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Mass from an Extrinsic Point of View

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Abstract. We express the q-th Gauss–Bonnet–Chern mass of an immersed submanifold of Euclidean space as a linear combination of two terms: the total $(2q)$-th mean curvature and the integral, over the entire manifold, of the inner product between the $(2q+1)$-th mean curvature vector and the position vector of the immersion. As a consequence, we obtain, for each q, a geometric inequality that holds whenever the positive mass theorem (for the q-th Gauss–Bonnet–Chern mass) holds.

Key words: Gauss–Bonnet–Chern mass; asymptotically Euclidean submanifolds; positive mass theorem; Hsiung–Minkowski identities

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1 Introduction

In this article, we explore the concept of mass from an extrinsic point of view. Our approach is based on the introduction of a class of immersions in Euclidean space whose members we call *asymptotically Euclidean immersions* (Definition 3.5) and on an integral identity (Theorem 3.6) that can be seen as a version of the classical Hsiung-Minkowski formulas [16, 19] for a class of non-compact immersed submanifolds. Through this identity, we deduce a geometric inequality (Corollary 3.7) that must be satisfied whenever the positive mass conjecture for the GBC mass (and ADM mass, in particular) is valid.

We also explore the necessary conditions for vector fields to generate the asymptotic charges of interest (Proposition 2.7) and present two conjectures. In one of them, we conjecture that asymptotically Euclidean spaces admit an asymptotically Euclidean isometric immersion (Conjecture 3.8); in the other, we conjecture that the aforementioned geometric inequality must hold whenever an asymptotically Euclidean immersion satisfies a natural hypothesis (Conjecture 3.9). If both conjectures are valid, the positive mass conjecture for the GBC mass (and ADM mass, in particular) can be directly deduced from our results.

2 Mass of asymptotically Euclidean spaces

In this section, we establish the background needed for understanding our method. As far as we are aware, the results presented in Proposition 2.7 and Corollary 2.8 are novel in the literature. They are fundamental ingredients for the employment of our line of action, since they establish sufficient conditions for a vector field to generate the asymptotic charges we are interest in.

We begin by remembering the concept of an asymptotically Euclidean end.

Definition 2.1 (asymptotically Euclidean end)**.** An asymptotically Euclidean end of order τ, with $\tau > 0$, is a Riemannian manifold (E^n, g), $n \geq 3$, for which there exists a diffeomorphism

$\Psi\colon E \to \mathbb{R}^n \setminus \overline{B}_1(0)$, introducing coordinates in E, say $\Psi(x) = \big(x^1, x^2, \dots, x^n\big)$, such that, in these coordinates, the following asymptotic condition holds:

$$|g_{ij} - \delta_{ij}| + \rho|g_{ij,k}| + \rho^2|g_{ij,kl}| = O(\rho^{-\tau}), \qquad \text{as } \rho \to \infty, \tag{2.1}$$

for all $i, j, k, l \in \{1, 2, \dots, n\}$, where the g_{ij}'s are the coefficients of g with respect to the coordinates $\Psi(x)$, $g_{ij,k} = \partial g_{ij}/\partial x^k$, $g_{ij,kl} = \partial g_{ij}/\partial x^k \partial x^l$, and $\rho = |\Psi(x)|$ denotes the distance function to the origin with respect to the Euclidean metric induced on the end.

Our model mass concept is the ADM mass of an end (E, g), introduced by Arnowitt, Deser and Misner in [2].

Definition 2.2 (ADM mass)**.** The ADM mass of an asymptotically Euclidean end (E^n, g) is defined by

$$\mathrm{m}_{\mathrm{ADM}}(E, g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\rho\to\infty} \int_{S_\rho} (g_{ij,i} - g_{ii,j})\nu^j \, \mathrm{d}S_\rho, \tag{2.2}$$

where ω_{n-1} is the volume of the unit sphere of dimension $n-1$, S_ρ is the Euclidean coordinate sphere of radius ρ, $\mathrm{d}S_\rho$ is the volume form induced on S_ρ by the Euclidean metric, and ν is the outward pointing unit normal to S_ρ (with respect to the Euclidean metric).

