Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843 - 7265 (print)
Volume **2** (2007), 113 – 122

CHARACTERIZATION OF THE ORDER RELATION ON THE SET OF COMPLETELY n-POSITIVE LINEAR MAPS BETWEEN C^*-ALGEBRAS

Maria Joiţa, Tania-Luminiţa Costache and Mariana Zamfir

Abstract. In this paper we characterize the order relation on the set of all nondegenerate completely n-positive linear maps between C^*-algebras in terms of a self-dual Hilbert module induced by each completely n-positive linear map.

1 Introduction and preliminaries

Completely positive linear maps are an often used tool in operator algebras theory and quantum information theory. The theorems on the structure of completely linear maps and Radon-Nikodym type theorems for completely positive linear maps are an extremely powerful and veritable tool for problems involving characterization and comparison of quantum operations (that is, completely positive linear maps between the algebras of observables (C^*-algebras) of the physical systems under consideration).

Given a C^*-algebra A and a positive integer n, we denote by $M_n(A)$ the C^*-algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A.

Let A and B be two C^*-algebras. A linear map $\rho\colon A \to B$ is *completely positive* if the linear maps $\rho^{(n)}\colon M_n(A) \to M_n(B)$ defined by

$$\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$$

are all positive, for any positive integer n.

The set of all completely positive linear maps from A to B is denoted by $CP_\infty(A, B)$.

An $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ of linear maps from A to B can be regarded as a linear map ρ from $M_n(A)$ to $M_n(B)$ defined by

$$\rho([a_{ij}]_{i,j=1}^n) = [\rho_{ij}(a_{ij})]_{i,j=1}^n.$$

2000 Mathematics Subject Classification: 46L05; 46L08
Keywords: Hilbert module, C^*-algebra, completely n -positive linear map, extreme points
This research was supported by CNCSIS grant code A 1065/2006.

We say that $[\rho_{ij}]_{i,j=1}^n$ is a *completely n-positive linear map* from A to B if ρ is a completely positive linear map from $M_n(A)$ to $M_n(B)$.

The set of all completely n-positive linear maps from A to B is denoted by $CP_\infty^n(A,B)$.

In [8], Suen showed that any completely n-positive linear map from a C^*-algebra A to $L(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H, is of the form $[V^*T_{ij}\Phi(\cdot)V]_{i,j=1}^n$, where Φ is a representation of A on a Hilbert space K, $V\in L(H,K)$ and $[T_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi(A)')$ ($\Phi(A)'$ denotes the commutant of $\Phi(A)$ in $L(K)$). In [4] we characterized the order relation on the set of all completely n-positive linear maps from A to $L(H)$ in terms of the representation associated with each completely n-positive linear map by Suen's construction.

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

A *Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E\times E\to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:

1. $\langle\xi,\eta\rangle^*=\langle\eta,\xi\rangle$ for every $\xi,\eta\in E$;
2. $\langle\xi,\xi\rangle\geq 0$ for every $\xi\in E$;
3. $\langle\xi,\xi\rangle=0$ if and only if $\xi=0$,

and which is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\|=\sqrt{\|\langle\xi,\xi\rangle\|}$.

Given two Hilbert C^*-modules E and F over a C^*-algebra A, the Banach space of all bounded module morphisms from E to F is denoted by $B_A(E,F)$. The subset of $B_A(E,F)$ consisting of all adjointable module morphisms from E to F (that is, $T\in B_A(E,F)$ such that there is $T^*\in B_A(F,E)$ satisfying $\langle\eta,T\xi\rangle=\langle T^*\eta,\xi\rangle$ for all $\xi\in E$ and for all $\eta\in F$) is denoted by $L_A(E,F)$. We will write $B_A(E)$ for $B_A(E,E)$ and $L_A(E)$ for $L_A(E,E)$.

In general $B_A(E,F)\neq L_A(E,F)$. So the theory of Hilbert C^*-modules is different from the theory of Hilbert spaces.

Any C^*-algebra A is a Hilbert C^*-module over A with the inner product defined by $\langle a,b\rangle=a^*b$ and the C^*-algebra of all adjointable module morphisms on A is isomorphic with the multiplier algebra $M(A)$ of A (see, for example, [5]).

The Banach space $E^\sharp$ of all bounded module morphisms from E to A becomes a right A-module with the action of A on $E^\sharp$ defined by $(aT)(\xi)=a^*(T\xi)$ for all $a\in A, T\in E^\sharp$ and $\xi\in E$. We say that E is *self-dual* if $E^\sharp=E$ as right A-modules.

