The Rogers–Ramanujan Identities and Cauchy's Identity

Dennis STANTON

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA
E-mail: stanton@math.umn.edu
URL: https://www-users.cse.umn.edu/~stant001/

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Abstract. The Rogers–Ramanujan identities are investigated using the Cauchy identity for Schur functions.

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Dedicated to Stephen Milne
for his 75th birthday

1 Introduction

Two of Steve Milne's most noteworthy works are on the Rogers–Ramanujan identities (see [2])

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{\left(q^1;q^5\right)_\infty \left(q^4;q^5\right)_\infty}, \qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{\left(q^2;q^5\right)_\infty \left(q^3;q^5\right)_\infty}. \tag{1.1}$$

With J. Lepowsky he proved (1.1) algebraically (see [6, 7]). The involution principle, with A. Garsia [3], gave an indirect bijection for MacMahon's combinatorial interpretation of the identities.

Stembridge [10] used symmetric function identities via Hall–Littlewood polynomials to prove and generalize the Rogers–Ramanujan identities. This was continued by Jouhet–Zeng [5] and S. Ole Warnaar [11]. A vast generalization to the Rogers–Ramanujan identities, corresponding to affine Lie algebras, was given in [4, 9]. Here the appropriate Hall–Littlewood polynomials are specializations of the Macdonald–Koornwinder polynomials.

The purpose of this note is explore a naive approach using the Cauchy identity for Schur functions. What would be required for a explicit bijective proof via the Cauchy identity is discussed in Section 2. Some related identities and a speculation are given in Sections 3 and 4, while Section 5 has two remarks.

All symmetric function facts can be found in Macdonald's book [8].

2 A proposal for a bijection

MacMahon's combinatorial interpretation of (1.1) uses integer partitions.

Proposition 2.1. *The first Rogers–Ramanujan identity is equivalent to the following two sets of integer partitions being equinumerous for any n:*

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(1) *integer partitions of n into parts congruent to 1 or 4 modulo 5,*

(2) *integer partitions of n whose parts differ by at least 2.*

There is no known direct bijection between these two finite sets of partitions. There is a similar statement for the second Rogers–Ramanujan identity, also with an unknown bijection.

In this paper, we use Schur functions, $s_\lambda(x_1, \dots, x_n, \dots)$, which are symmetric functions in variables $x_1, x_2, \dots$ indexed by integer partitions λ. A Schur function indexed by λ is the generating function of all column strict tableaux P of shape λ. For example, if $\lambda = (4, 2, 1)$, one such P is

$$P = \begin{matrix} 1 & 1 & 3 & 6 \\ 3 & 3 & & \\ 5 & & & \end{matrix}$$

whose weight is $x_1^2 x_3^3 x_5^1 x_6^1$. In this paper, the weights are always powers of q, so the weight of a column strict tableaux P is q^N, where N is the sum of the entries of P. Suppose the number of variables is finite, for example R variables. There are only R possible choices for entries in the first column, so if the indexing partition λ has more than R parts, the Schur function is zero. We will later use $R = 2$ case so that λ has at most 2 rows. The Schur function indexed by the empty partition is 1.

The Cauchy identity for Schur functions $s_\lambda(x_1, \dots, x_n, \dots)$ provides a start for a Rogers–Ramanujan bijection. The Cauchy identity is

$$\sum_\lambda s_\lambda(x_1, \dots, x_n, \dots) s_\lambda(y_1, \dots, y_m, \dots) = \prod_{i,j} (1 - x_i y_j)^{-1}. \tag{2.1}$$

Moreover, it is known that the Robinson–Schensted–Knuth correspondence is a direct bijection for (2.1), see [2, Chapter 11.3].

Choose $(x_1, \dots, x_n, \dots) = \big(1, q^5, q^{10}, q^{15}, \dots\big)$, $(y_1, y_2) = \big(q^1, q^4\big)$ so that the right side of (2.1) is the product side of the first Rogers–Ramanujan identity $\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$, while the left side is restricted to partitions with at most two rows

$$\sum_{\lambda \text{ at most 2 rows}} s_\lambda\big(1, q^5, q^{10}, q^{15}, \dots\big) s_\lambda\big(q^1, q^4\big).$$

Proposition 2.2. *The Robinson–Schensted–Knuth correspondence provides a direct bijection between*

(1) *integer partitions of n into parts congruent to 1 or 4 modulo 5,*

(2) *pairs of column strict tableaux (P, Q) of the same shape with at most two rows, P having entries congruent to 0 modulo 5, Q having entries 1 or 4, whose sum of entries is n.*

