g}$ endowed with two nested standard parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ corresponding to subsets $\Sigma_{\mathfrak{p}} \subset \Sigma_{\mathfrak{q}}$ of simple (restricted) roots. Then we define $\mathfrak{g}_{(0,0)} \subset \mathfrak{g}$ to be the sum of the Cartan subalgebra and all (restricted) root spaces of roots for which both the $\Sigma_{\mathfrak{p}}$-height and the $\Sigma_{\mathfrak{q}}$-height is zero. For integers i', i'' which have the same sign and are not both zero, we define $\mathfrak{g}_{(i',i'')}$ to be the direct sum of all the restricted root spaces for roots of $\Sigma_{\mathfrak{p}}$-height i' and $\Sigma_{\mathfrak{q}}$-height $i' + i''$.

We remark that the set $\Sigma_{\mathfrak{q}} \setminus \Sigma_{\mathfrak{p}}$ of simple roots gives rise to a second parabolic subalgebra $\tilde{\mathfrak{p}} \subset \mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \tilde{\mathfrak{p}}$. The integers i'' in Definition 2.1 correspond to the grading of $\mathfrak{g}$ induced by that parabolic. Since $\tilde{\mathfrak{p}}$ does not play a role in the geometric developments below, we avoid using it explicitly.

By construction we have a decomposition $\mathfrak{g} = \bigoplus_{i',i''} \mathfrak{g}_{(i',i'')}$ and we start by clarifying some elementary properties. Recall that the nilradical $\mathfrak{p}_+$ of a parabolic $\mathfrak{p}$ is the sum all positive grading components, while the reductive Levi-factor $\mathfrak{p}_0$ is the degree zero component of the grading determined by $\mathfrak{p}$.

Proposition 2.2. *The decomposition $\mathfrak{g} = \bigoplus_{i',i''} \mathfrak{g}_{(i',i'')}$ induced by $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ has the following properties:*

(i) $\left[\mathfrak{g}_{(i',i'')}, \mathfrak{g}_{(j',j'')}\right] \subset \mathfrak{g}_{(i'+i'',j'+j'')}$*, so one obtains a bigrading of the Lie algebra $\mathfrak{g}$. In particular, the bracket vanishes on $\mathfrak{g}_{(0,i'')} \times \mathfrak{g}_{(i',0)}$ if $i' > 0$ and $i'' < 0$ or $i' < 0$ and $i'' > 0$.*

(ii) $\mathfrak{p} = \bigoplus_{i' \geq 0, i''} \mathfrak{g}_{(i',i'')}$*,* $\mathfrak{p}_+ = \bigoplus_{i'>0,i''} \mathfrak{g}_{(i',i'')}$*, and* $\mathfrak{p}_0 = \bigoplus_{i''} \mathfrak{g}_{(0,i'')}$*.*

(iii) $\mathfrak{q} = \bigoplus_{i',i'' \geq 0} \mathfrak{g}_{(i',i'')}$ *and* $\mathfrak{q}_+ = \bigoplus_{i'+i''>0} g_{(i',i'')}$*, and* $\mathfrak{q}_0 = \mathfrak{g}_{(0,0)}$*.*

Proof. Let us denote by $\Sigma_{\mathfrak{p}} \subset \Sigma_{\mathfrak{q}}$ the subsets of simple (restricted) roots corresponding to the two parabolics. Then the first claim in (i) follows immediately from additivity of the Σ-height for any subset Σ. The second claim follows since we know that the bracket has values in $\mathfrak{g}_{(i',i'')}$ which by construction is zero under our assumptions on i' and i''. The claims in (ii) follow since the first index records the $\Sigma_{\mathfrak{p}}$-height. The claims in (iii) follow the sum of the two indices is the $\Sigma_{\mathfrak{q}}$-height and the only possibility to obtain $i'+i''=0$ is $i'=i''=0$ since both indices have to be either non-negative or non-positive. ■

The decomposition we obtain is a refinement of the one in formula (2.1) of [9]. In the notation used there, we get $\mathfrak{p}_- = \bigoplus_{i'<0,i''\leq 0} \mathfrak{g}_{(i',i'')}$, $\mathfrak{p}_0 \cap \mathfrak{q}_- = \bigoplus_{i''<0} \mathfrak{g}_{(0,i'')}$, $\mathfrak{p}_0 \cap \mathfrak{q}_+ = \bigoplus_{i''>0} \mathfrak{g}_{(0,i'')}$ for the components are not determined in Proposition 2.1.

2.2 The group level and invariant filtrations

It is usual in the theory of parabolic geometries, that gradings have to be viewed as auxiliary objects, since they are not invariant under the actions of parabolic subalgebras. To obtain objects that are invariant, one has to pass to associated filtrations. To do this, we first recall from [9] how to appropriately choose groups in the situation of nested parabolics. Given $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ as above, we first fix a group G with Lie algebra $\mathfrak{g}$ and choose a parabolic subgroup $P \subset G$ corresponding to $\mathfrak{p}$. It is well known that this means that P lies between the normalizer $N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G and its connected component of the identity. Finally, we choose a subgroup $Q \subset P$ corresponding to the Lie subalgebra $\mathfrak{q}$, which means that it lies between $N_P(\mathfrak{q})$ and its connected component of the identity. Since P and Q are parabolic subgroups, it is well known that the exponential map restricts to diffeomorphisms from $\mathfrak{p}_+$ onto a closed normal subgroup $P_+ \subset P$ and from $\mathfrak{q}_+$ onto a closed normal subgroup $Q_+ \subset Q$. From Proposition 2.2, we know that $\mathfrak{p}_+ \subset \mathfrak{q}_+$ so $P_+ \subset Q_+ \subset Q$.

Parabolic geometries of type (G,Q) come with principal Q-bundles, so via the construction of associated vector bundles, any representation of Q determines a natural vector bundle on each manifold endowed with such a geometry. Observe that by construction the subgroup $Q_+ \subset Q$ is nilpotent and it is well known that the quotient Q/Q_+ is reductive. Hence, representations of Q are not easy to understand in general. Special classes of natural bundles are obtained via considering a restricted class of representations of Q, which often are easier to understand. Classical examples of this situation (which are available for any parabolic subgroup) are *completely reducible bundles* that correspond to direct sums of irreducible representations of Q (on which Q_+ automatically acts trivially) and *tractor bundles* that correspond to restrictions to Q of representations of the group G. So in both these cases the relevant representation theory is well understood.

In the more restrictive situation of two nested parabolics $Q \subset P \subset G$, two more classes of natural bundles were introduced in [10, Definition 2.1]. These are *relative natural bundles* which correspond to representations of Q on which P_+ acts trivially and *relative tractor bundles* which correspond to restrictions to Q of representations of P on which P_+ acts trivially. So in particular any completely reducible representation of P gives rise to a relative tractor bundle on geometries of type (G,Q). There is a simple family of appropriate representations of P, which are just the filtration components of the filtration of $\mathfrak{g}$ determined by $\mathfrak{p}$.

Definition 2.3. Consider a real or complex semisimple Lie algebra $\mathfrak{g}$ endowed with the bigrading $\mathfrak{g} = \bigoplus_{i',i''} \mathfrak{g}_{(i',i'')}$ determined by two nested standard parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$. Then for each i', we define $\mathfrak{g}^{(i',*)} := \bigoplus_{j'\geq i',j''} \mathfrak{g}_{(j',j'')}$.

The main properties of these subspaces are easy to clarify.

Proposition 2.4. *Consider a real or complex semisimple Lie algebra $\mathfrak{g}$ endowed with the bigrading $\mathfrak{g} = \bigoplus_{i',i''} \mathfrak{g}_{(i',i'')}$ determined by two nested standard parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$. Then for each integer i' the subspace $\mathfrak{g}^{(i',*)} \subset \mathfrak{g}$ is invariant for the restriction of the adjoint action of G to P. Moreover, $[\mathfrak{p}_+, \mathfrak{g}^{(i',*)}] \subset \mathfrak{g}^{(i'+1,*)}$, so the subgroup P_+ acts trivially on the quotient $\mathbb{V}_{i'} := \mathfrak{g}^{(i',*)}/\mathfrak{g}^{(i'+1,*)}$. For each i'' such that $\mathfrak{g}_{(i',i'')} \neq 0$, let $\mathbb{V}_{i'}^{i''} \subset \mathbb{V}_{i'}$ be the image of the subspace $\bigoplus_{j' \geq i', j'' \geq i''} \mathfrak{g}_{(j',j'')} \subset \mathfrak{g}^{(i',*)}$ under the quotient projection. Then each of the spaces $\mathbb{V}_{i'}^{i''}$ is Q-invariant and they define a decreasing filtration of $\mathbb{V}_{i'}$ in the sense that for $i'' \leq \ell''$ we get $\mathbb{V}_{i'}^{i''} \supset \mathbb{V}_{i'}^{\ell''}$.*

