Surveys in Mathematics and its Applications

ISSN 1842-6298
Volume **2** (2007), 11 – 20

ADOMIAN DECOMPOSITION METHOD FOR NONLINEAR STURM-LIOUVILLE PROBLEMS

Sennur Somali and Guzin Gokmen

Abstract. In this paper the Adomian decomposition method is applied to the nonlinear Sturm-Liouville problem

$$-y'' + y(t)^p = \lambda y(t), \qquad y(t) > 0,\ t \in I = (0,1),\ y(0) = y(1) = 0,$$

where $p > 1$ is a constant and $\lambda > 0$ is an eigenvalue parameter. Also, the eigenvalues and the behavior of eigenfuctions of the problem are demonstrated.

1 Introduction

Recently a great deal of interest has been focused on the application of Adomian's decomposition method for the solution of many different problems. For example in [6], [12], [15]-[20] boundary value problems, algebraic equations and partial differential equations are considered. The Adomian decomposition method, which accurately computes the series solution, is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that are elegantly computed.

The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

There has been great interest in nonlinear Sturm-Liouville problems. Theory and algorithms which compute a given number of eigenvalues of the radial symmetric p-Laplacian are presented in [13], [8] and [9]. Asymptotic expansion of the eigenvalues of a specific problem is studied in [14]. In this article we explore the possibilities of the decomposition method in the nonlinear Sturm-Liouville problem [14].

Let us consider a general functional equation

$$y - N(y) = f, \tag{1}$$

2000 Mathematics Subject Classification: 34L16.
Keywords: Adomian decomposition method, nonlinear Sturm-Liouville problem.

where N is a nonlinear operator, f is a known function in which the solution y satisfying (1) is to be found. We assume that for every f, the problem (1) has a unique solution.

The Adomian's technique consists of approximating the solution of (1) as an infinite series

$$y = \sum_{n=0}^{\infty} y_n \tag{2}$$

and decomposing the nonlinear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n are Adomain polynomials of $y_0, y_1, ..., y_n$ (see [3], [4], [5]) given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left[N \left(\sum_{i=0}^{\infty} \mu^i y_i \right) \right]_{\mu=0}, \qquad n = 0, 1, 2, ... \tag{4}$$

Substituting (2) and (3) into (1) yields

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f. \tag{5}$$

Thus, we can identify

$$\begin{aligned} y_0 &= f, \\ y_{n+1} &= A_n(y_0, y_1, \ldots, y_n), \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{6}$$

We then define the M-term approximant to the solution y by

$$\phi_M[y] = \sum_{n=0}^{M} y_n$$

with

$$\lim_{M \to \infty} \phi_M[y] = y.$$

Convergence of the Adomian decomposition scheme was established by many authors by using fixed point theorems [1], [2], [5], [10].

2 Application to Sturm-Liouville Problems

We consider the nonlinear Sturm-Liouville problem

$$\begin{aligned} -y'' + y(t)^p &= \lambda y(t), \qquad t \in I = (0,1), \\ y(t) &> 0, \qquad t \in I, \\ y(0) &= y(1) = 0, \end{aligned} \tag{7}$$

where $p > 1$ is a constant and $\lambda > 0$ is an eigenvalue parameter. It is known by [7] and [11] that for each $\alpha > 0$, there exists a unique solution $(\lambda, y) = (\lambda(\alpha), y_\alpha) \in R_+ \times C^2(\bar{I})$ with $\|y_\alpha\|_2 = \alpha$.

The purpose of this paper is to explore the eigenvalues and the structure of positive eigenfunctions of (7) as indicated in [14] by diagrams that are obtained approximately by using the Adomian Decomposition method.

To begin with, (7) can be written in an operator form

$$\begin{aligned} Ly &= y(t)^p - \lambda y(t), \\ y(0) &= y(1) = 0, \end{aligned} \tag{8}$$

where $L = \frac{d^2}{dt^2}$ is the differential operator. Operating on both sides of (8) with the inverse operator of L (namely $L^{-1}[.] = \int\limits_0^t \int\limits_0^x [.] ds dx$) and using the first boundary condition $y(0) = 0$ yields

$$y(t, \lambda) = at + L^{-1} \left(y(t)^p - \lambda y(t) \right), \tag{9}$$

where $a = y'(0) \neq 0$ is not given but will be determined by using the other boundary condition. Substituting (2) and (3) into (9) gives

