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REPRESENTATION THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS IN HILBERT SPACES AND ITS APPLICATIONS

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Abstract. In this survey we recall the results obtained in [16] where we gave a representation theorem for the solutions of stochastic differential equations in Hilbert spaces. Using this representation theorem and the deterministic characterizations of exponential stability and uniform observability obtained in [16], [17], we will prove a result of Datko type concerning the exponential dichotomy of stochastic equations.

1 Introduction

In [16] V. Ungureanu established a representation theorem (see Theorem 3) for the mild solutions of linear stochastic differential equations. More precisely, in [16] a Lyapunov equation is associated to the discussed linear stochastic differential equation and it is established a relation between the mean square of the mild solution of the stochatic equation and the mild solution of the Lyapunov equation.

This representation theorem is a powerful tool which allow us to obtain deterministic characterizations of different properties of solutions of linear differential stochastic equations.

The aim of this survey is to illustrate how problems like uniform exponential stability, uniform observability or uniform exponential dichotomy of stochastic equations can be solved by using the result obtained in [16].

The survey is organized as it follows.

In the second section we recall basic facts concerning linear stochastic differential equations and Lyapunov equations, which we need in the sequel.

The representation theorem is stated in the third section.

In section 4 we introduce a solution operator associated to the Lyapunov equation associated to the stochastic differential equation and we establish some of its

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properties.

In section 5 we use the representation theorem to obtain deterministic characterizations of the uniform exponential stability, respectively uniform observability properties of the considered stochastic differential equation. We also recall an uniform exponential stability result obtained under uniform observability conditions and a result which give necessary and sufficient conditions for the uniform exponential stability of stochastic equations with periodic coefficients. We note that the characterizations of the uniform exponential stability obtained by the authoress of this survey are different to those obtained by G. Da Prato and I. Ichikawa in [3].

In the last section we introduce a notion of uniform exponential dichotomy for stochastic equations, which is slowly different to that introduced in [18]. Using the solution operator introduced in section 4 we derive deterministic characterizations of the uniform exponential dichotomy (see Theorem 20, which is a result of Datko's type or Theorem 19). Finally we obtained necessary (see Theorem 22) or sufficient (Theorem 23) conditions for the uniform exponential dichotomy by using Lyapunov functions.

2 Notations and preliminaries

Let H, V be separable real Hilbert spaces. We will denote by $L\left(H\right)$ the Banach space of all linear and bounded operators from H into V. Let $\mathcal{E}$ be the Banach subspace of $L(H)$ formed by all self adjoint operators. The operator $A \in \mathcal{E}$ is nonnegative and we will write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We will use the notation $L^+(H)$ for the cone of all nonnegative operators from $\mathcal{E}$. Let $P \in L^+(H)$ and $A \in L(H)$. We denote by $P^{1/2}$ the square root of P and by $|A|$ the operator $(A^*A)^{1/2}$. We put $\|A\|_1 = Tr(|A|) \leq \infty$ and we denote by $C_1(H)$ the set $\{A \in L(H)/ \|A\|_1 < \infty\}$ (the *trace class* of operators)(see [5], [6]).

If E is a Banach space we also denote by $C(J, E)$ the space of all mappings $G(t) : J \to E$ that are continuous. For each interval $J \subset \mathbf{R}_+ (\mathbf{R}_+ = [0, \infty))$ we will denote by $C_s(J, L(H))$ the space of all mappings $G(t) : J \to L(H)$ that are strongly continuous.

Let $(\Omega, F, \mathcal{F}_t, t \in [0, \infty), P)$ be a stochastic basis and let us denote $L^2_s(H) = L^2(\Omega, \mathcal{F}_s, P, H)$. In this paper we consider stochastic differential equations of the form

$$\begin{aligned} dy(t) &= A(t)y(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t) \\ y(s) &= \xi \in L^2_s(H), \end{aligned} \tag{1}$$

where and w_i's are independent real Wiener processes relative to $\mathcal{F}_t$.and the coefficients $A(t)$ and $G_i(t)$ satisfy the hypotheses:

P1 : *a)* $A(t)$, $t \in [0,\infty)$ *is a closed linear operator on* H *with constant domain* D *dense in* H.

b) there exist $M > 0$, $\eta \in (\frac{1}{2}\pi, \pi)$ *and* $\delta \in (-\infty, 0)$ *such that* $S_{\delta,\eta} = \{\lambda \in C; |\arg(\lambda - \delta)| < \eta\} \subset \rho(A(t))$, *for all* $t \geq 0$ *and*

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda - \delta|}$$

for all $\lambda \in S_{\delta,\eta}$ *where we denote by* $\rho(A)$, $R(\lambda, A)$ *the resolvent set of* A *and respectively the resolvent of* A.

c) there exist numbers $\alpha \in (0,1)$ *and* $\widetilde{N} > 0$ *such that*
$\left\|A(t)A^{-1}(s) - I\right\| \leq \widetilde{N}\,|t-s|^{\alpha}, t \geq s \geq 0.$

P2 : $G_i \in C_s(\mathbf{R}_+, L(H)), i = 1, ..., m,\ D(s) \in C_s(\mathbf{R}_+, L^+(H))$.