Proof. The grading of $\mathfrak{g}$ induced by $\mathfrak{p}$ induces a filtration, whose components by construction are exactly the space $\mathfrak{g}^{(i',*)}$. It is well known this components can be obtained algebraically from $\mathfrak{p}$ as the nilradical, the elements of its derived series and annihilators under the Killing form, compare with [7, Corollary 3.2.1]. Hence, any element of $N_G(\mathfrak{p})$ normalizes each of the filtration components which shows that each $\mathfrak{g}^{(i',*)}$ is P-invariant. In particular, $\mathfrak{p}_+ = \mathfrak{g}^{(1,*)}$ and the second claim follows from Proposition 2.2. Applying this argument to Q, we conclude that each of the spaces $\bigoplus_{j' \geq i', j'' \geq i''} \mathfrak{g}_{(i',j')}$ is Q-invariant and from this the second part follows readily. ■

2.3 Some basic relative tractor bundles

We can now obtain the distinguished relative tractor bundles by passing to associated bundles. Recall that, given a parabolic geometry $(p\colon \mathcal{G} \to M, \omega)$, the associated bundle induced by the restriction of the adjoint representation of G to Q is the *adjoint tractor bundle* $\mathcal{A}M = \mathcal{G} \times_Q \mathfrak{g}$. By Proposition 2.4, we see that for each i', we get a natural subbundle corresponding to the subspace $\mathfrak{g}^{(i',*)} \subset \mathfrak{g}$ which is P-invariant and hence Q invariant. We denote this subbundle by $\mathcal{A}^{(i',*)}M \subset \mathcal{A}M$. Of course we can then form the quotient $\mathcal{V}_{i'}M := \mathcal{A}^{(i',*)}M/\mathcal{A}^{(i'+1,*)}M$ which is isomorphic to $\mathcal{G} \times_Q \big(\mathfrak{g}^{(i',*)}/\mathfrak{g}^{(i'+1,*)}\big) = \mathcal{G} \times_Q \mathbb{V}_{i'}$. Hence, by Proposition 2.4, each $\mathbb{V}_{i'}$ is a relative tractor bundle. Moreover, the Q-invariant subspaces $\mathbb{V}_{i'}^{i''} \subset \mathbb{V}_{i'}$ give rise to a filtration of $\mathcal{V}_{i'}M$ by smooth subbundles $\mathcal{V}_{i'}^{i''}M$, each of which is a relative natural bundle.

In what follows, we will mainly be interested in the case where $i' < 0$. The relevance of this condition is that by construction $\mathfrak{g}^{(0,*)} = \mathfrak{p} \supset \mathfrak{q}$. It is well known that, via the Cartan connection ω, the associated bundle $\mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$ is isomorphic to TM. Now the subspace $\mathfrak{p} \subset \mathfrak{g}$ is P-invariant and hence Q-invariant, so $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is a Q-invariant subspace, too. Hence, it defines a natural subbundle in TM, which is called the *relative tangent bundle* $T_\rho M$ in [10]. For $i' < 0$, we get in the same way a Q-invariant subspace $\mathfrak{g}^{(i',*)}/\mathfrak{q}$ which gives rise to a smooth subbundle of TM that contains $T_\rho M$. We denote this bundle by $T_P^{i'}M$ to distinguish it form the filtration component coming from the initial parabolic geometry. By construction, we then can naturally identify $\mathcal{V}_{-1}M$ with $T_P^{-1}M/T_\rho M$ and, for $i' < -1$, $\mathcal{V}_{i'}M = T_P^{i'}M/T_P^{i'+1}M$, so we always obtain natural subquotients of the tangent bundle. This will be crucial for the later developments. In what follows it will be convenient to define $T_P^0 M := T_\rho M$, which in particular allows us to uniformly write $\mathcal{V}_{i'}M = T_P^{i'}M/T_P^{i'+1}M$ for any $i' \leq -1$.

A particularly important special case is that the relative tangent bundle $T_\rho M \subset TM$ is an involutive distribution on M. In this case, we can find *local leaf spaces* for this distribution. More explicitly, for any $x \in M$ there is an open neighborhood $U \subset M$ of x and a surjective submersion $\psi\colon U \to N$ onto some smooth manifold N such that, for any $y \in U$, $\ker(T_y\psi)$ coincides with the fiber of $T_\rho M$ at y. In particular, this implies that $T\psi$ induces an isomorphism $(TM/T_\rho M)|_U \cong \psi^*TN$. Characterizing involutivity of $T_\rho M$ and the question of which part of the given geometric structure on M descends to local leaf spaces is a key part of the theory of twistor spaces for parabolic geometries that was originally developed in [5]. Recall that the curvature of the Cartan connection ω can be interpreted as a two-form $\kappa \in \Omega^2(M, \mathcal{A}M)$ and since TM naturally is a quotient of $\mathcal{A}M$, there is a projection $\tau \in \Omega^2(M, TM)$ of κ, called the *torsion* of the Cartan connection ω. By [10, Proposition 4.6], involutivity of $T_\rho M$ is equivalent to the fact that τ maps $T_\rho M \times T_\rho M$ to $T_\rho M$.

There is a strengthening of this condition that plays an important role in the theory of relative BGG sequences, namely that κ vanishes on $T_\rho M \times T_\rho M$. (In [10], this is phrased as vanishing of the relative curvature.) By [10, Theorem 4.11], this implies that the relative BGG sequence determined by any relative tractor bundle is a complex and a fine resolution of a sheaf on M. This sheaf can be identified with the sheaf of those local sections of the relative tractor bundle which are parallel for the relative tractor connection, which will be discussed in Section 3.1 below. It is shown in [10, Section 4.7] that this implies that locally the resolved sheaf can be written as a pullback of a sheaf on a local leaf space. In the special case of a so-called correspondence space (which means that the geometry descends to the leaf space), that sheaf is identified explicitly in [10, Theorem 4.11]. In general, this sheaf is not identified in [10], we will present a solution to this question under some assumptions below.

3 Relative tractor connections and resolved sheaves

3.1 Restrictions of the adjoint tractor connection

A crucial feature of tractor bundles is that they carry canonical linear connections known as tractor connections. There is a uniform description of these tractor connections, based on the so-called fundamental derivative. For a geometry $(p\colon \mathcal{G} \to M, \omega)$ of type (G, Q) consider any representation $\mathbb{W}$ of Q and the corresponding natural vector bundle $\mathcal{W}M := \mathcal{G} \times_Q \mathbb{W}$. Then smooth sections of $\mathcal{W}M$ are in bijective correspondence with smooth functions $\mathcal{G} \to \mathbb{W}$ which are Q-equivariant in the sense that $f(u \cdot g) = g^{-1} \cdot f(u)$ for any $g \in Q$, where in the left-hand side we use the principal right action of Q on $\mathcal{G}$ and in the right-hand side the given representation on $\mathbb{W}$. In this way, sections of the adjoint tractor bundle $\mathcal{A}M$ get identified with Q-equivariant smooth functions $f\colon \mathcal{G} \to \mathfrak{g}$. But via the Cartan connection ω, smooth functions $\mathcal{G} \to \mathfrak{g}$ are in bijective correspondence with vector fields on $\mathcal{G}$, and the equivariancy condition on f is equivalent to the corresponding vector field $\xi \in \mathfrak{X}(\mathcal{G})$ being invariant under the principal right action of Q, i.e., $(r^g)^*\xi = \xi$ for any $g \in Q$.

Differentiating an equivariant function with respect to an invariant vector field produces an equivariant function, so we can view this as defining an operator $D\colon \Gamma(\mathcal{A}M) \times \Gamma(\mathcal{W}M) \to \Gamma(\mathcal{W}M)$ called the *fundamental derivative*, see [7, Section 1.5.8]. This has strong invariance properties and to emphasize the analogy to a covariant derivative, it is usually written as $(s, \sigma) \mapsto D_s\sigma$.

Now for a tractor bundle, we start with a representation $\mathbb{V}$ of G, so the infinitesimal representation defines a bundle map $\bullet\colon \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$ and we denote the corresponding tensorial map on sections by the same symbol. In the case that $\mathcal{V}M = \mathcal{A}M$, the infinitesimal representation is just the adjoint representation of $\mathfrak{g}$, so for $s, t \in \Gamma(\mathcal{A}M)$, we get $s \bullet t = \{s, t\}$ where $\{\ ,\ \}$ is induced by the Lie bracket on $\mathfrak{g}$. Recall from Section 2.3 that there is a natural projection $\Pi\colon \mathcal{A}M \to TM$. The induced operation on sections relates nicely to the picture of Q-invariant vector fields since any such vector field is projectable, and for $s \in \Gamma(\mathcal{A}M)$, $\Pi(s) \in \mathfrak{X}(M)$ is just the projection of the Q-invariant vector field corresponding to s. It turns out that for a tractor bundle $\mathcal{V}M$, $s \in \Gamma(\mathcal{A}M)$ and $\sigma \in \Gamma(\mathcal{V}M)$ the combination $D_s\sigma + s \bullet \sigma$ depends only on $\Pi(s)$ so this descends to an operation $\Gamma(TM) \times \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{V}M)$, see again [7, Section 1.5.8]. This turns out to be a linear connection, the *tractor connection* $\nabla^{\mathcal{V}}$. In particular, we get the defining equation for the adjoint tractor connection: $\nabla^{\mathcal{A}}_{\Pi(s)}t = D_s t + \{s, t\}$.

