

Some Spherical Function Values for Hook Tableaux Isotypes and Young Subgroups

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Abstract. A Young subgroup of the symmetric group \mathcal{S}_N , the permutation group of $\{1, 2, \dots, N\}$, is generated by a subset of the adjacent transpositions $\{(i, i+1) \mid 1 \leq i < N\}$. Such a group is realized as the stabilizer G_n of a monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$ with $\lambda = (d_1^{n_1}, d_2^{n_2}, \dots, d_p^{n_p})$ (meaning d_j is repeated n_j times, $1 \leq j \leq p$, and $d_1 > d_2 > \cdots > d_p \geq 0$), thus is isomorphic to the direct product $\mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \cdots \times \mathcal{S}_{n_p}$. The interval $\{1, 2, \dots, N\}$ is a union of disjoint sets $I_j = \{i \mid \lambda_i = d_j\}$. The orbit of x^λ under the action of \mathcal{S}_N (by permutation of coordinates) spans a module V_λ , the representation induced from the identity representation of G_n . The space V_λ decomposes into a direct sum of irreducible \mathcal{S}_N -modules. The spherical function is defined for each of these, it is the character of the module averaged over the group G_n . This paper concerns the value of certain spherical functions evaluated at a cycle which has no more than one entry in each interval I_j . These values appear in the study of eigenvalues of the Heckman–Polychronakos operators in the paper by V. Gorin and the author [arXiv:2412:01938]. In particular, the present paper determines the spherical function value for \mathcal{S}_N -modules of hook tableau type, corresponding to Young tableaux of shape $[N - b, 1^b]$.

Key words: spherical functions; subgroups of the symmetric group; hook tableaux; alternating polynomials

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*To the memory of G. de B. Robinson (1906–1992)
my first algebra professor*

1 Introduction

There is a commutative family of differential-difference operators acting on polynomials in N variables whose symmetric eigenfunctions are Jack polynomials. They are called Heckman–Polychronakos operators, defined by $\mathcal{P}_k := \sum_{i=1}^N (x_i \mathcal{D}_i)^k$, $k = 1, 2, \dots$, in terms of Dunkl operators

$$\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1, j \neq i}^N \frac{f(x) - f(x(i, j))}{x_i - x_j};$$

$x(i, j)$ denotes x with x_i and x_j interchanged, and κ is a fixed parameter, often satisfying $\kappa > -\frac{1}{N}$ (see Heckman [4], Polychronakos [6]; these citations motivated the name given the operators in [3]). The symmetric group on N objects, that is, the permutation group of $\{1, 2, \dots, N\}$, is denoted by \mathcal{S}_N and acts on $\mathbb{R}[x_1, \dots, x_N]$ by permutation of the variables. Specifically, for a polynomial $f(x)$ and $w \in \mathcal{S}_N$ the action is $wf(x) = f(xw)$, $(xw)_i = x_{w(i)}$, $1 \leq i \leq N$. This is a representation of \mathcal{S}_N . The operators \mathcal{P}_k commute with this action and thus the structure

of eigenfunctions and eigenvalues is strongly connected to the decomposition of the space of polynomials into irreducible \mathcal{S}_N -modules. The latter are indexed by partitions of N , that is, $\tau = (\tau_1, \dots, \tau_\ell)$ with $\tau_i \in \mathbb{N}$, $\tau_1 \geq \tau_2 \geq \dots \geq \tau_\ell > 0$ and $\sum_{i=1}^{\ell} \tau_i = N$. The corresponding module is spanned by the standard Young tableaux of shape τ . The general details are not needed here. The types of polynomial modules of interest here are spans of certain monomials: for $\alpha \in \mathbb{Z}_+^N$, let $x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}$. Suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, then set $V_\lambda = \text{span}_{\mathbb{F}}\{x^\beta \mid \beta = w\lambda, w \in \mathcal{S}_N\}$, that is, β ranges over the permutations of λ , and \mathbb{F} is an extension field of \mathbb{R} containing at least κ . The space V_λ is invariant under the action of \mathcal{S}_N . The eigenvector analysis of \mathcal{P}_k is based on the triangular decomposition $\mathcal{P}_k V_\lambda \subset V_\lambda \oplus \sum_{\nu \prec \lambda} \oplus V_\nu$, where $\nu \prec \lambda$ is the dominance order, $\sum_{i=1}^j \nu_i \leq \sum_{i=1}^j \lambda_i$ for $1 \leq j \leq N$, and $\sum_{i=1}^N \nu_i = \sum_{i=1}^N \lambda_i$. Part of the analysis is to identify irreducible \mathcal{S}_N -submodules of V_λ . This depends on the number of repetitions of values among $\{\lambda_i \mid 1 \leq i \leq N\}$. To be precise let $\lambda = (d_1^{n_1}, d_2^{n_2}, \dots, d_p^{n_p})$ (that is, d_j is repeated n_j times, $1 \leq j \leq p$), with $d_1 > d_2 > \dots > d_p \geq 0$ and $N = \sum_{i=1}^p n_i$. Let $G_{\mathbf{n}}$ denote the stabilizer group of x^λ , so that $G_{\mathbf{n}} \cong \mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \dots \times \mathcal{S}_{n_p}$. The representation of \mathcal{S}_N realized on V_λ is the induced representation $\text{ind}_{G_{\mathbf{n}}}^{\mathcal{S}_N}$. This decomposes into irreducible \mathcal{S}_N -modules and the number of copies (the multiplicity) of a particular isotype τ in V_λ is called a *Kostka* number (see Macdonald [5, p. 101]).

The operator \mathcal{P}_k arose in the study of the Calogero–Sutherland quantum system of N identical particles on a circle with inverse-square distance potential: the Hamiltonian is

$$\mathcal{H} = - \sum_{j=1}^N \left(\frac{\partial}{\partial \theta_j} \right)^2 + \frac{\kappa(\kappa-1)}{2} \sum_{1 \leq i < j \leq N} \frac{1}{\sin^2(\frac{1}{2}(\theta_i - \theta_j))};$$

the particles are at $\theta_1, \dots, \theta_N$ and the chordal distance between two points is $|2 \sin(\frac{1}{2}(\theta_i - \theta_j))|$. By changing variables $x_j = \exp i\theta_j$, the Hamiltonian is transformed to

$$\mathcal{H} = \sum_{j=1}^N \left(x_j \frac{\partial}{\partial x_j} \right)^2 - 2\kappa(\kappa-1) \sum_{1 \leq i < j \leq N} \frac{x_i x_j}{(x_i - x_j)^2}$$

(for more details see Chalykh [2, p. 16]).

In [3], Gorin and the author studied the eigenvalues of the operators \mathcal{P}_k restricted to submodules of V_λ of given isotype τ . It turned out that if the multiplicity of the isotype τ in V_λ is greater than one, then the eigenvalues are not rational in the parameters and do not seem to allow explicit formulation. However, the sum of all the eigenvalues (for any fixed k) can be explicitly found, in terms of the character of τ . In general, this may not have a relatively simple form but there are cases allowing a closed form. The present paper carries this out for hook isotypes, labeled by partitions of the form $[N - b, 1^b]$ (the Young diagram has a hook shape). The formula for the sum is quite complicated with a number of ingredients. For given (n_1, \dots, n_p) define the intervals associated with λ , $I_j = [\sum_{i=1}^{j-1} n_i + 1, \sum_{i=1}^j n_i]$ for $1 \leq j \leq p$ (notation $[a, b] := \{a, a+1, \dots, b\} \subset \mathbb{N}$). The formula is based on considering cycles $g_{\mathcal{A}}$ corresponding to subsets $\mathcal{A} = \{a_1, \dots, a_\ell\}$ of $[1, p]$, which are of length ℓ with exactly one entry from each interval I_{a_j} . Any such cycle can be used and the order of a_1, \dots, a_ℓ does not matter. The degrees d_1, \dots, d_p enter the formula in a shifted way:

$$\tilde{d}_i := d_i + \kappa(n_{i+1} + n_{i+2} + \dots + n_p), \quad 1 \leq i \leq p.$$

Let $h_m^{\mathcal{A}} := h_m(\tilde{d}_{a_1}, \tilde{d}_{a_2}, \dots, \tilde{d}_{a_\ell})$, the complete symmetric polynomial of degree m (the generating function is $\sum_{k \geq 0} h_k(c_1, c_2, \dots, c_q) t^k = \prod_{i=1}^q (1 - c_i t)^{-1}$, see [5, p. 21]). In [3], we used an ‘‘averaged character’’ (spherical function, in the present paper). Denote the character of the representation τ of \mathcal{S}_N by χ^τ , then

$$\chi^\tau[\mathcal{A}; \mathbf{n}] := \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \chi^\tau(g_{\mathcal{A}} h),$$

where $g_{\mathcal{A}}$ is an ℓ -cycle labeled by \mathcal{A} as above, and $\#G_{\mathbf{n}} = \prod_{i=1}^p n_i!$. In formula (1.1), the inner sum is over k -subsets of $[1, p]$, the list of labels of the partition $\{I_1, \dots, I_p\}$, so a typical subset is $\{a_1, a_2, \dots, a_k\}$, $g_{\mathcal{A}}$ is a cycle with one entry in each I_{a_i} and $h_{k+1-\ell}^{\mathcal{A}} = h_{k+1-\ell}(\tilde{d}_{a_1}, \tilde{d}_{a_2}, \dots, \tilde{d}_{a_\ell})$.