It is known that if $\tau > (n-2)/2$ and the scalar curvature of (E, g) is integrable, then the limit (2.2) exists, is finite, and is a geometric invariant, that is, two coordinate systems satisfying (2.1) yield the same value for it [4, 5] (see also [30]).

A complete Riemannian manifold (M^n, g), $n \geq 3$, is said to be asymptotically Euclidean of order τ if there exists a compact subset K of M such that $M \setminus K$ has finitely many connected components and, for any connected component E of $M \setminus K$, it occurs that (E, g) is an asymptotically Euclidean end of order τ.

If (M^n, g) is an asymptotically Euclidean Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is integrable, then its ADM mass, denoted by $\mathrm{m}_{\mathrm{ADM}}(M, g)$, is defined as the sum of the ADM masses of its ends.

One of the most important results in mathematical general relativity is the positive mass theorem (PMT):

Theorem 2.3. *If (M^n, g) is an asymptotically flat Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is nonnegative and integrable, then each of its ends has nonnegative ADM mass. Moreover, if the ADM mass of at least one of its ends is zero, then (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.*

The PMT was settled by Schoen and Yau when $n \leq 7$ [34, 35, 36] and when (M, g) is conformally Euclidean [37], and by Witten when M is spin [40] (see also [33]). The general cases of the PMT were treated by Lohkamp [23, 24, 25, 26] and by Schoen and Yau [38]. Proofs for the case when (M, g) is an Euclidean graph (without the rigidity statement) were given by Lam [20, 21] for graphs of codimension one (see also [8]) and by Mirandola and Vitório [31] for graphs of arbitrary codimension. The case of Euclidean hypersurfaces (not necessarily graphs), including the rigidity statement, was treated in [18]. Note that, since Euclidean graphs (of arbitrary codimension) and Euclidean hypersurfaces are spin, these cases also follow from Witten's proof.

In [12], a new mass (actually, a family of masses) for asymptotically Euclidean manifolds, called the Gauss–Bonnet–Chern mass, was introduced. For a positive integer $q < n/2$, consider the q-th Gauss–Bonnet curvature, denoted $L_{(q)}$, and defined by

$$L_{(q)} = \frac{1}{2^q} \delta^{a_1 a_2 \dots a_{2q-1} a_{2q}}_{b_1 b_2 \dots b_{2q-1} b_{2q}} \prod_{s=1}^{q} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} = P^{ijkl}_{(q)} R_{ijkl},$$

where R is the Riemann curvature tensor of (M, g) and $P_{(q)}$, which has the same symmetries of R (see [12, Section 3]), is given by

$$P_{(q)}^{ijkl} = \frac{1}{2^q}\delta^{a_1a_2\cdots a_{2q-3}a_{2q-2}ij}_{b_1b_2\cdots b_{2q-3}b_{2q-2}b_{2q-1}b_{2q}}\left(\prod_{s=1}^{q-1} R_{a_{2s-1}a_{2s}}{}^{b_{2s-1}b_{2s}}\right) g^{b_{2q-1}k}g^{b_{2q}l}.$$

Definition 2.4 (GBC mass)**.** The q-th GBC mass of an asymptotically Euclidean end (E^n, g) is defined by

$$\mathrm{m}_q(E, g) = c(n, q)\lim_{\rho\to\infty}\int_{S_\rho} P_{(q)}^{ijkl} g_{jk,l}\nu_i \,\mathrm{d}S_\rho, \tag{2.3}$$

where

$$c(n, q) = \frac{(n-2q)!}{2^{q-1}(n-1)!\,\omega_{n-1}}$$

and S_ρ, $\mathrm{d}S_\rho$, ν and ω_{n-1} are as in the definition of the ADM mass.

Note that $L_{(1)}$ is just the scalar curvature and, as observed in [12], m_1 coincides with the ADM mass.