While things are not formulated in this way in [10] (see the proof of Theorem 3.1 for details), relative tractor connections can be constructed in a very similar fashion. Suppose we have a bundle $\mathcal{V}M$ that is induced by a representation of P, where $Q \subset P \subset G$ as in Section 2.2. Then for any $g \in Q$, the subspace $\mathfrak{p} \subset \mathfrak{g}$ is invariant under $\operatorname{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$, so it gives rise to a smooth subbundle in $\mathcal{A}M$ that is denoted by $\mathcal{A}^{\mathfrak{p}}M$ in [10]. The infinitesimal representation again induces a bundle map $\bullet\colon \mathcal{A}^{\mathfrak{p}}M \times \mathcal{V}M \to \mathcal{V}M$. As before, we conclude that the operator $\Gamma(\mathcal{A}^{\mathfrak{p}}M) \times \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{V}M)$ defined by $(s,\sigma) \mapsto D_s\sigma + s \bullet \sigma$ depends only on $\Pi(s)$, but now $\Pi(s)$ is a section of $\mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q}) = T_\rho M$. Hence, we do not obtain a full linear connection but only a *partial connection* $\nabla^{\rho,\mathcal{V}}\colon \Gamma\big(T_\rho M\big) \times \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{V}M)$, which still is linear over smooth functions in the first argument and satisfies the Leibniz rule in the second argument.

In the special case of the relative tractor bundles $\mathcal{V}_{i'}M$ constructed in Section 2.3, we prove that the relative tractor connection is induced by the adjoint tractor connection.

Theorem 3.1. *In the setting of Section 2.3, consider the subbundles $\mathcal{A}^{(i',*)}M$. Then for $s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ and $t \in \Gamma\big(\mathcal{A}^{(i',*)}M\big)$ with $i' \leq 0$, we get $\nabla^{\mathcal{A}}_{\Pi(s)}t \in \Gamma\big(\mathcal{A}^{(i',*)}\big)M$. Hence, the adjoint tractor connection restricts to an operation $\Gamma\big(T_\rho M\big) \times \Gamma\big(T^{i'}_P M\big) \to \Gamma\big(T^{i'}_P M\big)$ for any $i' < 0$.*

Thus, there is an induced partial connection on $\Gamma(\mathcal{V}_{i'}M)$ and this coincides with the relative tractor connection $\nabla^{\rho,\mathcal{V}_{i'}}$ as obtained in [10, Section 4.3].

Proof. Consider the defining formula $\nabla^{\mathcal{A}}_{\Pi(s)}t = D_s t + \{s,t\}$ for $s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ and $t \in \Gamma\big(\mathcal{A}^{(i',*)}M\big)$. By [7, Proposition 1.5.8], the fundamental derivative preserves natural subbundles, so the first summand in the right-hand side is a section of $\mathcal{A}^{(i',*)}M$. On the other hand, Proposition 2.4 together with $\mathfrak{p} = \mathfrak{g}^{(0,*)}$ shows that $\big[\mathfrak{g}^{(0,*)}, \mathfrak{g}^{(i',*)}\big] \subset \mathfrak{g}^{(i',*)}$, so the second summand of the right-hand side of the defining formula is a section of $\mathcal{A}^{(i',*)}M$, too.

Fixing $i' < 0$, we have thus obtained a partial connection on the bundle $\mathcal{A}^{(i',*)}M$ for which the subbundle $\mathcal{A}^{(i'+1,*)}M$ is parallel, so it descends to a well-defined partial connection on the quotient bundle $\mathcal{V}_{i'}M = \mathcal{A}^{(i',*)}M/\mathcal{A}^{(i'+1,*)}M$. Thus, it remains to show that this coincides with the relative tractor connection. The construction of the latter in [10] was based on a variant of the fundamental derivative for relative natural bundles: One defines the relative adjoint tractor bundle $\mathcal{A}_\rho M = \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{p}_+)$, so in our notation, this is just $\mathcal{A}^{(0,*)}M/\mathcal{A}^{(1,*)}M$. Given a representation $\mathbb{W}$ of Q on which P_+ acts trivially, one can first restrict the fundamental derivative $D_s\sigma$ to $s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ and then observe that $D_s\sigma = 0$ if $s \in \Gamma\big(\mathcal{A}^{(1,*)}M\big)$, to obtain a well-defined operator $D^\rho\colon \Gamma(\mathcal{A}_\rho M) \times \Gamma(\mathcal{W}M) \to \Gamma(\mathcal{W}M)$. If $\mathcal{V}M$ is a relative tractor bundle, then $\mathbb{V}$ is the restriction of a representation of P, so the infinitesimal representation descends to an operation $\bullet\colon \mathfrak{p}/\mathfrak{p}_+ \times \mathbb{V} \to \mathbb{V}$. Similarly as above, the relative tractor connection is then characterized by $\nabla^{\rho,\mathcal{V}}_{\Pi(s)}\sigma = D^\rho_s\sigma + s \bullet \sigma$, where now Π is induced by the canonical projection $\mathfrak{p}/\mathfrak{p}_+ \to \mathfrak{p}/\mathfrak{q}$.

But from this description the claim follows easily: Take sections $\sigma \in \Gamma(\mathcal{V}_{i'}M)$ and $\xi \in T_\rho M$. For our construction, we choose representative sections $t \in \Gamma\big(\mathcal{A}^{(i',*)}M\big)$ for σ and $s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ for ξ and project $D_s t + \{s,t\} \in \Gamma\big(\mathcal{A}^{(i',*)}M\big)$ to the quotient bundle $\mathcal{V}_{i'}M$. Denoting by $\underline{s}$ the projection of s to $\mathcal{A}_\rho M$ the fact that $\mathfrak{p}_+$ acts trivially on $\mathfrak{g}^{(i',*)}/\mathfrak{g}^{(i'+1,*)}$ implies that the projection of $D_s t$ to $\Gamma(\mathcal{V}_{i'}M)$ depends only on $\underline{s}$ and then by construction it coincides with $D^\rho_{\underline{s}}\sigma$. Similarly, since $\big[\mathfrak{g}^{(1,*)}, \mathfrak{g}^{(i',*)}\big] \subset \mathfrak{g}^{(i'+1,*)}$ we conclude that the projection of $\{s,t\}$ to $\Gamma(\mathcal{V}_{i'}M)$ depends only on $\underline{s}$ and coincides with $\underline{s} \bullet \sigma$. ■

Remark 3.2. The description of a relative tractor connection as induced by the adjoint tractor connection leads to explicit descriptions in terms of the structure underlying a parabolic geometry of type (G,Q). This works via the concept of *Weyl structures* which was introduced in [6], see also [7, Section 5]. Any parabolic geometry carries a family of Weyl structures which form an affine space modelled on the space $\Omega^1(M)$ of one-forms on the manifold M. Any Weyl structure provides a linear connection on any natural vector bundle called the Weyl connection. Once Weyl connections are understood there is a general theory providing explicit formulae for tractor connections, see [7, Section 5.1].

3.2 The relation to the Lie bracket

As we have noted above, sections of the adjoint tractor bundle $\mathcal{A}M$ can be identified with Q-invariant vector fields on the total space $\mathcal{G}$ of the Cartan bundle. Now the subspace of Q-invariant vector fields is closed under the Lie bracket of vector fields, whence we obtain an induced bilinear operation on $\Gamma(\mathcal{A}M)$, which we denote by the usual symbol $[\,,\,]$. As we have noted in Section 3.1, for $s \in \Gamma(\mathcal{A}M)$, $\Pi(s) \in \mathfrak{X}(M)$ is the projection of the Q-invariant vector field corresponding to s. Hence, standard properties of the Lie bracket imply that for $s, t \in \Gamma(\mathcal{A}M)$, we get $\Pi([s,t]) = [\Pi(s), \Pi(t)]$ with the Lie bracket of vector fields in the right-hand side.

It is rather easy to express the Lie bracket of sections of $\mathcal{A}M$ in terms of the operations introduced above and of the curvature κ of the Cartan connection ω, see [7, Corollary 1.5.8]: For $s, t \in \Gamma(\mathcal{A}M)$, one has

$$[s,t] = D_s t - D_t s - \kappa(\Pi(s), \Pi(t)) + \{s,t\}. \tag{3.1}$$

Using this and Theorem 3.1, we can now give an alternative description of the relative tractor connections on the relative tractor bundles $\mathcal{V}_{i'}M$. Recall from Section 2.3 that we have subbundles $TM \supset T_P^{i'}M \supset T_\rho M$ such that $\mathcal{V}_{-1}M = T_P^{-1}M/T_\rho M$ and $\mathcal{V}_{i'}M = T_P^{i'}M/T_P^{i'+1}M$ for $i' < -1$. Hence, for a smooth section σ of $\mathcal{V}_{i'}M$ one can always chose a representative vector field in $\Gamma\big(T_P^{i'}M\big)$ that projects onto σ.