Now suppose the multiplicity of τ in V_λ is μ , then there are $\mu \dim \tau$ eigenfunctions and eigenvalues of \mathcal{P}_k , and the sum of all these eigenvalues is [3, Theorem 5.4]

$$\dim \tau \sum_{\ell=1}^{\min(k+1,p)} (-\kappa)^{\ell-1} \sum_{\mathcal{A} \subset [1,p], \#\mathcal{A}=\ell} \chi^\tau[\mathcal{A}; \mathbf{n}] h_{k+1-\ell}^{\mathcal{A}} \prod_{i \in \mathcal{A}} n_i!. \quad (1.1)$$

The main result of the present paper is to establish an explicit formula for $\chi^\tau[\mathcal{A}; \mathbf{n}]$ with $\tau = [N - b, 1^b]$. Since the order of the factors of $G_{\mathbf{n}}$ in the character calculation does not matter (by the conjugate invariance of characters), it will suffice to take $\mathcal{A} = \{1, 2, \dots, \ell\}$, for $2 \leq \ell \leq p$. In the following, e_i denotes the elementary symmetric polynomial of degree i and the Pochhammer symbol is $(a)_n = \prod_{i=1}^n (a+i-1)$. The combined main results are the following.

Theorem 1.1. *Let $m := p - b - 1$, $\ell \leq p$ and $\mathcal{A} = \{1, 2, \dots, \ell\}$, then*

$$\begin{aligned} \chi^\tau[\mathcal{A}; \mathbf{n}] &= \frac{1}{\prod_{i=1}^{\ell} n_i} \\ &\times \left\{ \sum_{k=0}^{\min(m,\ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_1-1, n_2-1, \dots, n_\ell-1) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\} \\ &= \binom{b+m}{b} + \sum_{i=1}^{\min(b,\ell-1)} (-1)^i \binom{b+m-i}{b-i} e_i \left(\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_\ell} \right). \end{aligned} \quad (1.2)$$

These results come from Theorems 5.9, 5.19 and Proposition 6.1. There are no nonzero $G_{\mathbf{n}}$ -invariants if $m < 0$ (as will be seen).

In Section 2, we present general background on spherical functions, harmonic analysis, and subgroup invariants for finite groups. Section 3 concerns alternating polynomials, which span a module of isotype $[N - b, 1^b]$. There is the definition of sums of alternating polynomials which make up a basis for $G_{\mathbf{n}}$ -invariants. The main results, proving formula (1.2) for the case $p = b + 1$ are in Section 4, and for the cases $p \geq b + 2$ are in Section 5, with subsections for $p = b + 2$ and $p > b + 2$. The details are of increasing technicality. Section 6 deduces formula (1.3) from (1.2). Some specializations of the formulas are discussed.

2 Spherical functions

Suppose the representation τ of \mathcal{S}_N is realized on a linear space V furnished with an \mathcal{S}_N -invariant inner product, then there is an orthonormal basis for V in which the restriction to $G_{\mathbf{n}}$ decomposes as a direct sum of irreducible representations of $G_{\mathbf{n}}$. Suppose the multiplicity of $1_{G_{\mathbf{n}}}$ is μ , and the basis is chosen so that for $h \in G_{\mathbf{n}}$

$$\tau(h) = \begin{bmatrix} I_\mu & O & \dots & O \\ O & T^{(1)}(h) & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & T^{(r)}(h) \end{bmatrix},$$

where $T^{(1)}, \dots, T^{(r)}$ are irreducible representations of $G_{\mathbf{n}}$ not equivalent to $1_{G_{\mathbf{n}}}$ (see [1, Sections 3.6 and 10] for details). For $g \in \mathcal{S}_N$, denote the matrix of $\tau(g)$ with respect to the basis

by $\tau_{i,j}(g)$; since $\tau(h_1gh_2) = \tau(h_1)\tau(g)\tau(h_2)$, we find

$$\frac{1}{(\#G_{\mathbf{n}})^2} \sum_{h_1, h_2 \in G_{\mathbf{n}}} \tau_{i,j}(h_1gh_2) = \begin{cases} \tau_{i,j}(g), & 1 \leq i, j \leq \mu, \\ 0, & \text{else.} \end{cases}$$

Then $\{\tau_{i,j} \mid 1 \leq i, j \leq \mu\}$ is a basis for the $G_{\mathbf{n}}$ -bi-invariant elements of $\text{span}\{\tau_{k\ell}\}$. The spherical function for the isotype τ and subgroup $G_{\mathbf{n}}$ is defined by

$$\Phi^\tau(g) := \sum_{i=1}^{\mu} \tau_{ii}(g), \quad g \in \mathcal{S}_N$$

(our notation for $\chi^\tau[\mathcal{A}; \mathbf{n}]$). Sometimes the term ‘‘spherical function’’ is reserved for Gelfand pairs where the multiplicity $\mu = 1$. The character of τ is $\chi^\tau(g) = \text{tr}(\tau(g))$, then

$$\Phi^\tau(g) = \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \chi^\tau(hg).$$

The symmetrization operator acting on V is ($\zeta \in V$)

$$\rho\zeta := \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \tau(h)\zeta.$$

The operator ρ is a self-adjoint projection.

Suppose there is an orthogonal subbasis $\{\psi_i \mid 1 \leq i \leq \mu\}$ for V which satisfies $\tau(h)\psi_i = \psi_i$ for $h \in G_{\mathbf{n}}$ (thus $\rho\psi_i = \psi_i$) and $1 \leq i \leq \mu$, then the matrix element $\tau_{i,i}(g) = \langle \tau(g)\psi_i, \psi_i \rangle / \langle \psi_i, \psi_i \rangle$ and the spherical function $\Phi^\tau(g) = \sum_{i=1}^{\mu} \frac{1}{\langle \psi_i, \psi_i \rangle} \langle \tau(g)\psi_i, \psi_i \rangle$. We will produce a formula for $\Phi^\tau(g)$ which works with a non-orthogonal basis of $G_{\mathbf{n}}$ -invariant vectors $\{\xi_i \mid 1 \leq i \leq \mu\}$ in V . The Gram matrix M is given by $M_{ij} := \langle \xi_i, \xi_j \rangle$. For $g \in \mathcal{S}_N$, let $T(g)_{ij} := \langle \tau(g)\xi_j, \xi_i \rangle$.

Lemma 2.1. $\Phi^\tau(g) = \text{tr}(T(g)M^{-1})$.

Proof. Suppose $\{\zeta_i \mid 1 \leq i \leq \mu\}$ is an orthonormal basis for the $G_{\mathbf{n}}$ -invariant vectors in V , then there is a (change of basis) matrix $[A_{ij}]$ such that $\zeta_i = \sum_{j=1}^{\mu} A_{ji}\xi_j$ and

$$\begin{aligned} \langle \tau(g)\zeta_i, \zeta_i \rangle &= \left\langle \sum_{j=1}^{\mu} \tau(g)A_{ji}\xi_j, \sum_{k=1}^{\mu} A_{ki}\xi_k \right\rangle = \sum_{j,k} A_{ji}A_{ki}T(g)_{kj}, \\ \Phi^\tau(g) &= \sum_i \sum_{j,k} A_{ji}A_{ki}T(g)_{kj} = \sum_{j,k} (AA^*)_{jk}T(g)_{kj} = \text{tr}((AA^*)T(g)). \end{aligned}$$

Furthermore,

$$\delta_{ij} = \langle \zeta_i, \zeta_j \rangle = \sum_{k,r} A_{ki}A_{rj} \langle \xi_k, \xi_r \rangle = \sum_{k,r} A_{ki}A_{rj}M_{kr} = (A^*MA)_{ij}$$

and $A^*MA = I$, $M = (A^*)^{-1}A^{-1} = (AA^*)^{-1}$. ■

Corollary 2.2. Suppose $\rho\tau(g)\xi_i = \sum_{j=1}^{\mu} B_{ji}(g)\xi_j$, $1 \leq i, j \leq \mu$, then $\Phi^\tau(g) = \text{tr}(B(g))$.