$$\sum_{n=0}^{\infty} y_n = at + L^{-1}\Big(\sum_{n=0}^{\infty} A_n\Big) - L^{-1}\Big(\lambda \sum_{n=0}^{\infty} y_n\Big), \tag{10}$$

where A_n are the Adomian polynomials. Identifying the zeroth component $y_0(t)$ by at, the remaining components $y_n(t)$, $n \geq 1$ can be determined by using the recurrence relation

$$\begin{aligned} y_0(t) &= at, \\ y_{k+1} &= L^{-1}(A_k - \lambda y_k), \qquad k \geq 0, \end{aligned} \tag{11}$$

where A_k are Adomian polynomials (4) involving the nonlinear term $N(y) = y^p$ and given by

$$\begin{aligned} A_0 &= N(y_0) = y_0^p, \\ A_1 &= y_1 N'(y_0) = p y_0^{p-1} y_1, \\ A_2 &= y_2 N'(y_0) + \frac{1}{2} y_1^2 N''(y_0) = p y_0^{p-1} y_2 + \frac{1}{2} p(p-1) y_0^{p-2} y_1^2, \\ A_3 &= p y_0^{p-1} y_3 + p(p-1) y_0^{p-2} y_1 y_2 + \frac{1}{6} p(p-1)(p-2) y_0^{p-3} y_1^3, \\ &\vdots \end{aligned} \tag{12}$$

Combining (11) and (12) yields

$$\begin{aligned} y_0(t) &= at, \\ y_1(t) &= \frac{a^p t^{p+2}}{(p+1)(p+2)} - \lambda a \frac{t^3}{3!}, \\ y_2(t) &= \frac{p a^{2p-1} t^{2p+3}}{(p+1)(p+2)(2p+2)(2p+3)}, \\ &\quad - \frac{\lambda a^p t^{p+4}}{(p+3)(p+4)} \left(\frac{p}{3!} + \frac{1}{(p+1)(p+2)}\right) \\ &\quad + \lambda^2 a \frac{t^5}{5!}, \\ &\quad \vdots \end{aligned} \tag{13}$$

It is in principle, possible to calculate more components in the decomposition series to enhance the approximation. In view of (13), the solution $y(t)$ is readily obtained in a series form by

$$\begin{aligned} y(t;\lambda) &= \frac{a}{\sqrt{\lambda}} \sin(\sqrt{\lambda}) t \\ &\quad + \frac{a^p t^{p+2}}{(p+1)(p+2)} \\ &\quad + \frac{p a^{2p-1} t^{2p+3}}{(p+1)(p+2)(2p+2)(2p+3)} \\ &\quad - \frac{\lambda a^p t^{p+4}}{(p+3)(p+4)} \left(\frac{p}{3!} + \frac{1}{(p+1)(p+2)}\right) \\ &\quad + \cdots \end{aligned} \tag{14}$$

The other boundary condition

$$y(1, \lambda) = 0 \tag{15}$$

gives a nonlinear equation $F(\lambda, a) = 0$, from which it is possible to obtain the branching diagram of the problem (7).

Since (7) is an autonomous and from Lemma 2.1 in [14] we know that $y(t)$ satisfies for $p > 1$

$$y(t) = y(1-t), \qquad 0 \le t \le 1.$$

It follows that

$$y'(0,\lambda) = y'(1,\lambda) \tag{16}$$

and then we have a nonlinear system of equations as

$$\begin{aligned} F(\lambda,a) &= 0, \\ G(\lambda,a) &= 0, \end{aligned} \tag{17}$$

where $G(\lambda,a) = y'(0,\lambda) - y'(1,\lambda)$. Solving the system (17) numerically, we can obtain the values of λ and a.

2.1 The linear case: $p = 1$

The iterations are then determined in the following recursive way

$$\begin{aligned} y_0 &= at, \\ y_{k+1} &= -L^{-1}((\lambda-1)y_k), \qquad k = 0,1,2,\ldots. \end{aligned} \tag{18}$$

or equivalently

$$y_{k+1}(t;\lambda) = -\int_0^t \int_0^{t_1} (\lambda-1) y_k(s;\lambda) ds dt_1. \tag{19}$$