Throughout this paper we will assume that P1 and P2 hold.

It is known that if P1 holds then the family $\{A(t)\}_{t\in\mathbf{R}_+}$ generates the evolution operator $U(t,s), t \geq s \geq 0$ (see [3], [13]).

Let us consider $T > 0$. It is known (see [1]) that (1) has a unique mild solution in $C([s,T]; L^2(\Omega; H))$ that is adapted to $\mathcal{F}_t$; namely the solution of

$$y(t) = U(t,s)\xi + \sum_{i=1}^{m} \int_s^t U(t,r)G_i(r)y(r)dw_i(r). \tag{2}$$

By convenience, we denote by $y(t,s;\xi)$ the solution of (1) with the initial condition $y(s) = \xi$, $\xi \in L_s^2(H)$.

Lemma 1. *[3]There exists a unique mild (resp. classical) solution to (1).*

Now we consider the following Lyapunov equation:

$$\frac{dQ(s)}{ds} + A^*(s)Q(s) + Q(s)A(s) + \sum_{i=1}^{m} G_i^*(s)Q(s)G_i(s) + D(s) = 0, s \geq 0 \tag{3}$$

According with [3], we say that Q is a mild solution on an interval $J \subset \mathbf{R}_+$ of (3), if $Q \in C_s(J, L^+(H))$ and if for all $s \leq t$, $s, t \in J$ and $x \in H$ it satisfies

$$\begin{aligned} Q(s)x &= U^*(t,s)Q(t)U(t,s)x + \int_s^t U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)Q(r)G_i(r) \\ &\quad + D(s)]U(r,s)xdr. \end{aligned} \tag{4}$$

Lemma 2. *[3] Let $0 < T < \infty$ and let $R \in L^{+}(H)$. Then there exists a unique mild solution Q of (3)(denoted $Q(T, s; R)$) on $[0, T]$ such that $Q(T) = R$ and it is given by*

$$Q(s)x = U^{*}(T, s)RU(T, s)x \qquad (5)$$
$$+ \int_{s}^{T} U^{*}(r, s)[\sum_{i=1}^{m} G_i^{*}(r)Q(r)G_i(r) + D(r)]U(r, s)xdr$$

Moreover it is monotone in the sense that $Q(T, s; R_1) \leq Q(T, s; R_2)$ if $R_1 \leq R_2$.

3 The covariance operator of the mild solutions of linear stochastic differential equations and the Lyapunov equations

Let $\xi \in L^2(\Omega, H)$. We denote by $E(\xi \otimes \xi)$ the bounded and linear operator which act on H given by $E(\xi \otimes \xi)(x) = E(\langle x, \xi\rangle \xi)$.

The operator $E(\xi \otimes \xi)$ is called the covariance operator of ξ (see also [8]). The following result is known.

Theorem 3. *[16] Let V be another real separable Hilbert space and $B \in L(H, V)$. If $y(t, s; \xi)$, $\xi \in L_s^2(H)$ is the mild solution of (1) and $Q(t, s, R)$ is the unique mild solution of (3), where $D(s) = 0, s \in \mathbf{R}_+$, with the final value $Q(t) = R \geq 0$ then*

a) $\langle E[y(t, s; \xi) \otimes y(t, s; \xi)]u, u\rangle = TrQ(t, s; u \otimes u)E(\xi \otimes \xi)$ for all $u \in H$

b)
$$E\left\|By(t, s; \xi)\right\|^2 = TrQ(t, s; B^{*}B)E(\xi \otimes \xi).$$

If we replace the hypotheses P1, P2 with

H1 : $A, G_i \in C(R_+, L(H)), i = 1, ..., m,$

we have the following corollary.

Corollary 4. *[16]If the assumption H1 holds then the statements a) and b) of the Theorem 3 are true.*

We note that if A is time invariant ($A(t) = A$, for all $t \geq 0$), then the condition P1 can be replaced with the hypothesis

H2 : A is the infinitesimal generator of a C_0-semigroup

and the time invariant version of the above result is the following:

Proposition 5. *[16]If P2 and H2 hold, then the conclusions of the above theorem stay true. Particularly, if we replace P2 with the condition $G_i \in L(H)$, $i = 1, ..., m$ the statement b) becomes:*

$$E\left\|By(t, s; \xi)\right\|^2 = TrQ(t, s, 0; B^{*}B)E(\xi \otimes \xi) = TrQ(t - s; B^{*}B)E(\xi \otimes \xi)$$

4 The solution operators associated to the Lyapunov equations

Let us assume throughout this section that the therm D of the Lyapunov equation (3) satisfy the condition $D(s) = 0$ for all $s \geq 0$. Let $Q(T, s; R), R \in L^+(H), T \geq s \geq 0$ be the unique mild solution of the Lyapunov equation (3), which satisfies the condition $Q(T) = R$.