Proof. The expansion holds because $\{\xi_i \mid 1 \leq i \leq \mu\}$ is a basis for the invariants. Then

$$T(g)_{ij} = \langle \tau(g)\xi_j, \xi_i \rangle = \langle \rho\tau(g)\xi_j, \xi_i \rangle = \left\langle \sum_{k=1}^{\mu} B_{kj}(g)\xi_k, \xi_i \right\rangle = \sum_{k=1}^{\mu} B_{kj}(g)M_{ki} = (M^T B(g))_{ij}$$

and $\text{tr}(T(g)M^{-1}) = \text{tr}(M^T B(g)M^{-1}) = \text{tr}(B(g))$ (note $M^T = M$). ■

We will use the method provided by the Corollary to determine $\Phi^\tau(g)$. This formula avoids computing $T(g)$ and the inverse M^{-1} of the Gram matrix. When the multiplicity $\mu = 1$, the formula simplifies considerably: there is one invariant ψ_1 , $T(g) = \langle g\psi_1, \psi_1 \rangle$ and $M = [\langle \psi_1, \psi_1 \rangle]$ so that $\Phi^\tau(g) = \frac{\langle g\psi_1, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}$ (or $= c$ if $\rho g\psi_1 = c\psi_1$). We are concerned with computing the spherical function at a cycle g of length ℓ with no more than one entry from each interval I_j , where the factor \mathcal{S}_{n_j} acts only on I_j . The idea is to specify the $G_{\mathbf{n}}$ -invariant polynomials ξ , the effect of an ℓ -cycle on each of these, and then compute the expansion of $\rho g\xi$ in the invariant basis.

3 Coordinate systems and invariant sums of alternating polynomials

To clearly display the action of $G_{\mathbf{n}}$, we introduce a modified coordinate system. Replace

$$(x_1, x_2, \dots, x_N) = (x_1^{(1)}, \dots, x_{n_1}^{(1)}, x_1^{(2)}, \dots, x_{n_2}^{(2)}, \dots, x_1^{(p)}, \dots, x_{n_p}^{(p)}),$$

that is, $x_i^{(j)}$ stands for x_s with $s = \sum_{i=1}^{j-1} n_i + i$. We use $x_*^{(i)}$, $x_{>}^{(i)}$ to denote a generic $x_j^{(i)}$ with $1 \leq j \leq n_i$, respectively, $2 \leq j \leq n_i$. In the sequel, g denotes the cycle $(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(\ell)})$. Throughout, denote $m := p - b - 1$.

Notation 3.1. For $0 \leq j \leq \ell$, denote the elementary symmetric polynomial of degree j in the variables $n_1 - 1, n_2 - 1, \dots, n_\ell - 1$ by $e_j(n_* - 1)$. Set $\pi_\ell := \prod_{i=1}^\ell n_i$ and $\pi_p := \prod_{i=1}^p n_i$. For integers $i \leq j$, the interval $\{i, i+1, \dots, j\} \subset \mathbb{N}$ is denoted by $[i, j]$.

Definition 3.2. The action of the symmetric group \mathcal{S}_N on polynomials $P(x)$ is given by $wP(x) = P(xw)$ and $(xw)_i = x_{w(i)}$, $w \in \mathcal{S}_N$, $1 \leq i \leq N$.

Note $(x(vw))_i = (xv)_{w(i)} = x_{v(w(i))} = x_{vw(i)}$, $vwP(x) = (wP)(xv) = P(xvw)$. The projection onto $G_{\mathbf{n}}$ -invariant polynomials is given by

$$\rho P(x) = \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} P(xh).$$

We use the polynomial module of isotype $[N - b, 1^b]$ with the lowest degree. This module is spanned by alternating polynomials in $b + 1$ variables.

Definition 3.3. For $x_{i_1}, x_{i_2}, \dots, x_{i_{b+1}}$, let

$$\Delta(x_{i_1}, x_{i_2}, \dots, x_{i_{b+1}}) := \prod_{1 \leq j < k \leq b+1} (x_{i_j} - x_{i_k}).$$

Lemma 3.4. Suppose x_1, \dots, x_{b+2} are arbitrary variables and $f_j := \Delta(x_1, x_2, \dots, \widehat{x_j}, \dots, x_{b+2})$ denotes the alternating polynomial when x_j is removed from the list, then $\sum_{j=1}^{b+2} (-1)^j f_j = 0$.

Proof. Let $F(x) := \sum_{j=1}^{b+2} (-1)^j f_j$. Suppose $1 \leq i < b + 2$ and $(i, i + 1)$ is the transposition of x_i and x_{i+1} , then $(i, i + 1)f_j = -f_j$ if $j \neq i, i + 1$, $(i, i + 1)f_i = f_{i+1}$ and $(i, i + 1)f_{i+1} = f_i$. Thus $(i, i + 1)F(x) = -F(x)$ and this implies $F(x)$ is divisible by the alternating polynomial in x_1, \dots, x_{b+2} which is of degree $\frac{1}{2}(b + 2)(b + 1)$, but F is of degree $\leq \frac{1}{2}b(b + 1)$ and hence $F(x) = 0$. ■

We briefly discuss the relation between hook tableaux and alternating polynomials. The irreducible representation of \mathcal{S}_N corresponding to the partition $[N - b, 1^b]$ has the basis of standard Young tableaux of this shape. For a given tableau T and an entry i (with $1 \leq i \leq N$) the content

$c(i, T) := \text{col}(i, T) - \text{row}(i, T)$ (the labels of the column, row of T containing i). The Jucys–Murphy elements $\omega_j := \sum_{i=1}^{j-1} (i, j)$, $1 \leq j \leq N$, mutually commute and satisfy $\omega_j T = c(j, T)T$; this is the defining property of the representation. There is a general identity $\omega_{j+1} = \sigma_j \omega_j \sigma_j + \sigma_j$ where $\sigma_j := (j, j+1)$ for $1 \leq j < N$. The tableau T_0 with first column containing $\{1, 2, \dots, b+1\}$ satisfies $c(i, T_0) = 1 - i$ for $1 \leq i \leq b+1$ and $= i - b - 1$ for $b+2 \leq i \leq N$. We use the notation of Lemma 3.4 so that $f_{b+2} := \Delta(x_1, \dots, x_{b+1})$.

Proposition 3.5. $\omega_i f_{b+2} = c(i, T_0) f_{b+2}$ for $1 \leq i \leq N$.

Proof. Since $(i, j) f_{b+2} = -f_{b+2}$ for $1 \leq i < j \leq b+1$, it follows that $\omega_i f_{b+2} = -(i-1) f_{b+2}$ for $1 < i \leq b+1$, while $\omega_1 = 0$ implies $\omega_1 f_{b+2} = 0 = c(1, T_0) f_{b+2}$. The claim $\omega_{b+2} f_{b+2} = f_{b+2}$ needs more detail. Consider the term for $(i, b+2)$ in $\omega_{b+2} f_{b+2}$ (for $1 \leq i \leq b+1$)

$$(i, b+2) \Delta(x_1, \dots, x_{b+1}) = \Delta(x_1, \dots, x_{b+2}^{(i)}, \dots, x_{b+1}) = (-1)^{b+1-i} f_i,$$

the sign comes from moving x_{b+2} by $b+1-i$ adjacent transpositions to the last argument of $\Delta(\ast)$. Thus

$$\sum_{i=1}^{b+1} (i, b+2) f_{b+2} = \sum_{i=1}^{b+1} (-1)^{b+1-i} f_i = f_{b+2}$$

by the lemma (multiply each term by $(-1)^{b+1}$). Now let $i \geq b+2$ and suppose $\omega_i f_{b+2} = (i-b-1) f_{b+2}$, then $\omega_{i+1} f_{b+2} = (\sigma_i \omega_i \sigma_i + \sigma_i) f_{b+2} = (\omega_i + 1) f_{b+2} = (i-b) f_{b+2}$. By induction, $\omega_i f_{b+2} = (i-b-1) f_{b+2}$ for $b+2 \leq i \leq N$ and this completes the proof. \blacksquare

The \mathcal{S}_N -module spanned by f_{b+2} is of isotype $[N-b, 1^b]$.

Usually \mathbf{x} denotes a $(b+1)$ -tuple as a generic argument of Δ .

Definition 3.6. Suppose $\mathbf{x} = (x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)}, \dots, x_{i_{b+1}}^{(j_{b+1})})$, then

$$\mathcal{L}(\mathbf{x}) := (j_1, j_2, \dots, j_{b+1}) \quad \text{and} \quad \Delta(\mathbf{x}) := \Delta(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)}, \dots, x_{i_{b+1}}^{(j_{b+1})}).$$

The arguments in $\mathcal{L}(\mathbf{x})$ can be assumed to be in increasing order, up to a change in sign of $\Delta(\mathbf{x})$ (for example, if σ is a transposition, then $\Delta(\mathbf{x}\sigma) = -\Delta(\mathbf{x})$).