Proposition 2.2 offers some advantages and disadvantages for a bijection. On the plus side, it changes the problem to a problem on tableaux, for which there is a well developed machinery of bijections. These more refined objects may be easier to sort than integer partitions. Conversely, the simple answer required, partitions whose parts differ by at least two, may not be apparent from this detailed view. The $\lambda = \varnothing$ term corresponds to the $n = 0$ term in the sum side of the first Rogers–Ramanujan identity, namely 1. Here are the pairs of column strict tableaux (P, Q) which correspond to the $n = 1$ term in the sum side of the first Rogers–Ramanujan identity

$$\frac{q^1}{1-q} = \frac{q^1 + q^2 + q^3 + q^4 + q^5}{1 - q^5}.$$

If $\lambda = 1$, the possible choices for (P, Q) are $(x, 1)$, $(x, 4)$, where x is a multiple 5. Their generating function is $\frac{q^1+q^4}{1-q^5}$. For $\lambda = 2$, we may take $((0, x), (1, 1))$, $((0, x), (1, 4))$ where x is a multiple 5,

Table 1. Column strict pairs (P, Q) for $n = 1$.

λ	(P, Q)	generating function
1	$((x),\ (1)$ or $(4))$	$(q^1 + q^4)/(1 - q^5)$
2	$((0, x),\ (1, 1)$ or $(1, 4))$	$(q^2 + q^5)/(1 - q^5)$
3	$((0, 0, x),\ (1, 1, 1))$	$q^3/(1 - q^5)$

whose generating function is $\frac{q^2+q^5}{1-q^5}$. For the remaining term, we take $\lambda = 3$, $((0, 0, x), (1, 1, 1))$, where x is a multiple 5. whose generating function is $\frac{q^3}{1-q^5}$.

The $n = 2$ term on the sum side is

$$\frac{q^4}{(1-q)(1-q^2)} = \frac{q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 2q^9 + 3q^{10} + 2q^{11} + 3q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16}}{(1-q^5)(1-q^{10})}.$$

We list, in Table 2, 25 classes of pairs (P, Q) which correspond to these 25 numerator terms. Each class has a generating function of $q^A/(1 - q^5)(1 - q^{10})$, for an A between 4 and 16. The denominator factors occur because the generating function for partitions with at most 2 parts, each part a multiple of 5, is $1/(1 - q^5)(1 - q^{10})$.

Table 2. Column strict pairs (P, Q) for $n = 2$.

λ	(P, Q)	A
2	$((y, x),\ (1, 1)$ or $(1, 4))$ $y \geq 5$	12, 15
2	$((y, x),\ (4, 4))$	8
3	$((0, y, x),\ (1, 1, 4))$ or $(1, 4, 4)$ or $(4, 4, 4)$	6, 9, 12
4	$((0, 0, y, x),\ (1, 1, 1, 1)$ or $(1, 1, 1, 4)$ or $(1, 1, 4, 4)$ or $(1, 4, 4, 4)$ or $(4, 4, 4, 4))$	4, 7, 10, 13, 16
5	$((0, 0, 0, y, x),\ (1, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 4)$ or $(1, 1, 1, 4, 4)$ or $(1, 1, 4, 4, 4))$	5, 8, 11, 14
6	$((0, 0, 0, 0, y, x),\ (1, 1, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 4)$ or $(1, 1, 1, 1, 4, 4))$	6, 9, 12
7	$((0, 0, 0, 0, 0, y, x),\ (1, 1, 1, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 1, 4)$ or $(1, 1, 1, 1, 1, 4, 4))$	7, 10, 13
8	$((0, 0, 0, 0, 0, 0, y, x),\ (1, 1, 1, 1, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 1, 1, 4)$ or $(1, 1, 1, 1, 1, 1, 4, 4))$	8, 11, 14
$(1, 1)$	$(\text{transpose}(y, x),\ x > y,\ \text{transpose}(1, 4))$	10

In general, we want to obtain the n-th term in the Rogers–Ramanujan sum

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^5;q^5)_n} \prod_{j=1}^{n} \sum_{p=0}^{4} q^{jp}.$$

Problem 2.3. *Can one choose pairs of column strict tableaux (P, Q) of the same shape such that*

(1) *the entries of P are multiples of 5,*

(2) *the entries of Q are 1 and 4,*

(3) *and whose generating function is*

$$F_n(q) = q^{n^2} \left(\prod_{j=1}^{n} \sum_{p=0}^{4} q^{jp} \right) / (q^5;q^5)_n?$$

Solving Problem 2.3 gives a Rogers–Ramanujan bijection when combined with Proposition 2.2. The pair of column strict tableaux (P,Q) correspond to an integer partition μ whose parts are 1 or 4 modulo 5. But they also correspond to an integer partition λ whose parts differ by two. For example, the $n=2$ term has the generating function

$$q^4\left(\frac{1+q+q^2+q^3+q^4}{1-q^5}\right)\left(\frac{1+q^2+q^4+q^6+q^8}{1-q^{10}}\right). \tag{2.2}$$

This means, after subtracting 1 from the second part of λ and 3 from the first part of λ, the resulting columns have length 1 or 2. The 5 terms in the numerator factors of (2.2) are the mod 5 values of the multiplicities of 1 and 2. It will take substantially more insight to resolve Problem 2.3 for an arbitrary n.