In the same article, it is shown that if $\tau > \tau_q$ and $L_{(q)}$ is integrable, then the limit (2.3) exists, is finite, and is a geometric invariant, where here and throughout the text,

$$\tau_q = \frac{(n-2q)}{(q+1)}.$$

As in the $q = 1$ case, if (M^n, g) is an asymptotically Euclidean manifold of order $\tau > \tau_q$ whose q-th Gauss–Bonnet curvature is integrable, then its q-th GBC mass, denoted by $\mathrm{m}_q(M, g)$, is defined as the sum of the q-th GBC masses of its ends.

The following is a version of the PMT for the GBC mass.

Conjecture 2.5. *Let n and q be integers such that $n \geq 3$ and $0 < q < n/2$. If (M^n, g) is an asymptotically Euclidean Riemannian manifold of order $\tau > \tau_q$ whose q-th Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable, then the q-th GBC mass of each of its ends is nonnegative. Moreover, if the GBC mass of at least one of its ends is zero, then (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.*

Conjecture 2.5, without the rigidity statement, was proved for graphs of codimension one in [12]; the case of graphs of arbitrary codimension and flat normal bundle was done by Li, Wei and Xiong when $q = 2$ [22] and by the authors when $0 < q < n/2$ [10, 11]. This conjecture, including the rigidity statement, is known to be true for conformally flat manifolds [13].

2.1 Mass in terms of the Lovelock tensor

Let (E^n, g), $n \geq 3$, be an asymptotically Euclidean end of order $\tau > \tau_1$, and let G be its Einstein tensor, that is,

$$G = \mathrm{Ric} - \frac{1}{2}(\mathrm{Sc})g,$$

where Ric and Sc denote, respectively, the Ricci tensor and the scalar curvature of (E, g). Throughout the text, we denote by X the vector field given by

$$X = x^i\frac{\partial}{\partial x^i}, \tag{2.4}$$

where $\Psi(x) = \left(x^1, x^2, \ldots, x^n\right)$ is a coordinate system satisfying (2.1).

Is is known that the ADM mass of (E, g) can be computed as follows (see [3, 6, 7]):

$$\mathrm{m}_{\mathrm{ADM}}(E, g) = -\frac{1}{(n-1)(n-2)\omega_{n-1}} \lim_{\rho\to\infty} \int_{S_\rho} G(X, \nu_g)\, \mathrm{d}S^g_\rho, \tag{2.5}$$

where ω_{n-1} and S_ρ are as in (2.2), ν_g is the outward unit normal vector to S_ρ with respect to the metric g, and $\mathrm{d}S^g_\rho$ is the volume form induced on S_ρ by g. The equivalence between formulas (2.2) and (2.5) can be shown by reducing the general case to the case of harmonic asymptotics via a density theorem (see, for example, [17]). Proofs without the use of a density theorem were given in [15, 29].

As we will see in a moment, a formula similar to (2.5) holds for the GBC mass. To state this result, we need to recall the so-called Lovelock curvature tensors.

Let (M^n, g), $n \geq 3$, be a Riemannian manifold and let $q < n/2$ be a positive integer. The q-th Lovelock curvature tensor, denoted by $G_{(q)}$, is defined by

$$G_{(q)ij} = -\frac{1}{2^{q+1}} g_{ik} \delta^{kb_1b_2\ldots b_{2q-1}b_{2q}}_{ja_1a_2\ldots a_{2q-1}a_{2q}} \prod_{s=1}^{q} R_{b_{2s-1}b_{2s}}{}^{a_{2s-1}a_{2s}}. \tag{2.6}$$

Note that $G_{(1)}$ is just the Einstein tensor.

Proposition 2.6. *The Lovelock curvature tensor $G_{(q)}$ satisfies the following:*

(i) It is symmetric, that is,

$$G_{(q)ij} = G_{(q)ji}. \tag{2.7}$$

(ii) It is divergent-free, that is,

$$\nabla^i G_{(q)ij} = 0. \tag{2.8}$$

(iii) Its trace satisfies the equation

$$\mathrm{tr}_g\, G_{(q)} = -\frac{n-2q}{2} L_{(q)}. \tag{2.9}$$

Proof. Identities (2.7) and (2.8) follow from [28, Theorem 1] (see also [27]). Identity (2.9) is a straightforward computation. ■