Proposition 3.7. If $h \in G_{\mathbf{n}}$, then $\mathcal{L}(\mathbf{x}h) = \mathcal{L}(\mathbf{x})$, and

$$\rho \Delta(\mathbf{x}) = \prod_{r=1}^{b+1} n_{j_r}^{-1} \sum \{ \Delta(\mathbf{y}) \mid \mathcal{L}(\mathbf{y}) = \mathcal{L}(\mathbf{x}) \}.$$

Proof. The second statement follows from the multiplicative property of ρ and from

$$\frac{1}{n_j!} \sum_{h \in \mathcal{S}_{n_j}} f(x_*^{(j)} h) = \frac{1}{n_j} \sum_{i=1}^{n_j} f(x_i^{(j)}),$$

where $x_*^{(j)}$ denotes an arbitrary $x_i^{(j)}$, and $(n_j-1)!$ elements h fix $x_i^{(j)}$. \blacksquare

It follows from Lemma 3.4 that a basis for the $G_{\mathbf{n}}$ -invariant polynomials is generated from $\Delta(\mathbf{x})$ with $\mathbf{x} = (x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)}, \dots, x_{i_{b+1}}^{(p)})$, that is, the last coordinate is in I_p .

In the following, we specify invariants by the indices omitted from $\mathcal{L}(\mathbf{x})$; this is actually more convenient.

Definition 3.8. Suppose $S \subset [1, p-1]$ with $\#S = m$, then define the invariant polynomial

$$\xi_S := \sum \{ \Delta(\mathbf{x}) \mid \mathbf{x} = (\dots, x_{i_j}^{(j)}, \dots, x_{i_p}^{(p)}), j \notin S \}.$$

That is, the coordinates of \mathbf{x} have b distinct indices from $[1, b+m] \setminus S$. The basis has cardinality $\mu = \binom{b+m}{b}$ (see [5, p. 105, Example 2(b)]). The underlying task is to compute the coefficient $B_{S,S}(g)$ in $\rho g \xi_S = \sum_{S'} B_{S',S}(g) \xi_{S'}$. This requires a decomposition of ξ_S .

Definition 3.9. Suppose $S \subset [1, p-1]$ with $\#S = m$ and $E \subset ([1, \ell] \cup \{p\}) \setminus S$ say $\mathbf{x} \in X_{S,E}$ if $j \in E$ implies $\mathbf{x}_j = x_1^{(j)}$, $j \in [\ell+1, p] \setminus S$ implies $\mathbf{x}_j = x_k^{(j)}$ with $1 \leq k \leq n_j$ and $j \in [1, \ell] \setminus (S \cup E)$ implies $\mathbf{x}_j = x_k^{(j)}$ with $2 \leq k \leq n_j$ (consider \mathbf{x} as a $(b+1)$ -tuple indexed by $[1, b+m] \setminus S \cup \{p\}$). Furthermore, let

$$\phi_{S,E} := \sum \{ \Delta(\mathbf{x}) \mid \mathbf{x} \in X_{S,E} \}.$$

Thus $\xi_S = \sum_E \phi_{S,E}$ and we will analyze $\rho g \phi_{S,E}$. It turns out only a small number of sets E allow $\rho g \phi_{S,E} \neq 0$, and an even smaller number have a nonzero coefficient $B_{S,(S,E)}$ in the expansion $\rho g \phi_{S,E} = \sum_{S'} B_{S',(S,E)} \xi_{S'}$, namely \emptyset (the empty set), $[1, \ell]$ and $[1, \min S - 1] \cup \{p\}$. Part of the discussion is to show this list is exhaustive.

Lemma 3.10. For $\rho \Delta(\mathbf{x}) \neq 0$, it is necessary that there be no repetitions in $\mathcal{L}(\mathbf{x})$.

Proof. Suppose $j_a = j_b = k$, and $\Delta(\dots, x_{i_a}^{(k)}, \dots, x_{i_b}^{(k)}, \dots)$ appears in the sum; we can assume $b = a + 1$ by rearranging the variables, possibly introducing a sign factor. If $i_a = i_b$, then $\Delta(\mathbf{x}) = 0$, else $\Delta(\dots, x_{i_b}^{(k)}, x_{i_a}^{(k)}, \dots)$ also appears, by the action of the transposition $(i_a, i_b) \in \mathcal{S}_{n_k}$. These two terms cancel out because $\Delta(\mathbf{x}(i_a, i_b)) = -\Delta(\mathbf{x})$. ■

If some \mathbf{x} has coordinates $x_1^{(i)}$, $x_{>}^{(i+1)}$ and $i < \ell$, then $\rho g \Delta(\mathbf{x}) = 0$ by the lemma since $\mathbf{x}g = (\dots, x_1^{(i+1)}, x_{>}^{(i+1)}, \dots)$. This strongly limits the sets E allowing $\rho \Delta(\mathbf{x}g) \neq 0$.

4 The case $p = b + 1$

This is the least complicated situation and introduces some techniques used later. Here $\mathcal{L}(\mathbf{x}) = (1, 2, \dots, b, b+1)$. There is just one $G_{\mathbf{n}}$ -invariant polynomial (up to scalar multiplication):

$$\psi := \sum \{ \Delta(x_{i_1}^{(1)}, x_{i_2}^{(2)}, \dots, x_{i_p}^{(b+1)}) \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, \dots, 1 \leq i_{b+1} \leq n_{b+1} \}.$$

In Definition 3.9, the set $S = \emptyset$ and we write ϕ_E for $\phi_{\emptyset,E}$.

Proposition 4.1. If $1 \leq \#E < \ell$, then $\rho g \phi_E = 0$.

Proof. Suppose there are indices $k, k+1$ with $k \in E$, $k < \ell$ and $k+1 \notin E$. If $\Delta(\mathbf{x})$ is one of the summands of ϕ_E , then $\mathbf{x} = (\dots, x_1^{(k)}, x_{>}^{(k+1)}, \dots)$ and $\mathbf{x}g = (\dots, x_1^{(k+1)}, x_{>}^{(k+1)}, \dots)$ and by Lemma 3.10 $\rho \Delta(\mathbf{x}) = 0$. Otherwise $k \in E$ implies $k+1 \in E$ or $k = \ell$ which by hypothesis implies $\ell \in E$ and $1 \notin E$, then $\mathbf{x} = (x_{>}^{(1)}, \dots, x_1^{(\ell)}, \dots)$, $\mathbf{x}g = (x_{>}^{(1)}, \dots, x_1^{(1)}, \dots)$ and $\rho \Delta(\mathbf{x}) = 0$ as before. ■

It remains to compute $\rho g \phi_E$ for $E = \emptyset$ and $E = [1, \ell]$. Note $g \phi_{\emptyset} = \phi_{\emptyset}$.

Suppose $\mathbf{x} \in X_{\emptyset, \emptyset}$, then $\mathbf{x} = (x_{>}^{(1)}, \dots, x_{>}^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+1)})$, and since $\rho \Delta(\mathbf{x}g) = \rho \Delta(\mathbf{x}) = \frac{1}{\pi_p} \psi$ (by Proposition 3.7) and $\#X_{\emptyset, \emptyset} = \prod_{i=1}^{\ell} (n_i - 1) \prod_{j=\ell+1}^{b+1} n_j$, it follows that

$$\rho \phi_{\emptyset, \emptyset} = \frac{1}{\pi_{\ell}} \prod_{i=1}^{\ell} (n_i - 1) = \frac{1}{\pi_{\ell}} e_{\ell}(n_* - 1).$$

Now suppose $E = [1, \ell]$ and $\mathbf{x} \in X_{\emptyset, [1, \ell]}$ implies $\mathbf{x} = (x_1^{(1)}, \dots, x_1^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+1)})$, then

$$\mathbf{x}g = (x_1^{(\ell)}, x_1^{(1)}, \dots, x_1^{(\ell-1)}, x_*^{(\ell+1)}, \dots, x_*^{(b+1)}).$$

Applying $\ell - 1$ transpositions $(\ell - 1, \ell), (\ell - 2, \ell - 1), \dots, (1, 2)$ transforms \mathbf{x} to $\mathbf{x}g$ and thus $\Delta(\mathbf{x}g) = (-1)^{\ell-1} \Delta(\mathbf{x})$. So

$$\rho \Delta(\mathbf{x}g) = (-1)^{\ell-1} \rho \Delta(\mathbf{x}) = (-1)^{\ell-1} \frac{1}{\pi_p} \psi.$$

Since $\#X_{\emptyset, [1, \ell]} = \prod_{i=\ell+1}^{b+1} n_i$, it follows that $\rho \phi_{B, [1, \ell]} = \frac{(-1)^{\ell-1}}{\pi_\ell}$.

Proposition 4.2. *Suppose $p = b + 1$ and $2 \leq \ell \leq b + 1$, then*

$$\Phi^\tau(g) = \frac{1}{\pi_\ell} \{e_\ell(n_* - 1) + (-1)^{\ell-1}\}.$$

Proof. $\rho g \psi = \rho g \phi_{\emptyset, \emptyset} + \rho g \phi_{\emptyset, [1, \ell]} = \frac{1}{\pi_\ell} \{e_\ell(n_* - 1) + (-1)^{\ell-1}\} \psi.$ ■

This is the main formula (1.2) specialized to $m = 0$.

5 The cases $p > b + 1$

There is some simplification for $p = b + 2$ compared to $p \geq b + 3$. First, we set up some tools.