Using the Gronwall's inequality we deduce the following Lemma:

Lemma 6. *[16]a) If $R_1, R_2 \in L^+(H)$ and $\alpha, \beta > 0$ then*

$$Q(T, s; \alpha R_1 + \beta R_2) = \alpha Q(T, s; R_1) + \beta Q(T, s; R_2).$$

b) $Q(p, s; Q(t, p; R)) = Q(t, s; R)$ for all $R \in L^+(H), t \geq p \geq s \geq 0$.

The following lemma is known [19].

Lemma 7. *Let $T \in L(\mathcal{E})$. If $T(L^+(H)) \subset L^+(H)$ then $\|T\| = \|T(I)\|$, where I is the identity operator on H.*

If $R \in \mathcal{E}$ then there exist $R_1, R_2 \in L^+(H)$ such that $R = R_1 - R_2$ (we take for example $R_1 = \|R\| I$ and $R_2 = \|R\| I - R$).

Let us introduce the mapping $\mathcal{T}(t, s) : \mathcal{E} \to \mathcal{E}$,

$$\mathcal{T}(t, s)(R) = Q(t, s; R_1) - Q(t, s; R_2) \tag{6}$$

for all $t \geq s \geq 0$. The mapping $\mathcal{T}(t, s)$ called the solution operator associated to the Lyapunov equation (3) has the following properties (see [16]):

1. $\mathcal{T}(t, s)$ is well defined. Indeed if R_1', R_2' are another two nonnegative operators such as $R = R_1' - R_2'$ we have $R_1' + R_2 = R_1 + R_2'$. From lemmas L.2 and L.6 we have $Q(t, s; R_1' + R_2) = Q(t, s; R_1 + R_2')$ and $Q(t, s; R_1') + Q(t, s; R_2) = Q(t, s; R_1) + Q(t, s; R_2')$. The conclusion follows.

2. $\mathcal{T}(t, s)(-R) = -\mathcal{T}(t, s)(R), R \in \mathcal{E}$.

3. $\mathcal{T}(t, s)(R) = Q(t, s; R)$ for all $R \in L^+(H)$ and $t \geq s \geq 0$.

4. $\mathcal{T}(t, s)(L^+(H)) \subset L^+(H)$.

5. For all $R \in \mathcal{E}$ and $x \in H$ we have

$$\langle \mathcal{T}(t, s)(R)x, x \rangle = E \langle Ry(t, s; x), y(t, s; x) \rangle . \tag{7}$$

(It follows from the Theorem 3 and from the definition of $\mathcal{T}(t, s)(R)$.)

6. $\mathcal{T}(t,s)$ is a linear and bounded operator and $\|\mathcal{T}(t,s)\| = \|\mathcal{T}(t,s)(I)\|$.

   From 5. we deduce that $\mathcal{T}(t,s)$ is linear. If $R \in \mathcal{E}$, we use (7) and we get
$$\|\mathcal{T}(t,s)(R)\| \leq \|R\| \sup_{x\in H,\|x\|=1} E\|y(t,s;x)\|^2 = \|R\| \|Q(t,s;I)\|.$$
   Thus $\mathcal{T}(t,s)$ is bounded. Using 4. and Lemma 7 we obtain the conclusion.

7. $\mathcal{T}(p,s)\mathcal{T}(t,p)(R) = \mathcal{T}(t,s)(R)$ for all $t \geq p \geq s \geq 0$ and $R \in \mathcal{E}$.

   It follows from Lemma 6 and the definition of $\mathcal{T}(t,s)$.

8. If $t \geq 0$ is fixed, then $\mathcal{T}(t,p)(R) \underset{p\to p_0}{\to} \mathcal{T}(t,p_0)(R)$ for any $R \in \mathcal{E}$.

   It is a direct consequence of Theorem 2.

   Let us introduce the following hypothesis

P3 $U(t,s)$ has an exponentially growth, that is there exist the positive constants m and a such that
$$\|U(t,s)\| \leq me^{a(t-s)}.$$

9 If P3 holds, then there exists an increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that
$$\mathcal{T}(t,s)(I) \leq f(t-s)I \tag{8}$$
for all $0 \leq s \leq t$.

Indeed, $\mathcal{T}(t,s)(I) = Q(t,s;I)$ and using (4), Gronwall's inequality and P3 we deduce that exists an increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\|\mathcal{T}(t,s)(I)\| \leq f(t-s)I$. Since $\mathcal{T}(t,s)(I) \in L^+(H)$, then the last inequality is equivalent with (8). The proof of the statement is complete.

If we change the definition of the mild solution of (3) by replacing the condition $Q \in C_s(J, L^+(H))$ with $Q \in C_s(J, \mathcal{E})$, then the statements of Lemma 2 stay true.