Let (E^n, g), $n \geq 3$, be an asymptotically Euclidean end of order $\tau > \tau_q$, where $q < n/2$ is a positive integer. It was shown in [39] that the q-th GBC mass of (E, g) can be computed as follows:

$$\mathrm{m}_q(E, g) = -b(n, q) \lim_{\rho\to\infty} \int_{S_\rho} G_{(q)}(X, \nu_g)\, \mathrm{d}S^g_\rho, \tag{2.10}$$

where X, ν_g and $\mathrm{d}S^g$ are as in (2.5) and

$$b(n, q) = \frac{(n-2q-1)!}{2^{q-1}(n-1)!\, \omega_{n-1}}.$$

Note that, when $q = 1$, formulas (2.10) and (2.5) coincide.

The following proposition establishes sufficient conditions for a vector field to generate the GBC mass.

Proposition 2.7. *Let (E^n, g), $n \geq 3$, be an asymptotically Euclidean end of order $\tau > \tau_q$, where $q < n/2$ is a positive integer. Let $\Psi(x) = (x_1, x_2, \ldots, x_n)$ be coordinates in E satisfying (2.1) and let X be the vector field given by (2.4). If Y is a vector field on E such that*

$$\big(Y^i - X^i\big)(x) = O\big(\rho^{-\tau+1}\big), \qquad i = 1, 2, \ldots, n,$$

as $\rho \to \infty$, then

$$\mathrm{m}_q(E, g) = -b(n, q) \lim_{\rho\to\infty} \int_{S_\rho} G_{(q)}(Y, \nu_g)\, \mathrm{d}S^g_\rho.$$

Proof. Let $\Psi(x) = \big(x^1, x^2, \ldots, x^n\big)$ be coordinates on E satisfying (2.1). We can use these coordinates to compare ν_g, the unit normal to S_ρ with respect to g, and $\mathrm{d}S^g_\rho$, the volume form induced on S_ρ by g, with their Euclidean counterparts ν_δ and $\mathrm{d}S^\delta_\rho$, respectively.

It holds

$$\nu^i_g - \nu^i_\delta = O(\rho^{-\tau}), \qquad \text{as } \rho \to \infty,$$

and

$$\mathrm{d}S^g_\rho = (1 + w)\, \mathrm{d}S^\delta_\rho$$

for some function $w\colon E \to \mathbb{R}$ satisfying

$$w = O(\rho^{-\tau}), \qquad \text{as } \rho \to \infty.$$

Furthermore, the components of the Riemman curvature tensor satisfy

$$R^l_{ijk} = O\big(\rho^{-\tau-2}\big), \qquad \text{as } \rho \to \infty.$$

Together with (2.6), this gives

$$G_{(q)ij} = O\big(\rho^{-q(\tau+2)}\big), \qquad \text{as } \rho \to \infty.$$

Thus, we have

$$\begin{aligned}
&\left|\int_{S_\rho} G_{(q)}(Y, \nu_g)\, \mathrm{d}S^g_\rho - \int_{S_\rho} G_{(q)}(X, \nu_g)\, \mathrm{d}S^g_\rho\right| \\
&\qquad \leq \int_{S_\rho} \big|G_{(q)ij}\big(Y^i - X^i\big)\big|\big(\big|\nu^i_g - \nu^i_\delta\big| + \big|\nu^i_\delta\big|\big)(1 + w)\, \mathrm{d}S^\delta_\rho \\
&\qquad \leq C(n)\rho^{-[q(\tau+2)+(\tau-1)-(n-1)]}(1 + \rho^{-\tau})^2,
\end{aligned}$$

where $C(n)$ is a constant that depends only on the dimension. Using that $\tau > \tau_q$, it follows that

$$\lim_{\rho\to\infty}\left|\int_{S_\rho} G_{(q)}(Y, \nu_g)\, \mathrm{d}S^g_\rho - \int_{S_\rho} G_{(q)}(X, \nu_g)\, \mathrm{d}S^g_\rho\right| = 0. \tag{2.11}$$

The proposition follows from (2.10) and (2.11). ■

The specialization of this proposition to the case of gradient fields is a key ingredient in the development of our extrinsic approach.