Definition 5.1. For an invariant basis element ξ and a polynomial ϕ , let $\text{coef}(\xi, \rho\phi)$ denote the coefficient of ξ in the expansion of $\rho\phi$ in the basis.

The main object is to determine

$$\Phi^\tau(g) = \sum_S \text{coef}(\xi_S, \rho g \xi_S). \tag{5.1}$$

Proposition 5.2. *Suppose S and E are given by Definition 3.9, $\ell \leq b + m$, $\text{coef}(\xi_S, \rho g \phi_{S, E}) \neq 0$, then $E = \emptyset$ or $S \cap [1, \ell] = \emptyset$ and $E = [1, \ell]$.*

Proof. Let $vE := \{j \in E \mid j + 1 \notin E\}$, the upper end-points of E . If $j \in vE$ and $j + 1 \in [1, b + m] \setminus (S \cup E)$, then $\mathcal{L}(\mathbf{x}g) = (\dots, j + 1, j + 1, \dots)$ and $\rho g \Delta(\mathbf{x}) = 0$ (Proposition 3.10). Thus $\mathbf{x} \in X_{S, E}$, $\rho g \Delta(\mathbf{x}) \neq 0$ and $k \in vE$ implies $k + 1 \in S \cup \{\ell\}$. Suppose $j \in vE$, $j + 1 \in S$ and $j < \ell$, then $x_1^{(j+1)}$ appears in $g\phi_E$. That is, if $x \in X_{S, E}$, then $j + 1$ is not an entry of $\mathcal{L}(\mathbf{x})$ but $j + 1$ appears in $\mathcal{L}(\mathbf{x}g)$ and thus $\text{coef}(\xi_S, \rho g \phi_{S, E}) = 0$. Another possibility is that there exists $i \notin S \cup E$, $1 \leq i < \ell$ and $i + 1 \in E$, in which case $i + 1$ does not appear in $\mathcal{L}(\mathbf{x}g)$ and $\text{coef}(\xi_S, \rho g \phi_{S, E}) = 0$. ■

The case $\ell = b + m + 1$ involves more technicalities.

Informally, consider S as the set of holes in $\mathcal{L}(\mathbf{x})$; no new holes can be adjoined or removed from $\mathcal{L}(\mathbf{x}g)$ because this would imply $\text{coef}(\xi_S, \rho \Delta(\mathbf{x}g)) = 0$. And of course $\mathcal{L}(\mathbf{x}g)$ can have no repetitions. This is the idea that limits the possible boundary points of E (that is, $j \in E$ and $j + 1 \notin E$ or $j - 1 \notin E$).

Recall $\phi_{S, E} := \sum \{\Delta(\mathbf{x}) \mid \mathbf{x} \in X_{S, E}\}$, and the task is to determine $\text{coef}(\xi_S, \rho g \phi_{S, E})$.

5.1 Case $p = b + 2$

Here $m = 1$ so the sets S are singletons $\{i\}$ with $1 \leq i \leq b+1$. Note that $m = 1$ is an underlying hypothesis throughout this subsection. Write ξ_i in place of $\xi_{\{i\}}$. Then $\{\xi_i \mid 1 \leq i \leq b+1\}$ is a basis for the $G_{\mathbf{n}}$ -invariants. The possibilities for E are \emptyset for any i , $[1, \ell]$ for $i > \ell$, and $[1, i-1] \cup \{b+2\}$ for $\ell = b+2$. Suppose $E = \emptyset$, then $\mathbf{x} \in X_{i, \emptyset}$ implies

$$\mathbf{x} = (x_{>}^{(1)}, \dots, x_{>}^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+2)})$$

with $x_{>}^{(i)}, x_*^{(i)}$ omitted if $i \leq \ell$ or $i > \ell$, respectively. From Proposition 3.7,

$$\rho\Delta(\mathbf{x}g) = \rho\Delta(\mathbf{x}) = \frac{n_i}{\pi_p} \xi_i.$$

Also $\#X_{i, \emptyset} = \prod_{j=1, j \neq i}^{\ell} (n_j - 1) \prod_{k=\ell+1}^{b+2} n_k$, or $\prod_{j=1}^{\ell} (n_j - 1) \prod_{k=\ell+1, k \neq i}^{b+2} n_k$, if $i \leq \ell$ or $i > \ell$, respectively.

Proposition 5.3. *Suppose $2 \leq \ell \leq b+1$ and $i > \ell$, then $\text{coef}(\xi_i, \rho\phi_{i, \emptyset}) = \frac{1}{\pi_{\ell}} e_{\ell}(n_* - 1)$.*

Proof. Multiply $\#X_{i, \emptyset}$ by $\frac{n_i}{\pi_p}$ with result $\frac{1}{\pi_{\ell}} \prod_{j=1}^{\ell} (n_j - 1)$. ■

Proposition 5.4. *Suppose $2 \leq \ell \leq b+2$ and $i \leq \ell$, then $\text{coef}(\xi_i, \rho g\phi_{i, \emptyset}) = \frac{1}{\pi_{\ell}} e_{\ell}(n_* - 1) \left(1 + \frac{1}{n_i - 1}\right)$.*

Proof. Multiply $\#X_{i, \emptyset}$ by $\frac{n_i}{\pi_p}$ with result

$$\prod_{j=1, j \neq i}^{\ell} \frac{n_j - 1}{n_j} = \left(\prod_{j=1}^{\ell} \frac{n_j - 1}{n_j} \right) \left(\frac{n_i}{n_i - 1} \right) = \frac{1}{\pi_{\ell}} \prod_{j=1}^{\ell} (n_j - 1) \left(1 + \frac{1}{n_i - 1}\right). \quad \blacksquare$$

Proposition 5.5. *Suppose $2 \leq \ell \leq b+2$ and $\ell < i$, then $\text{coef}(\xi_i, \rho g\phi_{i, [1, \ell]}) = \frac{1}{\pi_{\ell}} (-1)^{\ell-1}$.*

Proof. If $\mathbf{x} \in X_{i, [1, \ell]}$, then $\mathbf{x} = (x_1^{(1)}, \dots, x_1^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+2)})$ omitting $x_*^{(i)}$ and $\mathcal{L}(\mathbf{x}g) = (2, 3, \dots, \ell, 1, \ell+1, \dots, i-1, i+1, \dots, b+2)$. Applying a product of $\ell-1$ transpositions shows that $\Delta(\mathbf{x}g) = (-1)^{\ell-1} \Delta(\mathbf{x})$ and $\rho\Delta(\mathbf{x}g) = (-1)^{\ell-1} \frac{n_i}{\pi_p} \xi_i$. Multiply $(-1)^{\ell-1} \frac{n_i}{\pi_p}$ by $\#X_{i, [1, \ell]} = \prod_{s=\ell+1, s \neq i}^{b+2} n_s$ to obtain $(-1)^{\ell-1} \prod_{s=1}^{\ell} n_s^{-1}$. ■

Theorem 5.6. *Suppose $2 \leq \ell \leq b+1$, then*

$$\Phi^{\tau}(g) = \frac{1}{\pi_{\ell}} \left\{ (b+1)e_{\ell}(n_* - 1) + e_{\ell-1}(n_* - 1) + (-1)^{\ell-1}(b - \ell + 1) \right\}.$$

Proof. Break up the sum (5.1) into $i > \ell$ and $i \leq \ell$ sums:

$$\begin{aligned} \sum_{i=\ell+1}^{b+1} \text{coef}(\xi_i, \rho g\xi_i) &= \sum_{i=\ell+1}^{b+1} (\text{coef}(\xi_i, \rho g\phi_{i, \emptyset}) + \text{coef}(\xi_i, \rho g\phi_{[1, \ell]})) \\ &= \frac{1}{\pi_{\ell}} (b+1 - \ell) (e_{\ell}(n_* - 1) + (-1)^{\ell-1}), \\ \sum_{i=1}^{\ell} \text{coef}(\xi_i, \rho g\xi_i) &= \sum_{i=1}^{\ell} \text{coef}(\xi_i, \rho g\phi_{i, \emptyset}) = \frac{1}{\pi_{\ell}} \sum_{i=1}^{\ell} e_{\ell}(n_* - 1) \left(1 + \frac{1}{n_i - 1}\right) \\ &= \frac{1}{\pi_{\ell}} (\ell e_{\ell}(n_* - 1) + e_{\ell-1}(n_* - 1)). \end{aligned}$$

Add the two parts together. ■

Proposition 5.7. *Suppose $\ell = b + 2$ and $1 \leq i \leq b + 1$, then for $E = [1, i - 1] \cup \{b + 2\}$*

$$\text{coef}(\xi_i, \rho g \phi_{i,E}) = \frac{1}{\pi_p} (-1)^{i-1} \prod_{s=i+1}^{b+1} (n_s - 1).$$