It is clear that the convergence of the method depends on λ and the size of the norm $\| L^{-1} \|$ for the set $\{y_n\}$. In the linear case, the decomposition method is equivalent to a classical iterative method, but the a posterior calculations of $y(0)$ and $y'(0)$, by imposing to each $\sum_{n=0}^{M} y_n(t)$ to verify the boundary conditions, determines a set $\{y_n\}$ suitable for a good convergence. The recurrence relation (19) gives

$$\begin{aligned} y_0 &= at, \\ y_k(t;\lambda) &= (-1)^k \underbrace{\int_0^t \int_0^{t_{2k-1}} \ldots \int_0^{t_2} \int_0^{t_1}}_{2k \ times} (\lambda-1)^k y_0 ds dt_1 \ldots dt_{2k-2} dt_{2k-1}, \end{aligned} \tag{20}$$

that is,

$$y_k(t;\lambda) = a(-1)^k((\lambda-1)^k \frac{t^{2k+1}}{(2k+1)!}, \quad k = 0,1,\ldots. \tag{21}$$

The solution in a series form is thus given by

$$\begin{aligned} y(t;\lambda) &= a\sum_{k=0}^{\infty}(-1)^k(\lambda-1)^k\frac{t^{2k+1}}{(2k+1)!} \\ &= \frac{a}{\sqrt{\lambda-1}}\sum_{k=0}^{\infty}(-1)^k\frac{((\sqrt{\lambda-1})t)^{2k+1}}{(2k+1)!}. \end{aligned} \tag{22}$$

It is clear that the infinite series of the solution converges to

$$y(t;\lambda) = \frac{a}{\sqrt{\lambda-1}}\sin(\sqrt{\lambda-1})t, \qquad a \neq 0. \tag{23}$$

Using the boundary condition $y(1) = 0$, the eigenvalues are computed exactly

$$\lambda_n = 1 + n^2\pi^2, \qquad n = 0, 1, \dots.$$

So, the effectiveness and the usefulness of the Adomain method are demonstrated by finding exact eigenvalues and eigenfunctions of linear Sturm-Liouville problem.

2.2 The nonlinear case:$p = 2$

Branching diagram for equation (7) is shown for $M = 20$ in Figure 1. Solution curves (eigenfunctions) for the linear and nonlinear cases are shown respectively in Figure 2 and Figure 3 corresponding to various eigenvalues. All computations are performed in Mathematica 4.0. The nonlinear problem behaves quite differently from the linear problem. It can be easily seen the smoothness of the solution curves in the linear case and the shape of the positive solutions associated with various eigenvalues λ is almost "box" in nonlinear case which agrees with [14].
In the following table, the approximate eigenvalues in $[\pi^2, 4\pi^2)$ are given for $M = 20$ and the positive solutions corresponding to these eigenvalues are drawn as in Figure 3.

$a = y'(0)$	λ
2.2408×10^{-8}	9.8696
0.04597	9.88203
0.20987	9.92628
0.70005	10.0584
1.7354	10.3364
3.58172	10.8287
6.55366	11.6117
11.035	12.7717
17.4844	14.4005
26.4008	16.5798
38.3541	19.4179
50.5994	21.6492
82.3224	24.2605
92.7438	25.225

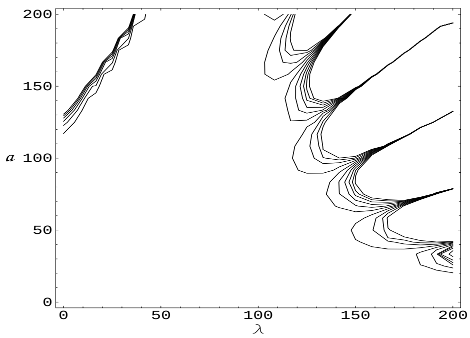

Figure 1:

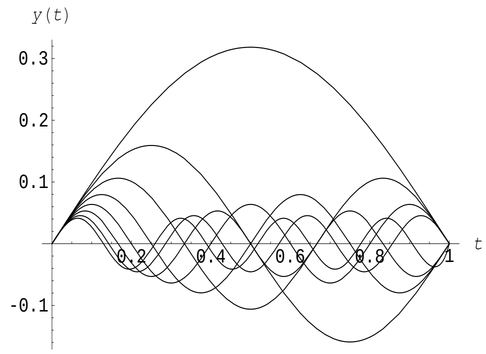

Figure 2:

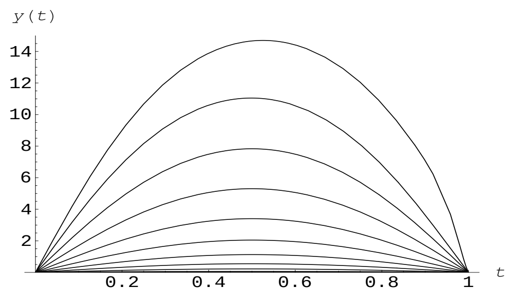

Figure 3:

References

[1] K. Abbaoui, Y. Cherruault, *Convergence of Adomian's method applied to non-linear equations*, Math. Comput. Modelling **20** (9) (1994) 60-73. MR1302630. Zbl 822.65027.