Corollary 2.8. *Under the same hypothesis as the ones in Proposition 2.7, if there exists* $f \in C^\infty(E)$ *such that*

$$\frac{\partial}{\partial x^i}\left(f - \frac{\rho^2}{2}\right) = O\big(\rho^{-\tau+1}\big), \qquad i = 1, 2, \dots, n, \tag{2.12}$$

as $\rho \to \infty$*, then*

$$\mathrm{m}_q(E, g) = -b(n, q) \lim_{\rho\to\infty} \int_{S_\rho} G_{(q)}\big(\nabla f, \nu_g\big)\, \mathrm{d}S^g_\rho.$$

Proof. Write $g^{ij} = \delta^{ij} + \theta^{ij}$ and take $Y = \nabla f$. We have

$$\begin{aligned} Y^i - X^i &= g^{ij}\frac{\partial f}{\partial x^j} - \frac{\partial}{\partial x^i}\left(\frac{\rho^2}{2}\right) = \delta^{ij}\frac{\partial f}{\partial x^j} + \theta^{ij}\frac{\partial f}{\partial x^j} - \frac{\partial}{\partial x^i}\left(\frac{\rho^2}{2}\right) \\ &= \frac{\partial}{\partial x^i}\left(f - \frac{\rho^2}{2}\right) + \theta^{ij}\frac{\partial f}{\partial x^j}. \end{aligned} \tag{2.13}$$

Note that (2.12) is equivalent to

$$\frac{\partial f}{\partial x^i} = x^i + O\big(\rho^{-\tau+1}\big), \qquad \text{as } \rho \to \infty. \tag{2.14}$$

Thus, since

$$\theta^{ij} = O(\rho^{-\tau}), \qquad \text{as } \rho \to \infty, \tag{2.15}$$

from (2.14) and (2.15) we find

$$\theta^{ij}\frac{\partial f}{\partial x^j} = O\big(\rho^{-\tau+1}\big), \qquad \text{as } \rho \to \infty. \tag{2.16}$$

The corollary follows from Proposition 2.7, (2.13) and (2.16). ■

3 Imersions in Euclidean spaces

Let $\psi\colon M^n \looparrowright \mathbb{R}^d$, $d > n$, be a smooth immersion and let $\bar\delta = \langle\cdot,\cdot\rangle$ be the canonical Euclidean metric on $\mathbb{R}^d$. All quantities related to this ambient space will also be denoted with an overbar, unless otherwise indicated.

Let B (which is normal-vector valued) be the second fundamental form of the immersion and let p be an integer such that $0 < p \le n$; if p is even, then the p-th mean curvature $S_{(p)} \in C^\infty(M)$ is defined by

$$S_{(p)} = \frac{1}{p!}\delta^{b_1\dots b_p}_{a_1\dots a_p}\big\langle B^{a_1}_{b_1}, B^{a_2}_{b_2}\big\rangle \cdots \big\langle B^{a_{p-1}}_{b_{p-1}}, B^{a_p}_{b_p}\big\rangle,$$

and, if p is odd, then the p-th mean curvature $S_{(p)} \in \Gamma\big(TM^\perp\big)$ is defined by

$$S_{(p)} = \frac{1}{p!}\delta^{b_1\dots b_p}_{a_1\dots a_p}\big\langle B^{a_1}_{b_1}, B^{a_2}_{b_2}\big\rangle \cdots \big\langle B^{a_{p-2}}_{b_{p-2}}, B^{a_{p-1}}_{b_{p-1}}\big\rangle B^{a_p}_{b_p},$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean metric on $\mathbb{R}^d$. We also set $S_{(0)} = 1$ and $S_{(n+1)} = 0$.