Theorem 3.3. *In the setting of Section 2.3, consider vector fields $\xi \in \Gamma\big(T_\rho M\big)$, $\eta \in \Gamma\big(T_P^{i'}M\big)$ and a section $\sigma \in \Gamma(\mathcal{V}_{i'}M)$. Let $\tau \in \Omega^2(M, TM)$ be the torsion of the Cartan connection ω.*

1. *The vector field $[\xi, \eta] + \tau(\xi, \eta)$ is a section of the subbundle $T_P^{i'}M \subset TM$.*
2. *If η is a representative for σ, then the projection of $[\xi,\eta] + \tau(\xi,\eta)$ to $\Gamma(\mathcal{V}_{i'}M)$ depends only on σ (and not on the choice of η) and coincides with $\nabla_\xi^{\rho,\mathcal{V}_{i'}}\sigma$.*

Proof. We can choose smooth representatives $s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ for ξ and $t \in \Gamma\big(\mathcal{A}^{(i',*)}M\big)$ for η, so $\Pi(s) = \xi$ and $\Pi(t) = \eta$. As we have noted above, this implies $\Pi([s,t]) = [\xi, \eta]$, so applying Π to formula (3.1) we obtain

$$[\xi, \eta] = \Pi(D_s t + \{s,t\}) + \Pi(D_t s) - \Pi(\kappa(\xi, \eta)).$$

The last summand in the right-hand side by definition is $-\tau(\xi, \eta)$, while naturality of the fundamental derivative implies that $D_t s \in \Gamma\big(\mathcal{A}^{(0,*)}M\big)$ and hence $\Pi(D_t s) \in \Gamma\big(T_\rho M\big)$. From Theorem 3.1, we know that the first term in the right-hand side is $\Pi\big(\nabla_\xi^{\mathcal{A}M} t\big)$ and that this is a section of $T_P^{i'}M$, so (1) follows. Theorem 3.1 also tells us that, if η represents σ, then projecting $\Pi\big(\nabla_\xi^{\mathcal{A}M} t\big)$ further to $\Gamma(\mathcal{V}_{i'}M)$, we obtain $\nabla_\xi^{\rho,\mathcal{V}_{i'}}\sigma$. Since $\Pi(D_t s)$ lies in the kernel of the latter projection, we obtain (2). ■

The relation to the Lie bracket is also crucial for obtaining an interpretation of the relative tractor bundles introduced in Section 2.3 in the case that the relative tangent bundle $T_\rho M$ is involutive (and additional assumptions are satisfied). As observed in Section 2.3, if $T_\rho M$ is involutive, then locally around any $x \in M$, we can find a local leaf space $\psi\colon U \to N$. This means that $U \subset M$ is open with $x \in U$ and ψ is a surjective submersion onto a smooth manifold such that for each $y \in U$ the kernel $\ker(T_y\psi) \subset T_yM$ coincides with the fiber of the relative tangent bundle in y. In particular, this implies that we obtain an isomorphism $TU/T_\rho U \cong \psi^*TN$ induced by $T\psi|_U$. Hence, there is the hope that the subbundles $T_P^{i'}M \subset TM$ that contain $T_\rho M$ descend to subbundles in TN. We can nicely characterize when this happens in terms of the torsion of the Cartan geometry. Under stronger assumptions (which are satisfied in important examples), we can nicely characterize parallel sections for the relative tractor connection.

Theorem 3.4. *In the setting of Section 2.3, assume that $T_\rho M$ is involutive and consider a local leaf space $\psi\colon U \to N$ with connected fibers. Fix an index i' and consider the bundles $T^{i'}_P M \subset TM$ and $T^{i'}_P M/T_\rho M \subset TM/T_\rho M$.*

(1) *There is a smooth subbundle $T^{i'}N \subset TN$ that corresponds to $T^{i'}_P M/T_\rho M$ under the isomorphism $TM/T_\rho M \cong \psi^*TN$ induced by $T\psi$ if and only if the torsion τ satisfies $\tau\big(T_\rho M, T^{i'}_P M\big) \subset T^{i'}_P M$.*

(2) *If the condition in (1) is satisfied, then $T\psi$ induces an isomorphism from $\mathcal{V}_{i'}M|_U$ to $\psi^*\big(T^{i'}N/T^{i'+1}N\big)$. Assuming in addition that $\tau\big(T_\rho M, T^{i'}_P M\big) \subset T^{i'+1}_P M \subset T^{i'}_P M$, a section of $\mathcal{V}_{i'}M|_U$ is parallel for the relative tractor connection $\nabla^{\rho,\mathcal{V}_{i'}}$ if and only if it is the pullback of a section of $T^{i'}N/T^{i'+1}N$.*

Proof. (1) We first observe that by Theorem 3.3, the condition on τ in (1) is equivalent to the fact that $[\xi,\eta] \in \Gamma\big(T^{i'}_P M\big)$ for any $\xi \in \Gamma\big(T_\rho M\big)$ and $\eta \in \Gamma\big(T^{i'}_P M\big)$. To prove necessity of this condition, we assume that $T^{i'}N \subset TN$ is a subbundle with the required property. Then for a local section $\underline{\eta}$ of $T^{i'}N$ we can first pull back to a local section of $T^{i'}_P M/T_\rho M$ and then choose a representative section $\eta \in \Gamma\big(T^{i'}_P M\big)$ for this. This means that $T_y\psi(\eta(y)) = \underline{\eta}(\psi(y))$ on the domain of definition of η, so the vector fields η and $\underline{\eta}$ are ψ-related. Moreover, $\xi \in \Gamma\big(T_\rho M\big)$ is of course ψ-related to the zero vector field on N. This shows that $[\xi,\eta]$ has to be ψ-related to the zero vector field, too, and hence $[\xi,\eta] \in \Gamma\big(T_\rho M\big)$. Starting with a local frame of $T^{i'}N$ and adding a local frame of $T_\rho M$, we conclude that there are local frames for $T^{i'}_P M$ consisting of vector fields η such that $[\xi,\eta] \in \Gamma\big(T_\rho M\big)$ for any $\xi \in \Gamma\big(T_\rho M\big)$, which clearly implies that the bracket with ξ sends any section of $T^{i'}_P M$ to a section of this subbundle.

To prove sufficiency of the condition, put $n = \dim(M)$, $\ell = \operatorname{rank}\big(T^{i'}_P M\big)$ and $k = \operatorname{rank}\big(T_\rho M\big)$. We adapt an argument from the proof of the Frobenius theorem in [19, Section 3.18]. Recall that given a point $x \in U$ we can find a Frobenius chart (V, v) (for the involutive distribution $T_\rho M$) around x such that $V \subset U$. This means that $v(V) \subset \mathbb{R}^n$ can be written as $W_1 \times W_2$ for connected open subsets in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, such that for each $a \in W_2$, the pre-image $v^{-1}(W_1 \times \{a\})$ is an integral submanifold for the distribution $T_\rho M$. Since ψ is a submersion, $\psi(V) \subset N$ is open and by construction the $\mathbb{R}^{n-k}$-component of v descends to a map $\psi(V) \to W_2$. This is smooth by the universal property of surjective submersions and a local diffeomorphism by construction. Shrinking V if necessary, this becomes a diffeomorphism and hence can be used as a chart map on N. We implement this by viewing the local coordinates v^i for $i = k+1, \dots, n$ to be defined both on V and on $\psi(V)$, and then in these coordinates ψ is given by $\big(v^1, \dots, v^n\big) \mapsto \big(v^{k+1}, \dots, v^n\big)$.

Let us denote by ∂_i the coordinate vector field $\frac{\partial}{\partial v^i}$ for each i. Then by construction $\partial_1, \dots, \partial_k$ form a local frame for $T_\rho M$ and, shrinking V if necessary, we can extend them by $\eta_{k+1}, \dots, \eta_\ell \in \Gamma\big(T^{i'}_P M|_V\big)$ to a local frame for $T^{i'}_P M|_V$. Expanding these sections in terms of the ∂_i, we can leave out all summands with $i \le k$, so we can assume that there are smooth functions $a_{ij}\colon V \to \mathbb{R}$ for $1 \le i \le n-k$ and $1 \le j \le \ell - k$ such that $\eta_{j+k} = \sum_{i=1}^{n-k} a_{ij}\partial_{k+i}$. Viewing (a_{ij}) as a matrix of size $(n-k)\times(\ell-k)$, it follows from linear independence of the fields η_j that the matrix $(a_{ij}(y))$ has rank $\ell - k$ for each $y \in V$ and hence has $\ell - k$ linearly independent rows. Specializing to $y = x$, we can assume (permuting coordinates in W_2 if necessary) that the first $\ell - k$ rows of $(a_{ij}(x))$ are linearly independent and hence the top $(\ell-k)\times(\ell-k)$ submatrix is invertible. Again shrinking V if necessary, we can assume that the corresponding submatrix in $(a_{ij}(y))$ is invertible for any $y \in V$. Hence, we find smooth functions $b_{ij}\colon V \to \mathbb{R}$ for $1 \le i, j \le \ell - k$ such that $\sum_r a_{ir}(y) b_{rj}(y) = \delta_{ij}$ for any i, j and any $y \in V$.