3 Formulas

For clarity, here are the explicit generating functions of the Schur functions as products. These follow from the principle specialization of Schur functions, the hook-content formula.

Proposition 3.1. *Let $\lambda=(a+b,a)$. Then*

$$s_\lambda\left(1,q^5,q^{10},q^{15},\dots\right)=\frac{q^{5a}}{\left(q^5;q^5\right)_a\left(q^5;q^5\right)_b\left(q^{5(b+2)};q^5\right)_a},\qquad s_\lambda\left(q^1,q^4\right)=q^{5a+b}\sum_{k=0}^{b}q^{3k}.$$

There is a weighted version using two new parameters x and y.

Theorem 3.2. *Choosing $y_1=xq^1$, $y_2=yq^4$, we have*

$$\frac{1}{\left(xq;q^5\right)_\infty\left(yq^4;q^5\right)_\infty}=\sum_{a,b\geq 0}\frac{q^{5a}}{\left(q^5;q^5\right)_a\left(q^5;q^5\right)_b\left(q^{5(b+2)};q^5\right)_a}x^ay^aq^{5a+b}\sum_{k=0}^{b}x^{b-k}y^kq^{3k}.$$

Theorem 3.2 independently follows from the finite identity

$$\sum_{a=0}^{M}\begin{bmatrix}N\\a\end{bmatrix}_q\left(q^a-q^{N-a}\right)=\frac{(q;q)_N}{(q;q)_M(q;q)_{N-M-1}}\qquad\text{for } 0\leq M\leq N-1.$$

Finally, a simple subclass of (P,Q) has a product formula. A proof of a more general result is given in Theorem 4.1.

Proposition 3.3. *We have*

$$\sum_{\lambda \text{ at most 1 row}}s_\lambda\left(1,q^5,q^{10},q^{15},\dots\right)s_\lambda\left(q^1,q^4\right)=\frac{1}{1-q^3}\left(\frac{1}{\left(q^1;q^5\right)_\infty}-\frac{q^3}{\left(q^4;q^5\right)_\infty}\right). \tag{3.1}$$

4 Rogers–Ramanujan mod $2k+3$

The same steps as in Section 2 can be done for higher moduli $2k+3$, the integer partitions whose parts avoid $\pm i$ and 0 mod $2k+3$, $1\leq i\leq 2k+2$. Set

$$\begin{aligned}&(x_1,\dots,x_n,\dots)=\left(1,q^{2k+3},q^{2(2k+3)},q^{3(2k+3)},\dots\right),\\&(y_1,\dots,y_{2k})=\left(q^1,\dots,q^{2k+2}\right)\qquad\text{with } q^i \text{ and } q^{2k+3-i} \text{ deleted,}\end{aligned} \tag{4.1}$$

so that

$$\sum_{\lambda \text{ at most } 2k \text{ rows}} s_\lambda(x_1,\dots,x_n,\dots)s_\lambda(y_1,\dots,y_{2k}) = \prod_{\substack{j=1\\ j\not\equiv \pm i,0 \bmod 2k+3}}^{\infty} \left(1-q^j\right)^{-1}.$$

The product side of the Rogers–Ramanujan identities (1.1) are the $k=1$ and $i=2,1$ special cases.

There is always a version of the subclass formula (3.1) as a sum of infinite products using (4.1).

Theorem 4.1. *Let $k\ge 1$, $1\le i\le 2k+2$ and $(y_1,\dots,y_{2k}) = \left(q^1,\dots,q^{2k+2}\right)$, with q^i and q^{2k+3-i} deleted. Then*

$$\sum_{\lambda \text{ at most } 1 \text{ row}} s_\lambda\left(1,q^{2k+3},q^{2(2k+3)},q^{3(2k+3)},\dots\right)s_\lambda(y_1,\dots,y_{2k}) = \sum_{\substack{p=1\\ p\neq i,2k+3-i}}^{2k+2} \frac{A_p}{\left(q^p;q^{2k+3}\right)_\infty},$$

where

$$A_p = \prod_{\substack{j=1\\ j\neq p,i,2k+3-i}}^{2k+2} \frac{1}{1-q^{-p+j}}.$$

Proof. If $\lambda = N$ has a single part we have

$$s_\lambda\left(1,q^{2k+3},q^{2(2k+3)},q^{3(2k+3)},\dots\right) = \frac{1}{\left(q^{2k+3};q^{2k+3}\right)_N},$$

$$s_\lambda(y_1,\dots,y_{2k}) = \text{ the coefficient of } t^N \text{ in } \prod_{\substack{j=1\\ j\neq i,2k+3-i}}^{2k+2} \left(1-tq^j\right)^{-1}.$$