Next, we turn to the so-called Newton transformations. The 0-th Newton transformation is defined as

$$T_{(0)} = g,$$

where $g = \psi^*\bar{\delta}$ denotes the induced metric. If $p \geq 2$ is even, then the p-th Newton transformation $T_{(p)}$ is defined by

$$T_{(p)ij} = \frac{1}{p!} g_{ik} \delta^{kb_1 b_2 \dots b_p}_{j a_1 a_2 \dots a_p} \langle B^{a_1}_{b_1}, B^{a_2}_{b_2} \rangle \cdots \langle B^{a_{p-1}}_{b_{p-1}}, B^{a_p}_{b_p} \rangle, \tag{3.1}$$

and, if p is odd, then the p-th Newton transformation $T_{(p)}$ is defined by

$$T_{(p)ij} = \frac{1}{p!} g_{ik} \delta^{kb_1 b_2 \dots b_{p-1}}_{j a_1 a_2 \dots a_{p-1}} \langle B^{a_1}_{b_1}, B^{a_2}_{b_2} \rangle \cdots \langle B^{a_{p-2}}_{b_{p-2}}, B^{a_{p-1}}_{b_{p-1}} \rangle B^{a_p}_{b_p}.$$

Note that, when p is odd, the p-th Newton transformation is normal-vector valued. Furthermore, by antisymmetry, it follows that $T_{(p)} \equiv 0$ when $p > n$.

The relation between the trace of Newton transformations and the higher-order mean curvatures is a fact well-known in the literature (see [14, Lemma 2.2], for example).

Proposition 3.1. *Let p be an integer such that $0 \leq p \leq n$. The Newton transformation $T_{(p)}$ satisfy*

$$\mathrm{tr}_g T_{(p)} = (n-p) S_{(p)}. \tag{3.2}$$

Moreover, if p is even, then

$$T_{(p)ij} B^{ij} = (p+1) S_{(p+1)}.$$

By a direct application of the Gauss equation, we obtain that the q-th Lovelock tensor (2.6) of the induced metric $g = \psi^*\bar{\delta}$ and the $2q$-th Newton transformation (3.1) of an immersion contain the same information.

Proposition 3.2. *Let q be a positive integer such that $q < n/2$. It holds*

$$G_q = -\frac{(2q)!}{2} T_{(2q)}. \tag{3.3}$$

The next lemma is an infinitesimal version of a Pohozaev–Schoen-type integral identity presented in [9, Propositon 3.2].

Lemma 3.3. *If K and V are, respectively, a symmetric bilinear form and a vector field on M, then the following identity holds:*

$$\mathrm{div}_g(\iota_V K) = \iota_V(\mathrm{div}_g K) + \frac{1}{2} g(K, \mathcal{L}_V g). \tag{3.4}$$

Throughout the text, we denote by $\bar{Z}$ the vector field on $\mathbb{R}^d$ given by

$$\bar{Z} = \bar{x}^\alpha \frac{\partial}{\partial \bar{x}^\alpha}, \qquad 1 \leq \alpha \leq d,$$

where $\left(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^d\right)$ is the standard coordinate system on $\mathbb{R}^d$. It is known that it is a conformal Killing gradient field and that it satisfies the following identity:

$$\bar{Z} = \frac{\overline{\nabla} \bar{\rho}^2}{2},$$

where $\bar{\rho}$ is the distance function to origin on $\mathbb{R}^d$.

The following proposition can be interpreted as an infinitesimal version of the integral identity of Theorem 3.6. It is inspired by a infinitesimal version of the flux formula presented in [1, equation (8.4)].

Proposition 3.4. *Let* $Y = \psi^*\bar{Z}^\top$*, where* $\bar{Z}^\top$ *denotes the tangent part of the vector field* $\bar{Z}$*. It holds*

$$\operatorname{div}_g(\iota_Y G_{(q)}) = -\frac{(2q)!}{2}\big[(n-2q)S_{2q} + (2q+1)\big\langle S_{2q+1}, \bar{Z}\big\rangle\big]. \tag{3.5}$$