Now we define $\tilde{\eta}_{k+j} := \sum_{i=1}^{\ell-k} b_{ij}\eta_{k+i}$ for $i = 1, \dots, \ell - k$. By construction, these are smooth sections of $T^{i'}_P M$ which in each point span the same subspace as the η's and hence also extend $\partial_1, \dots, \partial_k$ to a local frame of $T^{i'}_P M$ on V. Plugging in the expansions of the η's in terms of the coordinate vector fields, one immediately concludes that $\tilde{\eta}_{k+j} = \partial_{k+j} + \sum_{i=\ell+1}^{n} c_{ij}\partial_i$ for some smooth functions $c_{ij}\colon V \to \mathbb{R}$. Assuming that $[\xi,\eta] \in \Gamma\big(T_P^{i'}M\big)$ for any $\xi \in \Gamma\big(T_\rho M\big)$ and $\eta \in \Gamma\big(T_P^{i'}M\big)$ we conclude that for each $i = 1,\dots,k$, $[\partial_i, \tilde{\eta}_{k+j}]$ can be expanded in as a smooth linear combination of $\partial_1,\dots,\partial_k,\tilde{\eta}_{k+1},\dots,\tilde{\eta}_\ell$. But this bracket is given by $\sum_{r\geq \ell+1}(\partial_i\cdot c_{rj})\partial_r$ and in view of the form of the $\tilde{\eta}$'s this is only possible if $[\partial_i,\tilde{\eta}_{k+j}] = 0$ for any $i = 1,\dots k$ and $j = 1,\dots,\ell-k$. But this implies that all the functions c_{rj} satisfy $\partial_i \cdot c_{rj} = 0$ for all i, whence they are independent of the coordinates $v^1,\dots,v^k$ and hence constant along the fibers of ψ.

Given a point $x_0 \in U$ the image of the fiber of $T_P^{i'}M$ over x_0 is a linear subspace of $T_{\psi(x_0)}N$ of dimension $\ell - k$ and we consider the set $A \subset \psi^{-1}(\psi(x_0))$ consisting of all points x which lead to the same subspace. For $x \in A$, the above argument shows that $V \cap \psi^{-1}(\psi(x)) \subset A$, so A is open. Since A is evidently closed, connectedness of the fibers of ψ implies that $A = \psi^{-1}(\psi(x_0))$ which completes the proof of (1).

(2) The first statement is obvious by part (1). Under the additional assumption on τ, part (2) of Theorem 3.3 shows that for a representative $\eta \in \Gamma\big(T_P^{i'}M\big)$ for $\sigma \in \Gamma(\mathcal{V}_{i'}M)$ one obtains $\nabla_\xi^{\rho,\mathcal{V}_{i'}}\sigma$ as the projection of $[\xi,\eta]$. Hence, what we have to prove here is that $\nabla^{\rho,\mathcal{V}_{i'}}\sigma = 0$ if and only if for one or equivalently any representative η for ∇ we get $[\xi,\eta] \in \Gamma\big(T_P^{i'+1}M\big)$ for any $\xi \in \Gamma\big(T_\rho M\big)$. Now we can run the argument with a Frobenius chart as in the proof of (1) in two steps. First, we extend the local frame $(\partial_i)_{i=1,\dots,k}$ for $T_\rho M$ by vector fields $\eta_{k+1},\dots,\eta_{k+\ell'}$ to a local frame of $T_P^{i'+1}M$ in such a way that $\eta_{k+j} = \partial_{k+j} + \sum_{r>k+\ell'} a_r\partial_r$ for some smooth functions a_r. Then we can further extend by $\tilde{\eta}_{k+\ell'+1},\dots,\tilde{\eta}_{k+\ell}$ to a local frame for $T_P^{i'}M$. As before, we can obviously assume that the $\tilde{\eta}$'s are smooth linear combinations of the coordinate vector fields ∂_r with $r > k + \ell'$ only. As in part (1), we can then modify these to fields $\eta_{k+\ell'+1},\dots,\eta_{k+\ell}$ such that $\eta_{k+\ell'+j} = \partial_{k+\ell'+j} + \sum_{r>k+\ell} c_{rj}\partial_r$ for smooth functions c_{rj}.

In particular, as in part (1) the vector fields $\eta_{k+\ell'+1},\dots,\eta_{k+\ell}$ project to a local frame for $\mathcal{V}_{i'}M$ that descends to a local frame for $T^{i'}N/T^{i'+1}N$. Now in the domain of our Frobenius chart, take a section of $\mathcal{V}_{i'}M$ and a representative in $\Gamma\big(T_P^{i'}M\big)$. Expanding this in terms of our local frame, we can drop all summands except the last $\ell-\ell'$ ones without changing the projection to $\Gamma(\mathcal{V}_{i'}M)$. Hence, we find a representative of the form $\eta = \sum_{j=1}^{\ell-\ell'} c_j\eta_{k+\ell'+j}$ and $[\partial_i,\eta] = \sum_{j=1}^{\ell-\ell'}(\partial_i\cdot c_j)\eta_{k+\ell+j}$. The form of $\eta_{k+\ell+j}$ readily implies that this is a section of $T^{i'+1}M$ if and only if it is zero and hence to $\partial_i \cdot c_j = 0$ for any i and j. But this is equivalent to each c_j being independent of the first k coordinates. Hence, η descends to a local section of $T^{i'}N/T^{i'+1}N$ whose pullback visibly coincides with the original section of $\mathcal{V}_{i'}M$. Conversely, in our frame, any such pullback has coordinate functions that descend to N. ■

3.3 The distinguished class of relative tractor bundles

In the setting of Section 2.3, let us assume that we have given a parabolic geometry $(p\colon \mathcal{G}\to M,\omega)$ whose torsion τ satisfies $\tau\big(T_\rho M, T_P^{i'}M\big) \subset T_P^{i'}M$ for all $i' \leq 0$. Then $T_\rho M$ is involutive and by Theorem 3.4 for any local leaf space $\psi\colon U \to N$ for the induced foliation with connected fibers, we get a canonical filtration of the tangent bundle by the smooth subbundles $T^{i'}N$. As usual, one can form the associated graded of this filtration, which has the form

$$\operatorname{gr}(TN) = \bigoplus_{i'} \operatorname{gr}_{i'}(TN) \qquad \text{with} \qquad \operatorname{gr}_{i'}(TN) = T^{i'}N/T^{i'+1}N \quad \text{and} \quad T^0N = N\times\{0\}.$$

Performing natural constructions with vector bundles, one always obtains induced filtrations of the resulting bundles, and forming the associated graded is compatible with the constructions. For example, the natural filtration of the cotangent bundle T^*N has positive indices with the jth filtration component being the annihilator of $T^{-j+1}N$ so $TN = T^1N \supset T^2N \supset \cdots$. For the associated graded, this readily implies that $\operatorname{gr}_j(T^*N)$ is dual to $\operatorname{gr}_{-j}(TN)$. For filtered bundles $\big(E,E^i\big)$ and $\big(F,F^j\big)$, the components of the filtration on the tensor product $E\otimes F$ are defined by letting $(E \otimes F)^\ell$ be the span of the component $E^i \otimes F^j$ with $i+j=\ell$. For the associated graded, this means that $\mathrm{gr}_\ell(E \otimes F) = \bigoplus_{i+j=\ell} \mathrm{gr}_i(E) \otimes \mathrm{gr}_j(F)$. See [7, Section 3.1.1] for more details.

This allows us to formulate the description of the special class of relative tractor bundles we are aiming at.