Proposition 8. *[16]Let $R \in \mathcal{E}$ and $T > 0$. There exists a unique mild solution Q of (3) on $[0,T]$ such that $Q(T) = R$. It is given by (5). Moreover, $Q(T,s;R) = \mathcal{T}(T,s)(R)$.*

Proof. Let $R = R_1 - R_2 \in \mathcal{E}$, $R_1, R_2 \geq 0$. It is easy to see that $Q(T,s;R_1) - Q(T,s;R_2) \in C_s([0,T], \mathcal{E})$ satisfies the integral equation (5). If $Q' \in C_s([0,T], \mathcal{E})$ is another mild solution of (3) such that $Q'(T) = R$ then we denote $K(s) = Q(T,s;R_1) - Q(T,s;R_2) - Q'(s) \in C_s([0,T], \mathcal{E})$ and we have
$$\begin{aligned}\|K(s)\| &= \sup_{x\in H,\|x\|=1} \left| \sum_{i=1}^{m} \int_s^T \langle K(r)G_i(r)U(r,s)x, G_i(r)U(r,s)x\rangle \, dr \right| \\ &\leq \sum_{i=1}^{m} \int_s^T \|K(r)\| \|G_i(r)\| \|U(r,s)\|^2 \, dr.\end{aligned}$$

Now, we use the Gronwall's inequality and we obtain the conclusion. □

5 Uniform exponential stability and uniform observability

Definition 9. *[16] We say that (1) is uniformly exponentially stable if there exist the constants $M \geq 1$, $\omega > 0$ such that $E\left\|y(t,s;x)\right\|^2 \leq Me^{-\omega(t-s)}\left\|x\right\|^2$ for all $t \geq s \geq 0$ and $x \in H$.*

Using the representation Theorem 3 and the property 6. of the operator $\mathcal{T}(t,s)$ we obtain the following theorem:

Theorem 10. *[16] Let $Q(t,s,R)$ be the unique mild solution of (3)(where $D(s) = 0$ for all $s \geq 0$) such that $Q(t) = R, R \geq 0$. The following statements are equivalent:*
a) the equation (1) is uniformly exponentially stable
b) there exist the constants $M \geq 1$, $\omega > 0$ such that $Q(t,s;I) \leq Me^{-\omega(t-s)}I$ for all $t \geq s \geq 0$,
c) there exist the constants $M \geq 1$, $\omega > 0$ such that $\left\|\mathcal{T}(t,s)\right\| \leq Me^{-\omega(t-s)}$.

If $C \in C_s(\mathbf{R}_+, L(H))$, we consider the equation (1) and the observation relation

$$z(t) = C(t)y(t,s,x) \tag{9}$$

The system (1), (9) will be denoted $\{A, C; G_i\}$.
Since $y(.,s;x) \in C([s,T]; L^2(\Omega,H))$ for all $x \in H$ it follows that $C(.)y(.,s;x) \in C([s,T]; L^2(\Omega,V))$. We note that

$$t \to E\left\|C(t)y(t,s;x)\right\|^2 \text{ is continuous on } [s,T]. \tag{10}$$

Definition 11. *[12],[16]The system $\{A, C; G_i\}$ is uniformly observable if there exist $\tau > 0$ and $\gamma > 0$ such that for all $s \in \mathbf{R}_+$ and $x \in H$,*

$$E\int_s^{s+\tau}\left\|C(t)y(t,s;x)\right\|^2 dt \geq \gamma\left\|x\right\|^2$$

The following result is known and gives a characterization of the uniform exponential stability of uniformly observable differential stochastic equations in therms of Lyapunov equations:

Theorem 12. *[17]Let us assume that P3 holds, $C, C^* \in C_s(\mathbf{R}_+, L(H))$ and $D(s) = C^*(s)C(s), s \geq 0$ in (3). If $\{A, C; G_i\}$ is uniformly observable then the equation (1) is uniformly exponentially stable if and only if the equation (3) has a unique mild solution $\mathcal{Q}$ with the property that there exist the positive constants $\widetilde{m}$, $\widetilde{M}$ such that*

$$\widetilde{m}\left\|x\right\|^2 \leq \langle\mathcal{Q}(s)x, x\rangle \leq \widetilde{M}\left\|x\right\|^2 \tag{11}$$

for all $s \geq 0$ and $x \in H$.

Many results concerning stochastic uniform exponential stability (see the above theorem) or stochastic stabilizability (see [15], [12], [11]) are obtained under uniform observability conditions. Hence a deterministic characterization of the stochastic uniform observability is an important tool in solving problems which involve this property of stochastic differential equations.

Theorem 13. *[16] The system $\{A, C; G_i\}$ is uniformly observable iff there exist $\tau > 0$ and $\gamma > 0$ such that $\int_s^{s+\tau} Q(t, s; C^*(t)C(t))dt \geq \gamma I$ for all $s \in \mathbf{R}_+$, where I is the identity operator on H.*

Proof. By Theorem 3 b) we get $E\|C(t)y(t,s;x)\|^2 = \langle Q(t,s;C^*(t)C(t))x, x\rangle$ for all $x \in H$. Because $t \to E\|C(t)y(t,s;x)\|^2$ is continuous we deduce $\int_s^{s+\tau} E\|C(t)y(t,s;x)\|^2 dt < \infty$. From Definition 11 and Fubini's theorem it follows the conclusion. □

5.1 The uniform exponential stability of linear stochastic system with periodic coefficients

Let us assume that the following hypothesis holds:

P4 *There exists $\tau > 0$ such that $A(t) = A(t+\tau), G_i(t) = G_i(t+\tau), i = 1, ..., m$ for all $t \geq 0$.*