Proof. If $\mathbf{x} \in X_{i,E}$, then $\mathbf{x}g = (x_1^{(2)}, \dots, x_1^{(i)}, x_{>}^{(i+1)}, \dots, x_1^{(1)})$, $\mathcal{L}(\mathbf{x}g) = (2, \dots, i, i + 1, \dots, b + 1, 1)$ and $\Delta(\mathbf{x}g) = (-1)^b \Delta(\mathbf{y})$ with $\mathcal{L}(\mathbf{y}) = (1, 2, \dots, b + 1)$. Apply Lemma 3.4 to obtain

$$\sum_{j=1}^b (-1)^j \Delta(x_1, x_2, \dots, \widehat{x}_j, \dots, x_{b+2}) + (-1)^{b+2} \Delta(x_1, x_2, \dots, x_{b+1}) = 0$$

(the notation \widehat{x}_j means x_j is omitted). Use the term $j = i$ in the identity to obtain

$$\text{coef}(\xi_i, \rho g \Delta(\mathbf{y})) = \frac{n_i}{\pi_p} (-1)^{b+1-i}.$$

From $\#X_{i,E} = \prod_{s=i+1}^{b+1} (n_s - 1)$, it follows that

$$\text{coef}(\xi_i, \rho g \phi_{i,E}) = \frac{1}{\pi_p} (-1)^{i-1} \prod_{s=i+1}^{b+1} (n_s - 1). \quad \blacksquare$$

Proposition 5.8. *Suppose $\ell = b + 2$, then*

$$\sum_{i=1}^{b+1} \text{coef}(\xi_i, \rho g \phi_{i,[1,i-1] \cup \{p\}}) = \frac{1}{\pi_p} \left\{ \prod_{s=1}^{b+1} (n_s - 1) - (-1)^b \right\}.$$

Proof. The sum is

$$\begin{aligned} \frac{1}{\pi_p} \sum_{i=1}^{b+1} (-1)^{i-1} n_i \prod_{s=i+1}^{b+1} (n_s - 1) &= \frac{1}{\pi_p} \sum_{i=1}^{b+1} (-1)^{i-1} (n_i - 1 + 1) \prod_{s=i+1}^{b+1} (n_s - 1) \\ &= \frac{1}{\pi_p} \sum_{i=1}^{b+1} \left\{ (-1)^{i-1} \prod_{s=i}^{b+1} (n_s - 1) - (-1)^i \prod_{s=i+1}^{b+1} (n_s - 1) \right\} \\ &= \frac{1}{\pi_p} \prod_{s=1}^{b+1} (n_s - 1) - \frac{1}{\pi_p} (-1)^{b+1} \end{aligned}$$

by telescoping, leaving the first product with $i = 1$ and the last with $i = b + 1$. \blacksquare

Theorem 5.9. *Suppose $\ell = b + 2$, then*

$$\Phi^\tau(g) = \frac{1}{\pi_\ell} \{ (\ell - 1) e_\ell (n_* - 1) + e_{\ell-1} (n_* - 1) + (-1)^b \}.$$

Proof. Combine Propositions 5.4 and 5.8 (note $\pi_p = \pi_\ell$),

$$\begin{aligned} \sum_{i=1}^{b+1} \text{coef}(\xi_i, \rho g \xi_i) &= \frac{1}{\pi_\ell} \sum_{i=1}^{\ell-1} \prod_{j=1}^{\ell} (n_j - 1) \left(1 + \frac{1}{n_i - 1} \right) + \frac{1}{\pi_p} \prod_{s=1}^{\ell-1} (n_s - 1) - \frac{1}{\pi_p} (-1)^{b+1} \\ &= \frac{1}{\pi_\ell} \prod_{j=1}^{\ell} (n_j - 1) \left\{ \sum_{i=1}^{\ell-1} \left(1 + \frac{1}{n_i - 1} \right) + \frac{1}{n_\ell - 1} \right\} - \frac{1}{\pi_p} (-1)^{b+1} \\ &= \frac{1}{\pi_\ell} \{ (\ell - 1) e_\ell (n_* - 1) + e_{\ell-1} (n_* - 1) + (-1)^b \}. \quad \blacksquare \end{aligned}$$

Formula (1.2) with $m = 1$, $\ell = b + 2$ has the term $(-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} = (-1)^{\ell+2} = (-1)^b$. This completes the case $p = b + 2$.

5.2 Case $p > b + 2$

Write $p = b + m + 1$. Label the invariants by $S \subset [1, b + m]$, $\#S = m$,

$$\xi_S := \sum \left\{ \Delta(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)}, \dots, x_{i_b}^{(j_b)}, x_{i_p}^{(p)}) \mid \{j_1, \dots, j_b\} = [1, b + m] \setminus S, \right. \\ \left. 1 \leq i_s \leq n_{j_s}, 1 \leq s \leq b, 1 \leq i_p \leq n_p \right\},$$

and in $\Delta(\mathbf{x})$ take $j_1 < j_2 < \dots < j_b$. The following lemma generalizes the generating function for elementary symmetric polynomials.

Lemma 5.10. *Suppose y_1, y_2, \dots, y_r are variables and $q \leq r \leq s$, then*

$$\prod_{i=1}^r y_i \sum_{U \subset [1, s], \#U=q} \prod_{j \in U \cap [1, r]} \left(1 + \frac{1}{y_j}\right) = \sum_{k=0}^{\min(q, r)} \binom{s-k}{q-k} e_{r-k}(y_1, \dots, y_r).$$

Proof. The product

$$\prod_{j \in U \cap [1, r]} \left(1 + \frac{1}{y_j}\right) = \sum_{k=0}^q \sum \left\{ \prod_{j \in V} \left(\frac{1}{y_j}\right) \mid V \subset U \cap [1, r], \#V = k \right\}.$$

Any particular V with $\#V = k$ appears in $\binom{s-k}{q-k}$ different sets U . Then $e_k(y_1^{-1}, \dots, y_r^{-1})$ is a sum of $\prod_{j \in V} \left(\frac{1}{y_j}\right)$ over k -subsets of $[1, r]$, and thus the sum is $\sum_{k=0}^{\min(q, r)} \binom{s-k}{q-k} e_k(y_1^{-1}, \dots, y_r^{-1})$. Also

$$\left(\prod_{i=1}^r y_i \right) e_k(y_1^{-1}, \dots, y_r^{-1}) = e_{r-k}(y_1, \dots, y_r). \quad \blacksquare$$

The apparent singularity at $y_j = 0$ is removable.

Proposition 5.11. *If $S \subset [1, b + m]$, $\#S = m$ and $\ell \leq b + m + 1$, then*

$$\text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) = \prod \left\{ \frac{n_i - 1}{n_i} \mid 1 \leq i \leq \ell, i \notin S \right\}.$$

Proof. When $E = \emptyset$, then $\mathbf{x} \in X_{S, E}$ satisfies $\rho \Delta(\mathbf{x}g) = \rho \Delta(\mathbf{x}) = \left(\prod_{j=1, j \notin S}^{b+m+1} n_j^{-1} \right) \xi_S$. Furthermore, $\#X_{S, \emptyset} = \prod_{i=1, i \notin S}^{\ell} (n_i - 1) \times \prod_{j=\ell+1, j \notin S}^{b+m+1} n_j$ and the product of the two factors is

$$\prod_{i=1, i \notin S}^{\ell} \left(\frac{n_i - 1}{n_i} \right). \quad \blacksquare$$

Proposition 5.12. *For $\ell \leq b + m$,*

$$\sum_{S \subset [1, b+m], \#S=m} \text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) = \frac{1}{\pi_{\ell}} \sum_{k=0}^{\min(m, \ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_* - 1).$$

Proof. The sum equals

$$\sum_{S \subset [1, b+m], \#S=m} \prod_{s=1, s \notin S}^{\ell} \left(\frac{n_s - 1}{n_s} \right) = \frac{1}{\pi_{\ell}} \prod_{i=1}^{\ell} (n_i - 1) \sum_{S \subset [1, b+m], \#S=m} \prod_{j \in [1, \ell] \cap S} \left(\frac{n_j}{n_j - 1} \right) \\ = \frac{1}{\pi_{\ell}} \sum_{k=0}^{\min(m, \ell)} \binom{b+m-k}{m-k} e_{\ell-k}(n_* - 1)$$

by Lemma 5.10 with $r = \ell$, $s = b + m$, $q = m$ and $y_i = n_i - 1$. Also

$$\binom{b+m-k}{m-k} = \frac{(b+m-k)!}{b!(m-k)!} = \frac{(b+1)_{m-k}}{(m-k)!}. \quad \blacksquare$$

Proposition 5.13. *If $\ell \leq b$, $S \subset [\ell + 1, b + m]$, $\#S = m$, then $\text{coef}(\xi_S, \rho g \phi_{S, [1, \ell]}) = \frac{(-1)^{\ell+1}}{\pi_\ell}$.*