Corollary 3.5 (to Theorems 3.3 and 3.4)**.** *In the setting of Section 2.3, consider a parabolic geometry $(p\colon \mathcal{G} \to M, \omega)$ whose torsion τ satisfies $\tau\big(T_\rho M, T_P^{i'} M\big) \subset T_P^{i'} M$ for all $i' \leq 0$. Let $\psi\colon U \to N$ be a local leaf space for the involutive distribution $T_\rho M = T_P^0 M$ with connected fibers.*

(1) *For any i' we get $\mathcal{V}_{i'} M|_U = \psi^*(\mathrm{gr}_{i'}(TN))$. Applying any tensorial construction to the bundles $\mathcal{V}_{i'} M$, one obtains a relative tractor bundle whose restriction to U can be naturally identified with the pullback of the same construction applied to the bundles $\mathrm{gr}_{i'}(TN)$.*

(2) *Suppose that the geometry even satisfies $\tau\big(T_\rho M, T_P^{i'} M\big) \subset T_P^{i'+1} M$ for all $i' < 0$. Then for any relative tractor bundle constructed as in* (1) *a local section over U is parallel for the relative tractor connection if and only if it is the pullback of a section of the corresponding bundle over N.*

Proof. (1) The first part immediately follows from Theorems 3.3 and 3.4. Applying a tensorial construction vector bundles associated to a principal bundle, one always obtains the bundle associated to the representation obtained from the corresponding construction on the inducing representations. Hence, any tensorial construction with the bundles $\mathcal{V}_{i'} M$ leads to a relative tractor bundle. The last part then follows from the compatibility of tensorial constructions with the passage from filtered vector bundles to associated graded bundles as discussed above and with pullbacks.

(2) Recall the construction of the relative tractor connection in terms of the fundamental derivative (or its relative analog) and the infinitesimal representation as described in Section 3.1. Naturality of the fundamental derivative (see [7, Section 1.5.8]) then implies that applying a tensorial construction to the bundles $\mathcal{V}_{i'} M$, the relative tractor connection on the resulting relative tractor bundle coincides with the connection induced by the relative tractor connection on the bundles $\mathcal{V}_{i'} M$. Knowing this, the general claim follows from Theorem 3.4. ■

3.4 Example: Legendrean contact structures

Legendrean contact structures (also called Lagrangean contact structures following the article [20] in which they were first studied) are an example of parabolic contact structures, so they can be viewed as a refinement of contact structures. Explicitly, a Legendrean contact structure on a smooth manifold of odd dimension $2n+1$ is given by a contact structure $H \subset TM$ together with a decomposition $H = E \oplus F$ of H as a direct sum of two subbundles of rank n, which are Legendrean in the sense that the Lie bracket of two sections of one of the subbundles always is a section of H. Otherwise put, for any $x \in M$ the fibers $E_x, F_x \subset H_x \subset T_x M$ are maximal isotropic subspaces with respect to the non-degenerate bilinear form $\mathcal{L}_x\colon H_x \times H_x \to T_x M / H_x$ induced by the Lie bracket of vector fields. The interest in these structures mainly comes from their relation to the geometric theory of differential equations, see, e.g., [18] and to CR geometry, see, e.g., [12]. Indeed, Legendrean contact structures and partially integrable almost CR structures are two real forms of the same complex geometric structure.

The relation to parabolic geometries comes from the homogeneous model of such geometries, which is the partial flag manifold $F_{1,n+1}\big(\mathbb{R}^{n+2}\big)$ of lines contained in hyperplanes in $\mathbb{R}^{n+2}$. This is a homogeneous space of the group $G := \mathrm{PGL}(n+2, \mathbb{R})$ and the stabilizer of the standard flag $\mathbb{R} \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ descends to a parabolic subgroup $Q \subset G$. Indeed, $Q = P \cap \tilde{P}$ for two maximal parabolic subgroups of G, the stabilizers of $\mathbb{R} \subset \mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$. Correspondingly, there is a canonical homogeneous projection $G/Q \to G/P = \mathbb{R}P^{n+1}$ and it is easy to see that this identifies G/Q with the projectivized cotangent bundle $\mathcal{P}(T^*\mathbb{R}P^{n+1})$. This gives rise to a canonical contact structure on G/Q for which the vertical subbundle is well known to be Legendrean. The second Legendrean subbundle making G/Q into a Legendrean contact manifold is the vertical bundle of the canonical projection $G/Q \to G/\tilde{P} \cong \mathbb{R}P^{(n+1)*}$.

On the level of Lie algebras, we also have $\mathfrak{q} = \mathfrak{p} \cap \tilde{\mathfrak{p}} \subset \mathfrak{g} = \mathfrak{sl}(n+2,\mathbb{R})$, where $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ are the stabilizers of $\mathbb{R} \subset \mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$, respectively. The parabolics $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ are exchanged by an outer automorphism of $\mathfrak{g}$, so we only discuss $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$. Decomposing matrices of size $n+2$ into blocks of size 1, n, and 1, the bigrading induced by this pair has the form

$$\begin{pmatrix} \mathfrak{g}_{(0,0)} & \mathfrak{g}_{(1,0)} & \mathfrak{g}_{(1,1)} \\ \mathfrak{g}_{(-1,0)} & \mathfrak{g}_{(0,0)} & \mathfrak{g}_{(0,1)} \\ \mathfrak{g}_{(-1,-1)} & \mathfrak{g}_{(0,-1)} & \mathfrak{g}_{(0,0)} \end{pmatrix}. \tag{3.2}$$

In particular, we just get one representation $\mathbb{V}_{i'}$ in this case, namely $\mathbb{V}_{-1} = \mathfrak{g}/\mathfrak{p}$ and the Q-invariant filtration of this representation consists of the single invariant subspace $(\mathfrak{g}_{(-1,0)} \oplus \mathfrak{p})/\mathfrak{p}$. The corresponding bundles are easy to interpret, too.

General results on parabolic geometries imply that for any Legendrean contact structure $(M, H = E \oplus F)$ there is a normal Cartan geometry $(p\colon \mathcal{G} \to M, \omega)$ of type (G,Q) which is unique up to isomorphism, see [7, Section 4.2.3]. It is a general fact about Cartan geometries that $\mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q}) \cong TM$ and the characterizing property of the geometry (apart from normality) is that the Q-invariant subspaces $(\mathfrak{g}_{(-1,0)} \oplus \mathfrak{q})/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ and $(\mathfrak{g}_{(0,-1)} \oplus \mathfrak{q})/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ correspond to the subbundles E and F, respectively. Since $\mathfrak{g}_{(0,-1)} \oplus \mathfrak{q} = \mathfrak{p}$ we conclude that $T_\rho M = F$ in this case, and that our basic tractor bundle $\mathcal{V}_{-1}M$ is simply TM/F. The natural filtration of this bundle is given by the single invariant subspace $H/F \subset TM/F$.

Now we can collect our results in the case of Legendrean contact structures. We restrict to the case $n > 1$, i.e., $\dim(M) \geq 5$. The three-dimensional case can be easily dealt with directly. It is not very interesting from the point of view of relative BGG theory, however, since relative BGG sequences consist only of a single operator in this dimension.

In the current situation, relative tractor bundles correspond to representations of the general linear group $\mathrm{GL}(n+1,\mathbb{R})$, so they correspond to natural vector bundles on smooth manifolds of dimension $n+1$. The relative tractor bundle $\mathcal{V}M = TM/F$ corresponds to the standard representation of $\mathrm{GL}(n+1,\mathbb{R})$ on $\mathbb{R}^{n+1}$, so via tensorial constructions as in Section 3.3 we obtain all analogs of tensor bundles. It is clear that the latter are by far the most important natural vector bundles (in fact, we are not aware of interesting applications of other types of natural bundles). Thus, it should be clear that we obtain (up to isomorphism) all the important relative tractor bundles on Legendrean contact structures via the constructions of Section 3.3, but it is difficult to make a precise statement in this direction. Instead of trying to do this, we just prove a result on the resulting representations in the following theorem, which shows the richness of the class.

Theorem 3.6. *Let $(M, H = E \oplus F)$ be a Legendrean contact structure of dimension $2n+1 \geq 5$ such that the distribution $F \subset TM$ is involutive and consider the relative tractor bundle $\mathcal{V}M := TM/F$. Applying tensorial constructions to $\mathcal{V}M$ as in Section 3.3, we arrive at a class of relative tractor bundles that correspond to representations of $P/P_+ \cong \mathrm{GL}(n+1,\mathbb{R})$ which contains all one-dimensional representations and whose restrictions to $\mathfrak{sl}(n+1,\mathbb{R}) \subset \mathfrak{p}/\mathfrak{p}_+$ exhaust all finite-dimensional representations of this Lie algebra.*

For any of these relative tractor bundles and any local leaf space $\psi\colon U \to N$ for the distribution F with connected fibers, the induced relative BGG sequence defines a fine resolution of the sheaf of pullbacks of sections of a tensor bundle on N. This tensor bundle is obtained by applying the tensorial constructions from above to the tangent bundle TN.

Proof. We first need some information on the curvature of Legendrean contact structures that is available in the literature. For $n > 1$, [7, Proposition 4.3.2] completely describes the harmonic curvature of Legendrean contact structures: There are two harmonic curvature quantities of homogeneity one, which are sections of the bundles $\Lambda^2 E^* \otimes F$ and $\Lambda^2 F^* \otimes E$, respectively. These turn out to be induced by the Lie bracket of vector fields and they are exactly the obstructions to involutivity of the distributions E and F. So in particular, one of these components is exactly the obstruction to involutivity of $T_\rho M = F$. The third and last harmonic curvature quantity has homogeneity two and can be interpreted as a section of $E^* \otimes F^* \otimes L(E, E)$ (with some symmetries).