It is known (see [14], [2]) that if P1, P4 hold then we have

$$U(t+\tau, s+\tau) = U(t,s) \text{ for all } t \geq s \geq 0. \tag{12}$$

Proposition 14. *[16]If P4 holds and $Q(t,s;R)$ is the unique mild solution of (3)(with $D(s) = 0$) such that $Q(t) = R, R \geq 0$, then for all $t \geq s \geq 0$ and $x \in H$ we have*
a) $Q(t+\tau, s+\tau; R) = Q(t,s;R)$.
b) $\mathcal{T}(t+\tau, s+\tau) = \mathcal{T}(t,s)$
c)$\mathcal{T}(n\tau, 0) = \mathcal{T}(\tau, 0)^n$
d) $E\|y(t+\tau, s+\tau; x)\|^2 = E\|y(t,s;x)\|^2$

The next result (see its proof in [16]) gives necessary and sufficient conditions for uniform exponential stability of periodic equations.

Theorem 15. *If P4 holds, then the following assertions are equivalent:*
a) the equation (1) is uniformly exponentially stable;
b) $\lim_{n\to\infty} E\|y(n\tau, 0; x)\|^2 = 0$ uniformly for $x \in H$, $\|x\| = 1$;
c) $\rho(\mathcal{T}(\tau, 0)) < 1$.

It is not difficult to see that under the hypothesis H1 the Lyapunov equation 3 with final condition has a unique classical solution. Consequently the operator $\mathcal{T}(t,s)$ is well defined and has the properties 1.-9. stated in the last section. From Corollary 4 and Proposition 5 we obtain the following result:

Proposition 16. *Assume that P4 hold. If either H2 and P2 or H1 hold, then the statements of the above theorem stay true.*

The following example illustrate the theory (see also [16]).

Example 17. *Consider an example of equation (1)*

$$dy = e^{-\sin^2(t)} y dt + \sin(t) y dw(t), t \geq 0 \tag{13}$$

where $w(t)$ is a real Wiener process. It is clear that H1 and P4 (with $\tau = 2\pi$) hold. The Lyapunov equation associated to (13) is

$$dQ + (2e^{-\sin^2(t)} + \sin^2(t)) Q dt = 0 \text{ and}$$

$$\begin{aligned} Q(2\pi, 0; I) &= \exp(-\int_0^{2\pi} 2e^{-\sin^2(t)} + \sin^2(t) dt) I \\ &\leq e^{-\pi} \exp(-\int_0^{2\pi} 2e^{-\sin^2(t)} dt) I < I. \end{aligned}$$

Since

$$\rho(\mathcal{T}(2\pi,0)) \leq \|\mathcal{T}(2\pi,0)\| = \|\mathcal{T}(2\pi,0)(I)\| = \|Q(2\pi,0;I)\| < 1$$

we can deduce from the Proposition 16 that the solution of the stochastic equation (13) is uniformly exponentially stable.

6 Uniform exponential dichotomy of stochastic differential equations

In this section we will introduce the notion of uniform exponential dichotomy for linear differential stochastic equations, which is different to those introduced in [18]. Using the representation Theorem 3 and the solution operator $\mathcal{T}(t,s)$ introduced in section 2, we will give deterministic characterizations of this concept.

The obtained result are stochastic versions of those obtained in [10], [9] for deterministic case.

Let fix $s \geq 0$. We will assume that H_1 is a closed subspace of H. (An example of H_1 could be the closure of the linear subspace formed by all $x \in H$ with the property

$\sup_{t \geq s} E \|y(t,s;x)\|^2 < \infty$). Let P_1 be the projection of H on H_1 and $P_2 = I - P_1$ be the projection of H on $H_2 = H_1^{\perp}$. If $y(t,s;x)$ is the mild solution of (1), we will denote $y_1(t,s;x) = y(t,s;P_1x)$ and respectively $y_2(t,s;x) = y(t,s;P_2x)$.

Definition 18. We say that the pair (H_1, H_2) *induces* an *uniform exponential dichotomy for* the mild solution $y(t,s;x)$ of (1), iff there exists the constants N_1, N_2, $\gamma > 0$ such that

$$E(\|y_1(t,s;x)\|^2) \leq N_1 e^{-\gamma(t-\tau)} E(\|y_1(\tau,s;x)\|^2) \tag{14}$$

$$E(\|y_2(t,s;x)\|^2) \geq N_2 e^{\gamma(t-\tau)} E(\|y_2(\tau,s;x)\|^2) \tag{15}$$

for all $x \in H$ *and* $t \geq \tau \geq s$.

6.1 Characterizations of the exponential dichotomy

The following result is a direct consequence of Definition 18 and Theorem 3.