Proof. If $\mathbf{x} \in X_{S, [1, \ell]}$, then $\mathcal{L}(\mathbf{x}g) = (2, 3, \dots, \ell, 1, \ell + 1, \dots, b + m + 1)$ with $\{j \mid j \in S\}$ omitted. Thus

$$\Delta(\mathbf{x}g) = (-1)^{\ell-1} \Delta(\mathbf{x}) \quad \text{and} \quad \rho \Delta(\mathbf{x}g) = (-1)^{\ell-1} \prod_{i=1, i \notin S}^{b+m+1} n_i^{-1} \xi_S.$$

The number of summands in $\phi_{S, [1, \ell]}$ is

$$\#X_{S, [1, \ell]} = \prod_{s=\ell+1, s \notin S}^{b+m+1} n_s$$

and the required coefficient is the product with $(-1)^{\ell-1} \prod_{i=1, i \notin S}^{b+m+2} n_i^{-1}$, namely

$$(-1)^{\ell-1} \prod_{i=1}^{\ell} n_i^{-1}. \quad \blacksquare$$

Theorem 5.14. *Suppose $\ell \leq b + m$, then*

$$\Phi^\tau(g) = \frac{1}{\pi_\ell} \left\{ \sum_{k=0}^{\min(m, \ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_* - 1) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\}.$$

Proof. When $b < \ell \leq b + m$, then the sum (5.1) equals $\sum_S \text{coef}(\xi_S, \rho g \phi_{S, \emptyset})$, else if $2 \leq \ell \leq b$, then it equals

$$\sum_{S \subset [1, b+m]} \text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) + \sum_{S \subset [\ell+1, b+m]} \text{coef}(\xi_S, \rho g \phi_{S, [1, \ell]}).$$

There are $\binom{b+m-\ell}{m}$ subsets $S \subset [\ell+1, b+m]$. In both cases the sums evaluate to the claimed value, since $\frac{(b-\ell+1)_m}{m!} = \binom{b+m-\ell}{m}$ if $\ell \leq b$ and $= 0$ if $b+1 \leq \ell \leq b+m$. \blacksquare

For the case $\ell = b + m + 1$, the sets E which allow $\text{coef}(\xi_S, \rho g \phi_{S, E}) \neq 0$ are $E = \emptyset, [1, \min S - 1] \cup \{\ell\}$. Lemma 3.4 is used just as in the situation $m = 1$.

Proposition 5.15. *Suppose $\ell = b + m + 1$, then*

$$\sum_{S \subset [1, b+m], \#S=m} \text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) = \frac{1}{\pi_\ell} (n_\ell - 1) \sum_{k=0}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k}(n_1 - 1, \dots, n_{\ell-1} - 1).$$

Proof. By Proposition 5.11,

$$\text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) = \prod_{i=1, i \notin S}^{b+m+1} \frac{n_i - 1}{n_i} = \left(\prod_{i=1}^{b+m+1} \frac{n_i - 1}{n_i} \right) \prod_{j \in S} \left(\frac{n_j}{n_j - 1} \right)$$

and

$$\begin{aligned} \sum_{S \subset [1, b+m], \#S=m} \text{coef}(\xi_S, \rho g \phi_{S, \emptyset}) &= \frac{1}{\pi_\ell} \prod_{i=1}^{\ell} (n_i - 1) \sum_{S \subset [1, b+m]} \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right) \\ &= (n_\ell - 1) \frac{1}{\pi_\ell} \prod_{i=1}^{b+m} (n_i - 1) \sum_{S \subset [1, b+m]} \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right) \\ &= \frac{1}{\pi_\ell} (n_\ell - 1) \sum_{k=0}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k}(n_1 - 1, \dots, n_{\ell-1} - 1), \end{aligned}$$

by Lemma 5.10 with $r = s = b + m$, $q = m$, $(r = \ell - 1)$ and $y_i = n_i - 1$. \blacksquare

Proposition 5.16. *Suppose $S \subset [1, b+m]$, $\#S = m$ and $\ell = b+m+1$, and $E = [1, t-1] \cup \{\ell\}$ with $t := \min S$, then*

$$\text{coef}(\xi_S, \rho g \phi_{S,E}) = (-1)^{t+1} \frac{1}{\pi_\ell} \prod_{i=t}^{b+m} (n_i - 1) \prod_{j \in S} \left(1 + \frac{1}{n_j - 1}\right).$$

Proof. If $\min S = 1$, then $E = \{\ell\}$. Set $t := \min S$. A typical point in $X_{S,E}$ is

$$\mathbf{x} = (x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(t-1)}, x_{>}^{(t+1)}, \dots, x_{>}^{(\ell-1)}, x_1^{(\ell)})$$

omitting $\{x_*^{(j)} \mid j \in S\}$. Then

$$\mathbf{x}g = (x_1^{(2)}, x_1^{(3)}, \dots, x_1^{(t)}, x_{>}^{(t+1)}, \dots, x_{>}^{(\ell-1)}, x_1^{(1)}) \quad \text{and}$$

$$\Delta(\mathbf{x}g) = (-1)^b \Delta(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(t)}, x_{>}^{(t+1)}, \dots, x_{>}^{(\ell-1)})$$

omitting $S \setminus \{t\}$ terms (applying b adjacent transpositions). To apply Lemma 3.4, we relabel $(x_1^{(1)}, \dots, x_1^{(t)}, x_{>}^{(t+1)}, \dots, x_{>}^{(\ell-1)}, x_1^{(\ell)})$ (omit $S \setminus \{t\}$) as (y_1, \dots, y_{b+2}) with $y_i = x_1^{(i)}$ for $1 \leq i \leq t$ and $i = \ell$. Thus

$$\Delta(y_1, y_2, \dots, y_{b+1}) = \sum_{j=1}^b (-1)^{j+b+1} \Delta(y_1, y_2, \dots, \widehat{y}_j, \dots, y_{b+2}).$$

Apply ρ , then the term with $j = t$ becomes $(-1)^{t+b+1} (\prod_{i=1, i \notin S}^{\ell} n_i^{-1}) \xi_S$. Thus

$$\text{coef}(\xi_S, \rho \Delta(\mathbf{x}g)) = (-1)^b (-1)^{t+b+1} \left(\prod_{i=1, i \notin S}^{\ell} n_i^{-1} \right).$$

Multiply by $\#X_{S,E} = \prod_{i=t+1, i \notin S}^{b+m} (n_i - 1)$ to obtain

$$\text{coef}(\xi_S, \rho g \phi_{S,E}) = (-1)^{t+1} \frac{1}{\pi_\ell} \prod_{i=t}^{b+m} (n_i - 1) \prod_{j \in S} \left(1 + \frac{1}{n_j - 1}\right). \quad \blacksquare$$

The next step is to sum over S with the same $\min S$.

Proposition 5.17. *Suppose $\ell = b+m+1$, $1 \leq t \leq b+1$, and $E = [1, t-1] \cup \{\ell\}$, then*

$$\sum_{\min S=t} \text{coef}(\xi_S, \rho g \phi_{S,E}) = (-1)^t \frac{n_t}{\pi_\ell} \sum_{k=0}^{\ell-1-t} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e_{\ell-1-t-k}(n_{t+1}-1, \dots, n_{\ell-1}-1).$$

Proof. By Proposition 5.16,

$$\begin{aligned} \sum_{\min S=t} \text{coef}(\xi_S, \rho g \phi_{S,E}) &= (-1)^{t+1} \frac{1}{\pi_\ell} \prod_{i=t}^{b+m} (n_i - 1) \left(1 + \frac{1}{n_t - 1}\right) \\ &\quad \times \sum_{U \subset [t+1, b+m], \#U=m-1} \prod_{j \in U} \left(1 + \frac{1}{n_j - 1}\right) \\ &= (-1)^t \frac{n_t}{\pi_\ell} \sum_{k=0}^{m-1} \binom{b+m-t-k}{m-1-k} e_{\ell-1-t-k}(n_{t+1}-1, \dots, n_{\ell-1}-1) \end{aligned}$$

by Lemma 5.10 with $r = s = \ell - 1 - t$, $q = m - 1$ (note $b+m = \ell - 1$). In the inner sum $S = U \cup \{t\}$. The binomial coefficient is equal to the coefficient in the claim. \blacksquare

To shorten some ensuing expressions, introduce

$$e(k; u) := e_k(n_u - 1, n_{u+1} - 1, \dots, n_{\ell-1} - 1).$$

Proposition 5.18. *Suppose $\ell = b + m + 1$, then*

$$\sum_{t=1}^{b+1} (-1)^t n_t \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell-1-t-k; t+1) = \sum_{k=1}^m \frac{(b+1)_{m-k}}{(m-k)!} e(\ell-k; 1) + (-1)^b.$$

Proof. Write $n_t = (n_t - 1) + 1$ and use a simple identity for elementary symmetric functions

$$n_t e(\ell-1-t-k; t+1) = e(\ell-t-k; t) - e(\ell-t-k; t+1) + e(\ell-1-t-k; t+1),$$

then the sum becomes

$$\begin{aligned} & \sum_{k=0}^{m-1} \frac{(m-k)_b}{b!} e(\ell-1-k; 1) \\ & + \sum_{t=2}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell-t-k; t) \end{aligned} \quad (5.2)$$

$$\begin{aligned} & - \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell-t-k; t+1) \\ & + \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell-1-t-k; t+1), \end{aligned} \quad (5.3)$$