By partial fractions on t we see that the A_p satisfy

$$\prod_{\substack{j=1\\ j\neq i,2k+3-i}}^{2k+2} \left(1-tq^j\right)^{-1} = \sum_{\substack{p=1\\ p\neq i,2k+3-i}}^{2k+2} A_p(1-tq^p)^{-1},$$

so that

$$s_\lambda(y_1,\dots,y_{2k}) = \sum_{\substack{p=1\\ p\neq i,2k+3-i}}^{2k+2} A_p q^{pN}.$$

We then use

$$\sum_{N=0}^{\infty} \frac{q^{pN}}{\left(q^{2k+3};q^{2k+3}\right)_N} = \frac{1}{\left(q^p;q^{2k+3}\right)_\infty}.$$

to complete the proof. ■

Note that Theorem 4.1 for $k=1$ and $i=2$ is (3.1).

Speculation 4.2. *The 1 row and at most $2k$ rows cases are sums of products in the Rogers–Ramanujan infinite product. Perhaps this works for any number of rows $R\le 2k$. Is*

$$\sum_{\lambda \text{ at most } R \text{ rows}} s_\lambda(x_1,\dots,x_n,\dots)s_\lambda(y_1,\dots,y_{2k})$$

a sum of a product of R infinite products, each of the form, $1/\left(q^j;q^{2k+3}\right)_\infty$, $j\not\equiv \pm i$, $0 \bmod 2k+3$, with coefficients which are rational functions in q?

Note that Speculation 4.2 holds for $R=1$ and $R=2k$.

5 Other symmetric function Cauchy identities

Michael Schlosser has pointed out that the dual Cauchy identity

$$\sum_{\lambda} s_{\lambda}(x_1,\dots,x_n)s_{\lambda'}(y_1,\dots,y_m)=\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j)$$

can be similarly used with $m=2$ and $(x_1,x_2,\dots,x_n)=\left(1,q^3,\dots,q^{3(n-1)}\right)$, $(y_1,y_2)=\left(-q,-q^2\right)$ to obtain the Borwein product $\left(q^1;q^3\right)_n\left(q^2,q^3\right)_n$. Again column strict tableaux could be used to approach that problem, see [1].

There are other Cauchy identities. If x and y are arbitrary sets of variables, the Macdonald polynomials satisfy

$$\sum_{\lambda} P_{\lambda}(x;q,t)Q_{\lambda}(y;q,t)=\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{5.1}$$

Special cases of (5.1), restricted by rows, have been extensively used by Rains and S. Ole Warnaar [9].

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References

[1] Andrews G.E., On a conjecture of Peter Borwein, *J. Symbolic Comput.* **20** (1995), 487–501.

[2] Andrews G.E., The theory of partitions, *Cambridge Math. Lib.*, Cambridge University Press, Cambridge, 1998.

[3] Garsia A.M., Milne S.C., A Rogers–Ramanujan bijection, *J. Combin. Theory Ser. A* **31** (1981), 289–339.

[4] Griffin M.J., Ono K., Warnaar S.O., A framework of Rogers–Ramanujan identities and their arithmetic properties, *Duke Math. J.* **165** (2016), 1475–1527, arXiv:1401.7718.

[5] Jouhet F., Zeng J., New identities for Hall–Littlewood polynomials and applications, *Ramanujan J.* **10** (2005), 89–112, arXiv:math.CO/0110165.

[6] Lepowsky J., Milne S., Lie algebraic approaches to classical partition identities, *Adv. Math.* **29** (1978), 15–59.

[7] Lepowsky J., Milne S., Lie algebras and classical partition identities, *Proc. Nat. Acad. Sci. USA* **75** (1978), 578–579.

[8] Macdonald I.G., Symmetric functions and Hall polynomials, 2nd ed., *Oxf. Class. Texts Phys. Sci.*, The Clarendon Press, New York, 2015.

[9] Rains E., Warnaar S.O., Bounded Littlewood identities, *Mem. Amer. Math. Soc.* **270** (2021), vii+115 pages, arXiv:1506.02755.

[10] Stembridge J.R., Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities, *Trans. Amer. Math. Soc.* **319** (1990), 469–498.

[11] Warnaar S.O., Hall–Littlewood functions and the A_2 Rogers–Ramanujan identities, *Adv. Math.* **200** (2006), 403–434, arXiv:math.CO/0410592.