Theorem 19. *The mild solution of the equation (1) has an uniform exponential dichotomy induced by the pair* (H_1, H_2) *iff there exist the constants* N_1, N_2, $\gamma > 0$ *such that*

$$\begin{aligned} \langle P_1 \mathcal{T}(t,s)(I)P_1x, x\rangle &\leq N_1 e^{-\gamma(t-\tau)} \langle P_1 \mathcal{T}(\tau,s)(I)P_1x, x\rangle, & (16)\\ \langle P_2 \mathcal{T}(t,s)(I)P_2x, x\rangle &\geq N_2 e^{\gamma(t-\tau)} \langle P_2 \mathcal{T}(\tau,s)(I)P_2x, x\rangle & (17) \end{aligned}$$

for all $x \in H$ *and* $t \geq \tau \geq s$, *where* $\mathcal{T}(t,s)$ *is the solution operator associated to the Lyapunov equation (3).*

The next theorem is a result of Datko type [4] (see also the results obtained in [10] for deterministic systems) which is similar to that obtained in [18] for autonomous stochastic differential equations and different notion of dichotomy.

Theorem 20. *If P1, P2 and P3 hold, then the solution of (1) has an exponential dichotomy induced by the pair* (H_1, H_2) *iff there exist the positive constants* M_1,M_2 *and* M_3 *such that*

$$\int_\tau^\infty \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle\, dt \leq M_1 \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle \tag{18}$$

and

$$\int_s^\tau \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle\, dt \leq M_2 \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle, \tag{19}$$

$$\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle \leq M_3 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle \tag{20}$$

for all $x \in H$ *and* $\tau \geq s \geq 0$.

Proof. a) First we will prove the equivalence of (14) and 18. Let us prove the implication "(14) $\Rightarrow$ (18)". Integrating (16) with respect to t on the interval $[\tau, \infty)$ we get (18) with $M_1 = \frac{N_1}{\gamma}$. Now we will prove the converse.

Step 1. We will prove that there exists $K > 0$ such that

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le \frac{K}{t-\tau+1} \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle \tag{21}$$

for all $0 \le s \le \tau \le t, x \in H$. Let $x \in H$ and $c = \int\limits_0^1 \frac{1}{f(u)} du$,where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is given by (8).

We have $c \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle = \int\limits_{t-1}^{t} \frac{1}{f(t-r)} \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \, dr.$

Case 1. Let $s \le t-1$. If $r \in [t-1, t]$, then $r \ge s$ and we have $\mathcal{T}(t,s)(I) = \mathcal{T}(r,s)\left(\mathcal{T}(t,r)(I)\right)$. Using (8) we get

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le f(t-r) \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \tag{22}$$

Hence $c \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le \int\limits_{t-1}^{t} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \, dr.$

If $\tau \le t-1$ then $\int\limits_{t-1}^{t} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \, dr \le \int\limits_{\tau}^{\infty} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \, dr$ and using (18) we get

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le \frac{M_1}{c} \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle .$$

If $\tau > t-1$ then $t - \tau < 1$ and taking $r = \tau$ in (22) we obtain

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le f(1) \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle \tag{23}$$

Case 2. If $s > t-1$, then $t - \tau < 1$ and reasoning as above we obtain (23). Consequently, denoting $N = \min\{\frac{M_1}{c}, f(1)\}$ we get for all $s \le \tau \le t$

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le N \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle .$$

Since

$$\int\limits_{\tau}^{t} \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \, dr \le N \int\limits_{\tau}^{t} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \, dr$$

$$\le N \int\limits_{\tau}^{\infty} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle \, dr \le NM_1 \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle$$

Thus

$$(t-\tau) \langle \mathcal{T}(t,s)(I)P_1x, P_1x\rangle \le NM_1 \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle$$

for all $0 \le s \le \tau \le t, x \in H$. Summing the last two inequalities we have

$$(t - \tau + 1) \langle \mathcal{T}(t,s)(I)P_1x, P_1x \rangle \le N(M_1 + 1) \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x \rangle$$

and

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x \rangle \le \frac{N(M_1+1)}{(t-\tau+1)} \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x \rangle \tag{24}$$

for all $0 \le s \le s+1 \le \tau \le t, x \in H$.

Taking $K = N(M_1 + 1)$ we obtain (21).

Step 2. Let $\rho > 0$ be such that $\frac{K}{(\rho+1)} = \frac{1}{2}$, and let $t \ge \tau \ge s$. There exist $n \in \mathbf{N}$ and $r_0 \in \mathbf{R}_+$ such that $t - \tau = n\rho + r_0$, $0 \le r_0 < \rho$. Using the induction it is easy to see that if $t - \tau = n\rho + r_0, 0 \le r_0 < \rho$ then

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x \rangle \le (\frac{1}{2})^n K \langle \mathcal{T}(\tau + r_0,s)(I)P_1x, P_1x \rangle \tag{25}$$

Indeed for $n = 0$ the statement follows from (21). Assuming that (25) holds for $n \ge 0$ and we will prove the inequality for $n+1$. Using (21) and the induction hypothesis we get $\langle \mathcal{T}(t,s)(I)P_1x, P_1x \rangle = \langle \mathcal{T}(\tau + (n+1)\rho + r_0,s)(I)P_1x, P_1x \rangle \le$
$(\frac{1}{2}) \langle \mathcal{T}(\tau + (n)\rho + r_0,s)(I)P_1x, P_1x \rangle \le (\frac{1}{2})(\frac{1}{2})^n K \langle \mathcal{T}(\tau + r_0,s)(I)P_1x, P_1x \rangle$. The conclusion follows.