We will show that there is a three-term telescoping effect, after changing the summation variables in sums: (5.2) $t \rightarrow t+1$, (5.3) $k \rightarrow k+1$. This results in (displayed in same order)

$$\begin{aligned} & \sum_{k=0}^{m-1} \frac{(m-k)_b}{b!} e(\ell-1-k; 1) \\ & + \sum_{t=1}^{b+1} (-1)^t \sum_{k=0}^{m-1} \frac{(m-k)_{b-t}}{(b-t)!} e(\ell-1-t-k; t+1) \\ & - \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=-1}^{m-2} \frac{(m-k-1)_{b+1-t}}{(b+1-t)!} e(\ell-1-t-k; t+1) \\ & + \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell-1-t-k; t+1). \end{aligned}$$

Thus the coefficient of $e(\ell-1-t-k; t+1)$ is

$$\begin{aligned} & \sum_{t=1}^b (-1)^t \sum_{k=0}^{m-1} \frac{(m-k)_{b-t}}{(b-t)!} - \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=-1}^{m-2} \frac{(m-k-1)_{b+1-t}}{(b+1-t)!} \\ & + \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!}, \end{aligned}$$

the limits in the middle sum can be replaced by $0 \leq k \leq m-2$ since $e(\ell-t; t+1) = 0$ (at $k = -1$). If a pair (t, k) occurs in each sum, then the sum of these terms vanishes, by

a straightforward calculation. Exceptions are at $t = b + 1$ (where $\ell - 1 - b - 1 = m - 1$ and $e(\ell - 1 - t - k; t + 1) = e(m - 1 - k; b + 2)$) and at $1 \leq t \leq b$, $k = m - 1$

$$\begin{aligned} & (-1)^{b+2} \left\{ - \sum_{k=0}^{m-2} 1 + \sum_{k=0}^{m-1} 1 \right\} e(m - 1 - k; b + 2) = (-1)^b e(0; b + 2), \\ & \left\{ \sum_{t=1}^b (-1)^t + \sum_{t=1}^b (-1)^{t+1} \right\} e(\ell - t - m; t + 1) = 0, \end{aligned}$$

respectively. Thus

$$\begin{aligned} & \sum_{t=1}^{b+1} (-1)^t n_t \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e(\ell - 1 - t - k; t + 1) \\ &= \sum_{k=0}^{m-1} \frac{(m-k)_b}{b!} e(\ell - 1 - k; 1) + (-1)^b \\ &= \sum_{k=1}^m \frac{(b+1)_{m-k}}{(m-k)!} e(\ell - k; 1) + (-1)^b, \end{aligned}$$

(changing $k \rightarrow k - 1$) since $\frac{(m-k)_b}{b!} = \frac{(m-k-1+b)}{(m-k-1)} = \frac{(b+1)_{m-k-1}}{(m-k-1)!}$. ■

Theorem 5.19. *Suppose $\ell = b + m + 1 (= p)$, then*

$$\Phi^\tau(g) = \pi_\ell^{-1} \left\{ \sum_{k=0}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_* - 1) + (-1)^b \right\}.$$

Proof. Combining the values from Proposition 5.15 for $E = \emptyset$ and from Proposition 5.18 for $E = [1, \min S - 1] \cup \{p\}$,

$$\begin{aligned} \sum_S \text{coef}(\xi_S, \rho g \xi_S) &= \frac{1}{\pi_\ell} (n_\ell - 1) \sum_{k=0}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k}(n_1 - 1, \dots, n_{\ell-1} - 1) \\ &\quad + \frac{1}{\pi_\ell} \left\{ \sum_{k=1}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_1 - 1, \dots, n_{\ell-1} - 1) + (-1)^b \right\} \\ &= \frac{1}{\pi_\ell} \left\{ \sum_{k=0}^m \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_* - 1) + (-1)^b \right\}. \end{aligned}$$

In the second line the lower limit $k = 1$ can be replaced by $k = 0$ because

$$e_\ell(n_1 - 1, \dots, n_{\ell-1} - 1) = 0. \quad \blacksquare$$

Observe that formula (1.2) contains $(-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!}$ which becomes

$$(-1)^{\ell+1} \frac{(-m)_m}{m!} = (-1)^{m+\ell+1},$$

and $\ell = b + m + 1$. Thus we have proven the general formula for any ℓ with $2 \leq \ell \leq p = b + m + 1$.

6 An equivalent formula

Formula (1.2) can be expressed in terms of $e_k\left(\frac{1}{n_1}, \dots, \frac{1}{n_\ell}\right)$, as displayed in formula (1.3).

Proposition 6.1. *For $m \geq 0$ and $2 \leq \ell \leq b + m + 1 = p$,*

$$\begin{aligned} & \frac{1}{\pi_\ell} \left\{ \sum_{k=0}^{\min(m, \ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k}(n_*) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\} \\ &= \binom{b+m}{b} + \sum_{i=1}^{\min(b, \ell-1)} (-1)^i \binom{b+m-i}{b-i} e_i\left(\frac{1}{n_1}, \dots, \frac{1}{n_\ell}\right). \end{aligned}$$

Proof. From the generating function for elementary symmetric functions (we denote $e_i(n_1, n_2, \dots, n_\ell)$ by $e_i(n_*)$),

$$\begin{aligned} \sum_{j=0}^{\ell} t^j e_j(n_*) - 1 &= \prod_{i=1}^{\ell} (1 + t(n_i - 1)) = (1-t)^\ell \prod_{i=1}^{\ell} \left(1 + \frac{t}{1-t} n_i\right) \\ &= \sum_{i=1}^{\ell} (1-t)^{\ell-i} t^i e_i(n_*) = \sum_{i=1}^{\ell} \sum_{k=0}^{\ell-i} (-1)^k \binom{\ell-i}{k} t^{i+k} e_i(n_*) \\ &= \sum_{j=0}^{\ell} t^j \sum_{i=0}^j (-1)^{j-i} \binom{\ell-i}{j-i} e_i(n_*), \end{aligned}$$

and thus $e_j(n_* - 1) = \sum_{i=0}^j (-1)^{j-i} \binom{\ell-i}{j-i} e_i(n_*)$. The first formula equals

$$\pi_\ell^{-1} \left\{ \sum_{k=0}^{\min(m, \ell)} \sum_{i=0}^{\ell-k} \frac{(b+1)_{m-k}}{(m-k)!} (-1)^{\ell-k-i} \binom{\ell-i}{\ell-k-i} e_i(n_*) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\};$$

the coefficient of $e_i(n_*)$ is

$$\sum_{k=0}^{\min(m, \ell-i)} \frac{(b+1)_{m-k}}{(m-k)!} \frac{(i-\ell)_k}{k!} (-1)^{\ell-i} = (-1)^{\ell-i} \frac{(b+1+i-\ell)_m}{m!},$$

(by the Chu–Vandermonde sum) which leads to

$$\begin{aligned} & \pi_\ell^{-1} \left\{ \sum_{i=0}^{\ell} (-1)^{\ell-i} \frac{(b+1+i-\ell)_m}{m!} e_i(n_*) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\} \\ &= \pi_\ell^{-1} \sum_{i=1}^{\ell} (-1)^{\ell-i} \frac{(b+1+i-\ell)_m}{m!} e_i(n_*) = \sum_{j=0}^{\ell-1} (-1)^j \frac{(b+1-j)_m}{m!} \frac{e_{\ell-j}(n_*)}{\pi_\ell}; \end{aligned}$$

(with $j = \ell - i$) this is the second formula since

$$\frac{(b+1-j)_m}{m!} = \frac{(b+m-j)!}{(b-j)!m!} \quad \text{and} \quad \frac{e_{\ell-j}(n_*)}{\pi_\ell} = e_j\left(\frac{1}{n_1}, \dots, \frac{1}{n_\ell}\right). \quad \blacksquare$$

The second formula is more concise than the first one when b is relatively small. For example, when $b = 1$ (the isotype $[N-1, 1]$ and $p = m + 2$), the value is $p - 1 - e_1\left(\frac{1}{n_1}, \dots, \frac{1}{n_\ell}\right)$; this was already found in [3, Theorem 5.6].

Another interesting specialization of the first formula is for $n_i = 1$ for all i so that the spherical function reduces to the character ($\tau = [N - b, 1^b]$)

$$\chi^\tau(g) = \begin{cases} \frac{(b+1)_{m-\ell}}{(m-\ell)!} + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!}, & \ell \leq m, \\ (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!}, & m < \ell \leq N. \end{cases}$$

Observe that $\chi^\tau(g) = 0$ when $b \leq \ell - 1$ and $m \geq 1$. If $N = b + 1$, $m = 0$, then $\tau = \text{sign}$ whose value at an ℓ -cycle is $(-1)^{\ell+1}$.

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