[2] K. Abbaoui, Y. Cherruault, *New ideas for proving convergence of decomposition methods*, Comput. Math. Appl. **29 (7)** (1995) 103-108. MR1321262(95k:65057). Zbl 0832.47051.

[3] G.Adomian, *Stochastic Systems, Academic Press*, New York, 1983. MR0714710(86d:93001). Zbl 0523.60056.

[4] G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, New York, 1986. MR0872695(88j:60112). Zbl 0609.60072.

[5] G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer Academic Publishers, Dordrecht, 1989. MR0982493(91e:93081). Zbl 0659.93003.

[6] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Dordrecht, 1994. MR1282283(95e:00026). Zbl 0802.65122.

[7] H. Berestycki, *Le nombre de solutions de certains problemes semi-lineares elliptiques*, J. Funct. Anal. **40** (1981) 1-29. MR0607588(82k:35033).

[8] B. M. Brown, W. Reichel, *Computing eigenvalues and Fucik-spectrum of the radially symmetric p-Laplacian*, J. Comp. Appl. Math., **148**, (2002) 183-211. MR1946195(2004a:65145). Zbl 1020.65083.

[9] B. M. Brown, W. Reichel, *Eigenvalues of the radially symmetric p-Laplacian in R^n*, J. London Math. Soc. (2), **69(3)**, (2004) 657-675. MR2050039(2004m:34057). Zbl 1058.34114.

[10] Y. Cherruault, G. Adomian, *Decomposition methods: a new proof of convergence*, Math. Comput. Modelling **18 (12)** (1993) 103-106. MR1262286(94k:65083). Zbl 0805.65057.

[11] J. M. Fraile, J. Lopez-Gomez, and J. C. Sabina de Lis, *On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems*, J. Differential Equations **123** (1995) 180-212. MR1359917(96j:35073). Zbl 0847.35050.

[12] W. Al-Hayani, L. Casasus, *The Adomian decomposition method in turning point problems*, J. Comput. and Appl. Math. **177** (2005) 187-203. MR2118667(2005j:65076). Zbl 1062.65076.

[13] W. Reichel, W. Walter, *Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions*, J. Differential Equations, **156(1)** (1999) 50-70. MR1701814(2000e:34036). Zbl 0931.34059.

[14] T. Shibata, *Precise Spectral Asymptotics for Nonlinear Sturm-Liouville Problems*, J. Differential Equations, **180**, (2002) 374-394. MR1894177(2002m:34124). Zbl 0452.35038.

[15] A. M. Wazwaz, *A reliable modification of Adomian decomposition method*, Appl. Math. Comput. **102** (1999) 77-86. MR1682855(99m:65156). Zbl 0928.65083.

[16] A. M. Wazwaz, *A new algorithm for calculating Adomian polynomials for non-linear operators*, Applied Mathematics and Computation **111** (2000) 53-69. MR1745908. Zbl 1023.65108.

[17] A. M. Wazwaz, *The decomposition method for solving the diffusion equation subject to the classification of mass*, IJAM **3 (1)** (2000) 25-34. MR1774083(2001c:35103). Zbl 1052.35049.

[18] A. M. Wazwaz, *Exact solutions to nonlinear diffusion equations obtained by the decomposition method*, Appl. Math. Comput. **123** (2001) 109-122. MR1846715(2003d:35031). Zbl 1027.35019.

[19] A. M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Emden-Fowler equation*, Appl. Math. Comput. **161** (2005) 543-560. MR2112423(2005h:65125). Zbl 1061.65064.

[20] A. M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Bratu-type eqautions*, Appl. Math. Comput. **166** (2005) 652-663. MR2151056. Zbl 1073.65068.

Sennur Somali
Dokuz Eylul University
Faculty of Arts and Sciences
Department of Mathematics,
Tinaztepe Kampüsü, 35160 Buca, Izmir,
Turkey.
e-mail: sennur.somali@deu.edu.tr

Guzin Gokmen
Dokuz Eylul University
Faculty of Arts and Sciences
Department of Mathematics,
Tinaztepe Kampüsü, 35160 Buca, Izmir,
Turkey.
e-mail:guzin.gokmen@deu.edu.tr