Now

$$\langle \mathcal{T}(t,s)(I)P_1x, P_1x \rangle \le (\frac{1}{2})^{\frac{t-\tau}{\rho}} (\frac{1}{2})^{\frac{-r_0}{\rho}} K \langle \mathcal{T}(\tau + r_0,s)(I)P_1x, P_1x \rangle \le$$
$$2(\frac{1}{2})^{\frac{t-\tau}{\rho}} K_1 \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x \rangle$$

Taking $\gamma = -\frac{1}{\rho} ln \frac{1}{2}$ and $N_1 = 2K_1$ we obtain (16).

b) Now we will prove the equivalence between (15) and (19), (20). Since the implication "(15) $\Rightarrow$ (19), (20)" is obviously true, we only have to prove the converse.

Let $0 \le s \le r \le t$.

Case 1. If $0 \le s \le r - 1 \le r \le t$ then we use (22) to deduce the following inequalities

$$\begin{aligned} c \langle \mathcal{T}(r,s)(I)P_2x, P_2x \rangle &\le \int_{r-1}^{r} \frac{1}{f(r-p)} \langle \mathcal{T}(r,s)(I)P_2x, P_2x \rangle \, dp \\ &\le \int_{r-1}^{r} \langle \mathcal{T}(p,s)(I)P_2x, P_2x \rangle \, dp \\ &\le \int_{s}^{t} \langle \mathcal{T}(p,s)(I)P_2x, P_2x \rangle \, dp \end{aligned}$$

By (19) we obtain for all $0 \leq s \leq r-1 \leq r \leq t$

$$c \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \leq M_2 \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle . \tag{26}$$

Case 2. If $s \geq r-1$ and $r+1 \leq t$ we apply (20), (26) and

$$\begin{aligned} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle &\leq M_3 \langle \mathcal{T}(r+1,s)(I)P_2x, P_2x\rangle \\ &\leq M_3 \frac{M_2}{c} \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle \end{aligned} \tag{27}$$

for all $x \in H$.

Case 3. If $s \geq r-1$ and $r+1 > t$ then using (20) and (22) we get:

$$\begin{aligned} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle &\leq M_3 \langle \mathcal{T}(r+1,s)(I)P_2x, P_2x\rangle \\ &\leq M_3 f(1) \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle \end{aligned} \tag{28}$$

for all $x \in H$.

From (26), (27) and (28) it follows that there exists a positive constant P such that :

$$\langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \leq P \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle$$

for all $t \geq r \geq s \geq 0$ and $x \in H$. Replacing t with τ in the above inequality and integrating from r to t with respect to τ we obtain

$$\begin{aligned} (t-r) \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle &\leq P \int_r^t \langle \mathcal{T}(\tau,s)(I)P_2x, P_2x\rangle \, d\tau \\ &\leq M_2 P \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle \end{aligned}$$

Summing the last two inequalities we get

$$(t-r+1) \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \leq (M_2+1)P \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle$$

for all $t \geq r \geq s \geq 0$ and $x \in H$.

Thus, $C(t-r+1) \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \leq \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle$, where $C = \frac{1}{(M_2+1)P}$.

Arguing as in the last part of the proof of a) we obtain the conclusion $\square$

6.2 Uniform exponential dichotomy and Lyapunov functions

Definition 21. We say that $V : \mathbf{R}_+ \times H \to \mathbf{R}$ is a *Lyapunov function for the mild solution* of (1) if it satisfy the following conditions:

1) There exists $k > 0$ such that $|V(t,x)| \leq k\langle \mathcal{T}(t,s)(I)x, x\rangle$ for all $x \in H_1 \cup H_2, t \geq 0$.

2) $\int_\tau^t \langle \mathcal{T}(r,s)(I)x, x\rangle \, dr \leq V(\tau,x) - V(t,x)$ for all $s \leq \tau \leq t$ and $x \in H$.

Theorem 22. *If the mild solution $y(t,s;x)$ of (1) has an uniform exponential dichotomy induced by the pair (H_1,H_2) then there exists a Lyapunov function V such that:*

i) $V(t,x)\geq 0$ for all $x\in H_1$, $t\geq s$,

ii) $V(t,x)\leq 0$ for all $x\in H_2$, $t\geq s$.

Proof. Let

$$V(t,x)=2\int_t^{\infty}\langle\mathcal{T}(r,s)(I)P_1x,P_1x\rangle\,dr-2\int_s^t\langle\mathcal{T}(r,s)(I)P_2x,P_2x\rangle\,dr, x\in H.$$

It is clear that $V(.,.)$ satisfy the conditions i) and ii). We will prove that $V(.,.)$ is a Lyapunov functions.

We note that $\langle Rx,x\rangle\leq 2\left(\langle RP_1x,P_1x\rangle+\langle RP_2x,P_2x\rangle\right)$ for any $R\in L^+(H)$.