Hence, involutivity of F implies that the harmonic curvature vanishes upon insertion of two elements of F while the components of torsion type even vanish upon insertion of one element of F. These are exactly the assumptions needed to apply part (1) of [10, Proposition 4.18]. The proof of this proposition then shows that the full Cartan curvature has the same properties, which in our language means that $\tau\big(T_\rho M, TM\big) = 0$, so the assumptions of Corollary 3.5 are satisfied. On the other hand, [10, Proposition 4.18] says that the assumptions of part (1) of [10, Theorem 4.11] are satisfied. This shows that any relative BGG sequence is a fine resolution of the sheaf of local parallel sections for the relative tractor connection. Together with Corollary 3.5, this implies all claimed properties of the relative BGG sequence for $\mathcal{V}M = TM/F$. For bundles obtained via tensorial constructions, they then follow readily since relative tractor connections are compatible with tensorial constructions.

So it remains to verify the claim about the representations corresponding to our relative tractor bundles, i.e., those obtained by tensorial constructions from the standard representation. Putting $\mathfrak{s} := \mathfrak{sl}(n+1, \mathbb{R}) \subset \mathfrak{p}/\mathfrak{p}_+$ semi-simplicity of $\mathfrak{s}$ implies that it acts trivially on any one-dimensional representation of P/P_+. Hence, also the subgroup $\mathrm{SL}(n+1, \mathbb{R}) \subset P/P_+$ acts trivially so any such representation factorizes through the determinant representation $\det\colon \mathrm{GL}(n+1, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$. The determinant representation is realized via the top exterior power of the standard representation. The square of this line bundle corresponds to a representation with values in $\mathbb{R}^+$, so we can form powers of that line bundle with arbitrary non-zero real exponents. These together with their tensor products with $\Lambda^n TM$ exhaust all possible one-dimensional representations of P/P_+.

On the other hand, it is well known that any irreducible representation of $\mathfrak{s}$ can be realized in an iterated tensor product of copies of the standard representation and its dual via Young symmetrizers. By construction, any Young symmetrizer is P/P_+-equivariant, and hence each of these irreducible components is P/P_+-invariant. Thus, we see that any irreducible representation of $\mathfrak{s}$ is realized by a tensorial construction and since forming direct sums is no problem, we can arrive at an isomorphic copy of any finite-dimensional representation of $\mathfrak{s}$ from a tensorial construction. ■

Remark 3.7.

(1) It is clear that there are numerous examples of Legendrean contact structures for which F is involutive but the harmonic curvature component that is a section of the bundle $E^* \otimes F^* \otimes L(E, E)$ is non-zero. This follows, for example, from the relation of Legendrean contact structures to differential equations as used in [18]. In such cases, Theorem 3.6 applies to give a precise description of the sheaves resolved by relative BGG sequences. However, these geometries do not fall in the class of correspondence spaces as treated in part (2) of [10, Theorem 4.11]. Hence, in these cases no explicit description of the resolved sheaves was known before.

(2) The idea to use quotients of TM as basic relative tractor bundles originally arose in the context of Legendrean contact structures during the work on the thesis [22] of the third author under the direction of the first author. Legendrean contact structures play only

a minor role as an example in the thesis, however. The thesis mainly explores the fact that the machinery of relative BGG sequences extends beyond the case of Legendrean contact structures, namely to contact manifolds that are endowed with either a single involutive Legendrean distribution or with a Legendrean distribution with an arbitrary chosen complement in the contact distribution. In any case, the quotient of TM by the distinguished Legendrean distribution plays a pivotal role in these developments, since this can be shown to carry a canonical partial linear connection obtained from the Bott connection as in Section 3.2 or from a family of distinguished partial linear connections determined by a choice of contact form, compare to [7, Section 5.2.14] and to [21]. The latter are restrictions of Weyl connections, so one obtains descriptions of relative tractor bundles and relative tractor connections in terms of Weyl structures as discussed in Remark 3.2.

3.5 Example: Generalized path geometries

These geometries represent a kind of a dual situation to the case of Legendrean contact structures discussed in Section 3.4 above, in the sense that locally they have a projectivized tangent bundle as an underlying structure. They can be defined directly by a configuration of distributions as follows. On a smooth manifold M of dimension $2n+1$, one requires a distribution $H \subset TM$ of rank $n+1$ together with a decomposition $H = E \oplus V$ where E and V have rank 1 and n, respectively, such that the bundle map $H \otimes H \to TM/H$ induced by the Lie bracket of vector fields vanishes on $V \otimes V$ and restricts to an isomorphism on $E \otimes V$.

It then turns out that for $n \neq 2$, the distribution V is involutive and locally M is diffeomorphic to an open subset in the projectivized tangent bundle of a local leaf space for V. For $n = 2$, we will assume that this is the case, since this is necessary in order to obtain relative BGG resolutions. Assuming that one deals with a global projectivized tangent bundle, $M = \mathcal{P}TN$, the subbundle $E \subset TM$ defines a one-dimensional foliation on M and projecting the leaves to N, one obtains a family of paths (unparametrized curves) on N with exactly one path through each point in each direction. This is a classical path geometry and generalized path geometries can be considered as a local analog of that (where, for example, paths can be defined for an open set of directions only). The importance of these structures comes from the fact that they provide a way to geometrize systems of second order ODE, which is the starting point for their study in [14].

Similarly as in Section 3.4, the relation to parabolic geometries comes from the homogeneous model, which this time is the flag manifold $F_{1,2}\big(\mathbb{R}^{n+2}\big)$ of lines in planes in $\mathbb{R}^{n+2}$. So we again put $G := \mathrm{PGL}\big(n+2,\mathbb{R}\big)$ and the stabilizer Q of $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^{n+2}$ has the form $P \cap \tilde{P}$, where $G/P \cong \mathbb{R}P^{n+1}$ and $G/\tilde{P}$ is the Grassmannian $\mathrm{Gr}\big(2,\mathbb{R}^{n+2}\big)$. The distributions V and E are the vertical subbundles of the projections to G/P and $G/\tilde{P}$, respectively, and the integral submanifolds of E project exactly to the projective lines in G/P.

On the level of Lie algebras, the situation is closely parallel to Section 3.4, in particular the bigrading induced by $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ has the same form as in (3.2), but now with blocks of size 1, 1, and n, so now $\mathfrak{g}_{(-1,0)}$ is one-dimensional, while $\mathfrak{g}_{(0,-1)}$ and $\mathfrak{g}_{(-1,-1)}$ both have dimension n. In this case, the situation is less symmetric and we will make some remarks on the "other direction" $\mathfrak{q} \subset \tilde{\mathfrak{p}} \subset \mathfrak{g}$ below. We will also restrict to the case $n > 1$ here, for $n = 1$ there is no difference between generalized path geometries and Legendrean contact structures.

The canonical Cartan geometry $(\mathcal{G} \to M, \omega)$ associated to $(M, H = E \oplus V)$ here has the property that the Q-invariant subspaces $\big(\mathfrak{g}_{(-1,0)} \oplus \mathfrak{q}\big)/\mathfrak{q}$ and $\big(\mathfrak{g}_{(0,-1)} \oplus \mathfrak{q}\big)/\mathfrak{q}$ of $\mathfrak{g}/\mathfrak{q}$ induce the subbundles E and V of TM, respectively. As in Section 3.4, one concludes that $T_\rho M = V$ and that the basic relative tractor bundle in this case is $\mathcal{V}M = TM/V$ with the natural filtration given by $H/V \subset TM/V$. Collecting our results in this case, we obtain.

Theorem 3.8. *Let $(M, H = E \oplus V)$ be a generalized path geometry of dimension $2n+1 \geq 5$, where V is assumed to be involutive if $n = 2$ and consider the relative tractor bundle $\mathcal{V}M := TM/V$. Applying tensorial constructions to $\mathcal{V}M$ as in Section 3.3, we arrive at a class of relative tractor bundles that correspond to representations of $P/P_+ \cong \mathrm{GL}(n+1, \mathbb{R})$ which contain all one-dimensional representations and whose restrictions to $\mathfrak{sl}(n+1, \mathbb{R}) \subset \mathfrak{p}/\mathfrak{p}_+$ exhaust all finite-dimensional representations of this Lie algebra.*

For any of these relative tractor bundles and any local leaf space $\psi\colon U \to N$ for the distribution F with connected fibers, the induced relative BGG sequence defines a fine resolution of the sheaf of pullbacks of sections of a tensor bundle on N. This tensor bundle is obtained by applying the tensorial constructions from above to the tangent bundle TN.

Proof. According to [7, Section 4.4.3], there are only two harmonic curvature components for generalized path geometries with involutive V. One of these is a torsion of homogeneity 2 that is a section of $E^* \otimes (TM/H)^* \otimes V$. The other one is a curvature of homogeneity 3 that is a section of $V^* \otimes (TM/H)^* \otimes L(V, V)$ (with additional symmetry properties). Using this information, the proof is completed exactly as for Theorem 3.6. ■

Remark 3.9.