Proof. Since $Y = \psi^*\bar{Z}^\top$, we have

$$\psi_* Y = \bar{Z}^\top = \bar{Z} - \bar{Z}^\perp$$

and

$$\mathcal{L}_Y g = \mathcal{L}_Y \psi^*\bar{\delta} = \psi^*\big(\mathcal{L}_{\psi_* Y}\bar{\delta}\big) = \psi^*\big(\mathcal{L}_{(\bar{Z}-\bar{Z}^\perp)}\bar{\delta}\big) = \psi^*\big(\mathcal{L}_{\bar{Z}}\bar{\delta}\big) - \psi^*\big(\mathcal{L}_{\bar{Z}^\perp}\bar{\delta}\big). \tag{3.6}$$

Denote by $B_{\bar{Z}}$ the symmetric bilinear form on M defined by

$$B_{\bar{Z}}(V,W) = \big\langle B(V,W), \bar{Z}\big\rangle.$$

For any $V, W \in \Gamma(TM)$,

$$\big(\mathcal{L}_{\bar{Z}^\perp}\bar{\delta}\big)(V,W) = \big\langle D_V \bar{Z}^\perp, W\big\rangle + \big\langle V, D_W \bar{Z}^\perp\big\rangle = -2\big\langle B(V,W), \bar{Z}\big\rangle = -2B_{\bar{Z}}(V,W),$$

that is,

$$\mathcal{L}_{\bar{Z}^\perp}\bar{\delta} = -2B_{\bar{Z}}. \tag{3.7}$$

We also have

$$\mathcal{L}_{\bar{Z}}\bar{\delta} = 2\bar{\delta}. \tag{3.8}$$

Thus, (3.6)–(3.8) yield

$$g(G_{(q)}, \mathcal{L}_Y g) = 2\operatorname{tr}_g G_{(q)} + 2g(G_{(q)}, B_{\bar{Z}}). \tag{3.9}$$

Identity (3.5) follows from (2.8), (3.2)–(3.4) and (3.9). ■

3.1 Asymptotically Euclidean imersions

Next, we describe our main object of study: a class of smooth immersions that place certain smooth manifolds M^n into some Euclidean space $\mathbb{R}^d$, where $d > n$, in a special way.

Definition 3.5 (asymptotically Euclidean immersion)**.** Let $\psi\colon M^n \looparrowright \mathbb{R}^d$, $d > n$, be a smooth immersion of a smooth manifold M^n, $n \geq 3$. We say that ψ is an asymptotically Euclidean immersion of order τ, for some $\tau > 0$, if the following conditions hold:

(i) the Riemannian manifold $\big(M, \psi^*\bar{\delta}\big)$ is complete and asymptotically Euclidean of order τ;

(ii) if $E \subset M$ is such that $(E, \psi^*\bar{\delta})$ is an asymptotically Euclidean end of order τ, then, as $\rho \to \infty$,

$$\frac{\partial}{\partial x^i}\big(\psi^*\bar{\rho}^2 - \rho^2\big)(x) = O\big(\rho^{-\tau+1}\big), \qquad i = 1, 2, \dots, n,$$

for any coordinate system $\Psi(x) = \big(x^1, x^2, \dots, x^n\big)$ on E satisfying (2.1).

The following theorem is the main result of this article. It can be seen as a version of the Hsiung–Minkowski formulas [16, 19] to asymptotically Euclidean immersions.

Theorem 3.6. *Let n and q be integers such that $n \geq 3$ and $0 < q < n/2$. If $\psi\colon M^n \hookrightarrow \big(\mathbb{R}^d, \bar{\delta}\big)$, $d > n$, is an asymptotically Euclidean immersion of order $\tau > \tau_q$ such that $L_{(q)}$, the q-th Gauss-Bonnet curvature of the induced metric $\psi^*\bar{\delta}$, is integrable, then the functions S_{2q} and $\big\langle S_{2q+1}, \bar{Z}\big\rangle$ are integrable and the q-th Gauss–Bonnet–Chern mass of $\big(M, \psi^*\bar{\delta}\big)$ satisfies the following integral identity:*

$$\mathrm{m}_q\big(M, \psi^*\bar{\delta}\big) = a(n,q)\bigg[(n-2q)\int_M S_{2q}\,\mathrm{d}M + (2q+1)\int_M \langle S_{2q+1}, \bar{Z}\rangle\,\mathrm{d}M\bigg],$$