Thus, for all $s\leq\tau\leq t$ and $x\in H$,

$$V(\tau,x)-V(t,x)=2\int_\tau^t\langle\mathcal{T}(r,s)(I)P_1x,P_1x\rangle\,dr$$

$$+2\int_\tau^t\langle\mathcal{T}(r,s)(I)P_2x,P_2x\rangle\,dr\geq\int_\tau^t\langle\mathcal{T}(r,s)(I)x,x\rangle\,dr$$

and we proved 2). Now we will prove the first condition. We have $|V(t,x)|\leq 2\int_t^{\infty}\langle\mathcal{T}(t,r)(I)P_1x,P_1x\rangle\,dr+2\int_s^t\langle\mathcal{T}(r,s)(I)P_2x,P_2x\rangle\,dr, x\in H$. Using Theorem 20 ((18) and (19)) we get

$$|V(t,x)|\leq 2M_1\langle\mathcal{T}(t,s)(I)P_1x,P_1x\rangle+2M_2\langle\mathcal{T}(t,s)(I)P_2x,P_2x\rangle)$$

and the conclusion follows. We deduce that V is a Lyapunov function.

The proof is complete. □

Finally we give the converse of this theorem.

Theorem 23. *If there exists a Lyapunov function V such that the conditions i) and ii) of the above theorem hold, then $y(r,s;x)$ has an uniform exponential dichotomy.*

Proof. Using the condition i) and the property 2) of Lyapunov function V it follows that for all $t\geq\tau\geq s$

$$\int_\tau^t\langle\mathcal{T}(r,s)(I)P_1x,P_1x\rangle\,dr\leq V(\tau,P_1x).$$

Taking into account the property 1) in Definition 21 and passing to the limit for $t \to \infty$ we deduce that there exists $k > 0$ such that

$$\int_{\tau}^{\infty} \langle \mathcal{T}(r,s)(I)P_1x, P_1x\rangle\, dr \le k \langle \mathcal{T}(\tau,s)(I)P_1x, P_1x\rangle\,.$$

Now we apply 2) and 1) from Definition 21 and hypothesis ii), and we get

$$\int_{\tau}^{t} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle\, dr \le -V(t,P_2x) \le k \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle\,.$$

We note that if $\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle = 0$ it follows by property 9 of the operator $\mathcal{T}(t,s)(I)$ and the above inequality that $\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle = 0$ for all $t \ge s$. In this case the conclusion of the theorem follows. Hence we may assume, that $\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle = 0$ for all $t \ge s$ and taking $k_1 = 2k$ we have

$$\int_{\tau}^{t} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle\, dr < k_1 \langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle \tag{29}$$

Now we will prove condition (20) of Theorem 20. Let $t \ge s$ be fixed.

Let $s \le t$. If $t-1 \le s$ it is easy to see that (20) holds. Indeed, the function

$$t \to \frac{\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle}{\langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle}$$

is continuous on the compact interval $[s, s+1]$ and there exists $M_3 > 0$ such that

$$\frac{\langle \mathcal{T}(t,s)(I)P_2x, P_2x\rangle}{\langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle} \le M_3.$$

Condition (20) follows.

Let as assume that $t-1 > s$. Using a mean theorem and (29) it follows that there exists $\tau \in [t, t+1]$ such that

$$\begin{aligned} \langle \mathcal{T}(\tau,s)(I)P_2x, P_2x\rangle &= \int_{t}^{t+1} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle\, dr \\ &< k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle\,. \end{aligned} \tag{30}$$

Let t_1 be the smallest $\tau \in [t-1, t+1]$ which satisfy the condition

$$\langle \mathcal{T}(\tau,s)(I)P_2x, P_2x\rangle < k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle\,.$$

If $t_1 \leq t$ the conclusion follow from proprety 9 of $\mathcal{T}(t,s)$. Indeed

$$\mathcal{T}(t,s)(I) = \mathcal{T}(t_1,s)(\mathcal{T}(t,t_1)(I)) \leq f(t-t_1)\mathcal{T}(t_1,s)(I) \leq f\,(1)\,\mathcal{T}(t_1,s)(I)$$

and it is clear that we obtain (20).

Assume that $t_1 > t$. First we prove that $t_1 \neq t+1$. If $t_1 = t+1$ then

$$\langle \mathcal{T}(\tau,s)(I)P_2x, P_2x\rangle \geq k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle$$

for all $\tau \in [t,t+1)$ and integrating on $[t,t+1]$ with respect to τ we contradict (30).

Hence $t_1 < t+1$. On the other hand for $r \in [t_1-1,t_1) \subset [t-1,t+1]$ we have

$$\langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \geq k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle$$

and $\int\limits_{t_1-1}^{t_1} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle \geq k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle$. Since

$$\int\limits_{t_1-1}^{t_1} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle\, dr \leq \int\limits_{p}^{t+1} \langle \mathcal{T}(r,s)(I)P_2x, P_2x\rangle\, dr < k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle\,.$$

it follows

$$k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle < k_1 \langle \mathcal{T}(t+1,s)(I)P_2x, P_2x\rangle$$

that is absurd. Thus the hypothesis that $t_1 > t$ is false and the conclusion follows. □

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