(1) Similarly as in the case of Legendrean contact structures, Theorem 3.8 covers many cases in which no good description of the sheaves resolved by relative BGG sequences was known. Indeed, such a description was only known in the case of a correspondence space treated in part (2) of [10, Theorem 4.11]. In the case of generalized path geometries (with involutive subbundle V) these are exactly those geometries for which the harmonic curvature component of homogeneity 3 mentioned in the proof of Theorem 3.8 vanishes identically. This means that locally the family of paths can be realized as geodesics of a torsion-free linear connection on the tangent bundle of a local leaf space. Since the unparametrized geodesics depend only on the projective equivalence class of the connection, such correspondence spaces in dimension $2n+1$ are best described as induced from a projective structure in dimension $n+1$. Of course, there are many systems of second order ODE which do not admit a description via geodesics, and these lead to generalized path geometries for which the harmonic curvature component of homogeneity 3 is non-trivial.

(2) As remarked already, the situation is not as symmetric here as for Legendrean contact structures. However, from the point of view of this article, the pair $\mathfrak{q} \subset \tilde{\mathfrak{p}} \subset \mathfrak{g}$ is much less interesting. This leads to $T_\rho M = E$ and the basic relative tractor bundle TM/E. On the one hand, the fact that $T_\rho M$ has rank one, then implies that all relative BGG sequences consist of a single operator only. On the other hand, the class of relative tractor bundles provided by our construction is less rich then in the cases discussed in Theorems 3.6 and 3.8. The point here is that essentially $\tilde{P}/\tilde{P}_+ \cong S(\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})) \subset \mathrm{GL}(2n, \mathbb{R})$ in our case and the bundle TM/E corresponds to the tensor product of the dual of the standard representation of the $\mathrm{GL}(2, \mathbb{R})$ factor with the standard representation of the $\mathrm{GL}(n, \mathbb{R})$-factor. The latter give rise to relative tractor bundles of rank 2 respectively n, that cannot be obtained by tensorial constructions form TM/E. The tensor product decomposition does not descend to a local leaf space for E unless the initial geometry is torsion-free, in which case it induces a $(2, n)$-Grassmannian structure on the local leaf space.

(3) The relation of relative tractor bundles to TM/V in the case of generalized path geometries was observed during the work on the thesis [15] of the second author under the direction of the first author. Relative tractor bundles and the relative BGG machinery play only a minor role in the thesis, however. One of the results of the thesis is a complete description of Weyl structures for generalized path geometries and relative tractor connections play

an important role in the characterization of Weyl connections and in the description of tractor calculus in terms of Weyl connections. In particular, this also leads to a description of relative tractor bundles and relative tractor connections as discussed in Remark 3.2.

3.6 Explicit examples of relative BGG resolutions

Here we give a few explicit examples of relative BGG resolutions for the examples treated in the last two sections. Both cases are closely related to standard BGG sequences associated to projective structures in appropriate dimension. We do not go into the details of how to compute the relevant relative Lie algebra homology groups but refer to [9] and [10]. We focus on the case of generalized path geometries in dimension 7. This is quite close to the five-dimensional case discussed in [10] but shows a bit more varied behavior in simple cases. In the end, we briefly comment on the case of Legendrean contact structures.

We will use the Dynkin diagram notation for irreducible representations, see [7, Section 3.2] for general information. In our case, $\times\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\circ$ indicates the parabolic subalgebra $\mathfrak{p}$ and irreducible representations of $\mathfrak{p}$ (and of the group P) are indicated by putting over the nodes the coefficients in the expansion of the highest weight in terms of fundamental weights. In particular the last three of these numbers have to be non-negative integers. Similarly, $\times\!\!-\!\!\times\!\!-\!\!\circ\!\!-\!\!\circ$ denotes the parabolic subalgebra $\mathfrak{q}$ and with numbers over the nodes irreducible representations of $\mathfrak{q}$ and of the corresponding group Q. Here only the last two coefficients have to be non-negative integers. Determining the forms of the relative BGG sequences is a simple extension of part (1) in [9, Example 3.2], and we just state the results below. They have the same form as for standard BGG sequences for three-dimensional projective structures. In the picture of weights, this corresponds to dropping the leftmost node of all diagrams.

So take a generalized path geometry $(M, E \oplus V)$ with $\dim(M) = 7$, so V has rank 3. The fact that the basic relative tractor bundle $\mathcal{V}M = TM/V$ is induced by the P-irreducible quotient of the adjoint representation shows that it corresponds to the representation $\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{1}{\circ}$. The irreducible subbundle $(E \oplus V)/V \cong E$ corresponds to $\overset{-2}{\times}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}$ and the quotient of TM/V by this subbundle is isomorphic to $TM/(E \oplus V)$ and hence corresponds to $\overset{1}{\times}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{1}{\circ}$. Dropping the leftmost nodes in the diagrams exhibits that this is analogous to the standard tractor bundle and its composition series for three-dimensional projective structures.

One family of well-known BGG sequences in projective geometry is induced by the symmetric powers of the standard cotractor bundle, which is dual to the standard tractor bundle. Now $(TM/V)^*$ corresponds to $\overset{-2}{\times}\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}$ and hence $S^k(TM/V)^*$ corresponds to $\overset{-2k}{\times}\!\!-\!\!\overset{k}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}$. The relative BGG sequence induced by each of these bundles starts with the Q-irreducible quotient of the bundle, which is the density bundle $\overset{-2k}{\times}\!\!-\!\!\overset{k}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}$. The next bundle in the relative BGG sequence is $\overset{-k+1}{\times}\!\!-\!\!\overset{-k-2}{\times}\!\!-\!\!\overset{k+1}{\circ}\!\!-\!\!\overset{0}{\circ}$. It is easy to see that this is obtained by twisting $S^{k+1}V^*$ by an appropriate density bundle. In particular, the first operator in the relative BGG sequences has order $k+1$ in this case, and by Theorem 3.8 locally its kernel consists of all pullbacks of symmetric $\binom{0}{k}$-tensor fields on a local leaf space. One can also deduce from this, that all further operators in the relative BGG sequence have order one.

The full relative BGG sequence for $k = 1$, i.e., the bundle $(TM/V)^*$, has the form

$$\overset{-2}{\times}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ} \to \overset{0}{\times}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\overset{0}{\circ} \to \overset{1}{\times}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\overset{1}{\circ} \to \overset{2}{\times}\!\!-\!\!\overset{-5}{\times}\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\overset{0}{\circ}. \tag{3.3}$$

Up to a twist by a line bundle, the last two bundles can be interpreted as the kernel of the complete symmetrization in $V^* \otimes S^2V^*$ and as V^*, respectively.

Another well-known projective BGG sequence is a projectively invariant version of the Riemannian deformation sequence. For flat metrics it is also known as the Calabi complex or as the fundamental complex of linear elasticity theory, see [13]. In our setting, this corresponds to the relative BGG sequence induced by the relative tractor bundle $\Lambda^2(TM/V)^*$, which corresponds to the representation $\overset{-3}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}$. One then computes that the full relative BGG sequence has the form

$$\overset{-3}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ} \to \overset{-2}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\circ}\!-\!\overset{0}{\circ} \to \overset{0}{\times}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{2}{\circ} \to \overset{1}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}.$$

Up to a twist by a density bundle, the first two bundles in the sequence are induced by the representations V^* and S^2V^*, so the first operator is like a symmetrized covariant derivative in V-directions. By Theorem 3.8, locally its kernel is isomorphic to the pullbacks of 2-forms on a local leaf space. Up to a twist by a density bundle, the third bundle is induced by S^2V, and the second operator is of order two. This operator is the relative analog of the projectively invariant version of the operator that computes the infinitesimal change of curvature caused by an infinitesimal deformation of a Riemannian metric. The last bundle is (up to a twist) induced by V and the last operator roughly is a divergence in V-directions. To make the operators in all the BGG sequences more explicit, one has to use Weyl structures for path geometries, see [15].

The relative BGG sequences for Legendrean contact structures as discussed in Section 3.4 are also similar in form to standard BGG sequences for projective structures. If the original manifold M has dimension $2n+1$, then relative BGG sequences are related to BGG sequences for projective structures in dimension n. Here the basic relative tractor bundle TM/F contains the distinguished subbundle $H/F \cong E \subset TM/F$ of rank n, and is corresponds to the dual of the standard tractor bundle in projective geometry. So in particular, the relative BGG sequence determined by this bundle starts with a second order operator defined on a line bundle, and then extends similar to (3.3) (up to twists by appropriate line bundles) with S^2E^*, the kernel of the complete symmetrization in $E^* \otimes S^2E^*$, and E^*. The operators in the sequence have orders 2, 1, and 1. Again, they can be made more explicit in terms of (families of) distinguished connections, see [22].

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