where $\mathrm{d}M$ *is the Riemannian measure on* $\big(M, \psi^*\bar{\delta}\big)$ *and*

$$a(n,q) = \frac{(2n)!(n-2q-1)!}{2^q(n-1)!\,\omega_{n-1}}.$$

Proof. Let E be one of the ends of M. Consider the function $f\colon M \to \mathbb{R}$, with

$$f = \frac{\psi^*\bar{\rho}^2}{2}.$$

Since $\psi\colon M^n \looparrowright \mathbb{R}^d$ is an asymptotically Euclidean immersion of order $\tau > \tau_q$, the function f satisfies (2.12), and hence, by Corollary 2.8,

$$\mathrm{m}_q\big(E, \psi^*\bar{\delta}\big) = -b(n,q)\lim_{\rho\to\infty}\int_{S_\rho} G_{(q)}(\nabla f, \nu)\,\mathrm{d}S_\rho, \tag{3.10}$$

where the geometric quantities in the integral are computed with respect the induced metric $\psi^*\bar{\delta}$.

Note that

$$\nabla f = \psi^*\bar{Z}^\top.$$

Thus, applying equation (3.10) to all the ends of M together with the divergence theorem and identity (3.5), we find

$$\mathrm{m}_q\big(M, \psi^*\bar{\delta}\big) = a(n,q)\int_M \big[(n-2q)S_{2q} + (2q+1)\big\langle S_{2q+1}, \bar{Z}\big\rangle\big]\mathrm{d}M. \tag{3.11}$$

It remains to show that both S_{2q} and $\big\langle S_{2q+1}, \bar{Z}\big\rangle$ are integrable, so the right hand side of (3.11) can be broken in two. The integrability of S_{2q} follows from (2.9), (3.2), (3.3) and the integrability of $L_{(q)}$. Once we know that $S_{(2q)}$ is integrable, the integrability of $\big\langle S_{2q+1}, \bar{Z}\big\rangle$ follows from the fact that the left hand side of (3.11) is finite. ■

An immediate consequence of this theorem is the following.

Corollary 3.7. *Under the same hypothesis as the ones in Theorem 3.6,* $\mathrm{m}_q\big(M, \psi^*\bar{\delta}\big) \geq 0$ *if and only if*

$$(n-2q)\int_M S_{2q}\,\mathrm{d}M + (2q+1)\int_M \big\langle S_{2q+1}, \bar{Z}\big\rangle\,\mathrm{d}M \geq 0.$$

A famous theorem by Nash [32] states that any Riemannian manifold (M, g) can be isometrically immersed in some Euclidean space $\big(\mathbb{R}^d, \bar{\delta}\big)$. Encouraged by Nash's theorem, we make the following conjecture.

Conjecture 3.8. *If (M^n, g), $n \geq 3$, is an asymptotically Euclidean manifold of order $\tau > 0$, then there exists an asymptotically Euclidean isometric immersion $\psi\colon M \to \mathbb{R}^d$ of order τ (in the sense of Definition 3.5).*

Let n and q be integers such that $n \geq 3$ and $0 < q < n/2$. Suppose that Conjecture 3.8 is true for every asymptotically Euclidean manifold (M^n, g) of order $\tau > \tau_q$ (or at least for those whose Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable). Then, by Theorem 3.6, Conjecture 2.5 would be a direct consequence of the following conjecture.

Conjecture 3.9. *Let n and q be integers such that $n \geq 3$ and $0 < q < n/2$. Let $\psi\colon M^n \to \mathbb{R}^d$, $d > n$, be an asymptotically Euclidean immersion of order $\tau > \tau_q$ for which S_{2q} is integrable and nonnegative. It holds*

$$(n-2q)\int_M S_{2q}\,\mathrm{d}M + (2q+1)\int_M \langle S_{2q+1}, \bar{Z}\rangle\,\mathrm{d}M \geq 0,$$

with the equality holding if and only if $\left(M, \psi^*\bar{\delta}\right)$ is isometric to Euclidean space.

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