If E and F are self-dual, then $B_A(E,F)=L_A(E,F)$ [7, Proposition 3.4].

Suppose that A is a W^*-algebra. Then the A-valued inner product on E extends to an A-valued inner product on $E^\sharp$ and in this way $E^\sharp$ becomes a self-dual Hilbert A-module [7, Theorem 3.2].

Let E be a Hilbert C^*-module over a C^*-algebra A. The algebraic tensor product $E\otimes_{\text{alg}}A^{**}$, where A^{**} is the enveloping W^*-algebra of A, becomes a right A^{**}-module if we define $(\xi \otimes b)\, c = \xi \otimes bc$, for $\xi \in E$ and $b, c \in A^{**}$.

The map $[\cdot,\cdot] : (E \otimes_{\text{alg}} A^{**}) \times (E \otimes_{\text{alg}} A^{**}) \to A^{**}$ defined by

$$\left[\sum_{i=1}^{n} \xi_i \otimes b_i, \sum_{j=1}^{m} \eta_j \otimes c_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i^* \left\langle \xi_i, \eta_j\right\rangle c_j$$

is an A^{**}-valued inner product on $E \otimes_{\text{alg}} A^{**}$ and the quotient module $(E \otimes_{\text{alg}} A^{**})/N_E$, where $N_E = \{\zeta \in E \otimes_{\text{alg}} B^{**}; [\zeta,\zeta] = 0\}$, becomes a pre-Hilbert A^{**}-module with the inner product defined by

$$\left\langle \sum_{i=1}^{n} \xi_i \otimes b_i + N_E, \sum_{j=1}^{m} \eta_j \otimes c_j + N_E \right\rangle_{A^{**}} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i^* \left\langle \xi_i, \eta_j\right\rangle c_j.$$

The Hilbert C^*-module $\overline{E \otimes_{\text{alg}} A^{**}/N_E}$ obtained by the completion of $(E\otimes_{\text{alg}}A^{**})/N_E$ with respect to the norm induced by the inner product $\langle\cdot,\cdot\rangle_{A^{**}}$ is called the extension of E by the C^*-algebra A^{**}. Moreover, E can be regarded as an A-submodule of $\overline{E \otimes_{\text{alg}} A^{**}/N_E}$, since the map $\xi \mapsto \xi \otimes 1 + N_E$ from E to $\overline{E \otimes_{\text{alg}} B^{**}/N_E}$ is an isometric inclusion. The self-dual Hilbert A^{**}-module $\left(\overline{E \otimes_{\text{alg}} B^{**}/N_E}\right)^{\#}$ is denoted by $\widetilde{E}$, and we can consider E as embedded in $\widetilde{E}$ without making distinction [6, 7, 9].

If $T \in B_A(E,F)$, then T extends uniquely to a bounded module morphism $\widehat{T}$ from $\overline{E \otimes_{\text{alg}} A^{**}/N_E}$ to $\overline{F \otimes_{\text{alg}} A^{**}/N_F}$ such that

$$\widehat{T}\left(\sum_{i=1}^{m} \xi_i \otimes b_i + N_E\right) = \sum_{i=1}^{m} T\xi_i \otimes b_i + N_F$$

and $\|T\| = \left\|\widehat{T}\right\|$ and, by [7, Proposition 3.6], $\widehat{T}$ extends uniquely to a bounded module morphism $\widetilde{T}$ from $\widetilde{E}$ to $\widetilde{F}$ such that $\|T\| = \left\|\widetilde{T}\right\|$. Moreover, $\widetilde{TS} = \widetilde{T}\widetilde{S}$ for all $T \in B_A(E,F)$ and $S \in B_A(F,E)$, and if $T \in L_A(E,F)$, then $\widetilde{T^*} = \widetilde{T}^*$.

A representation of a C^*-algebra A on a Hilbert C^*-module E over B is a $*$-morphism Φ from A to $L_B(E)$. Moreover, any representation Φ of a C^*-algebra A on a Hilbert C^*-module E over a C^*-algebra B induces a representation $\widetilde{\Phi}$ of A on $\widetilde{E}$ defined by $\widetilde{\Phi}\,(a) = \widetilde{\Phi\,(a)}$ for all $a \in A$.

A completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^{n}$ from A to $L_B(E)$, where E is a Hilbert module over a C^*-algebra B, is nondegenerate if for some approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ for A, the nets $\{\rho_{ii}\,(e_\lambda)\,\xi\}_{\lambda\in\Lambda}$, $i \in \{1,...,n\}$ are convergent to ξ for all $\xi \in E$ [5]. Using the theory of Hilbert C^*-modules, in [3] we extended the construction of

Suen for unital completely n-positive linear maps between unital C^*-algebras. Thus, we showed that any completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ is of the form $[\rho_{ij}(\cdot)]_{i,j=1}^n = \left[\widetilde{V_\rho}^* T_{ij}^\rho \widetilde{\Phi_\rho}(\cdot)\widetilde{V_\rho}\Big|_E\right]_{j,i=1}^n$, where Φ_ρ is a representation of A on a Hilbert B-module E_ρ, V_ρ is an element in $L_B(E, E_\rho)$ and $T^\rho = \left[T_{ij}^\rho\right]_{j,i=1}^n$ is an element in $M_n\left(\widetilde{\Phi_\rho}(A)'\right)$ with the property that $\widetilde{V_\rho}^* T_{ij}^\rho \widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E \in L_B(E)$ for all $a \in A$ and for all $i, j \in \{1, 2, ..., n\}$. Moreover, $\{\Phi_\rho(a) V_\rho \xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ, $\sum_{i=1}^n T_{ii}^\rho = n\mathrm{id}_{\widetilde{E_\rho}}$ and $\langle T^\rho((\eta_k)_{k=1}^n), (\eta_k)_{k=1}^n\rangle_{B^{**}} \geq 0$ for all $\eta_1, ..., \eta_n \in E_\rho$ [3, Theorem 2.2]. The quadruple $(\Phi_\rho, V_\rho, E_\rho, T^\rho)$ is unique up to unitary equivalence. From the proof of [3, Theorem 2.2], we deduce that $(\Phi_\rho, V_\rho, E_\rho)$ is the KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction [5, Theorem 5.6] associated with the unital completely positive linear map $\widetilde{\rho} = \frac{1}{n}\sum_{i=1}^n \rho_{ii}$. If the completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ is nondegenerate, then $\widetilde{\rho}$ is nondegenerate and it is not difficult to check that the above construction associated with a unital completely n-positive linear map is still valid for nondegenerate completely n-positive linear maps. In this paper we characterize the order relation on the set of all nondegenerate completely n-positive linear maps between C^*-algebras in terms of a self-dual Hilbert module induced by each completely n-positive linear map.

2 The main results

For an element $T \in L_{B^{**}}(\widetilde{E})$ we denote by $T|_E$ the restriction of the map T on E.

Let $\rho \in CP_\infty^n(A, L_B(E))$. We denote by $C(\rho)$ the C^*-subalgebra of $M_n(L_{B^{**}}(\widetilde{E_\rho}))$ generated by $\{S = [S_{ij}]_{i,j=1}^n \in M_n(L_{B^{**}}(\widetilde{E_\rho})); \widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E \in L_B(E)$, $S_{ij}\widetilde{\Phi_\rho}(a) = \widetilde{\Phi_\rho}(a)S_{ij}, \forall a \in A, \forall i, j \in \{1, \ldots n\}\}$.

Lemma 1. *Let $S = [S_{ij}]_{i,j=1}^n$ be an element in $C(\rho)$ such that*

$$\langle S((\xi_i)_{i=1}^n), (\xi_i)_{i=1}^n\rangle_{B^{**}} \geq 0$$

for all $\xi_1, ..., \xi_n$ in E_ρ. Then the map $\rho_S = [(\rho_S)_{ij}]_{i,j=1}^n$ from $M_n(A)$ to $M_n(L_B(E))$ defined by

$$\rho_S([a_{ij}]_{i,j=1}^n) = \left[\widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a_{ij})\widetilde{V_\rho}\Big|_E\right]_{i,j=1}^n$$

is a completely n-positive linear map from A to $L_B(E)$.

Proof. It is not difficult to see that ρ_S is an $n\times n$ matrix of continuous linear maps from A to $L_B(E)$, the (i,j)-entry of the matrix ρ_S is the linear map $(\rho_S)_{ij}$ from A to $L_B(E)$ defined by

$$(\rho_S)_{ij}(a) = \widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E$$

To show that ρ_S is a completely n-positive linear map from A to $L_B(E)$ it is sufficient, by [2, Theorem 1.4], to prove that $\Gamma(\rho_S)\in CP_\infty(A,M_n(L_B(E)))$, where Γ is the isomorphism from $CP^n_\infty(A,L_B(E))$ onto $CP_\infty(A,M_n(L_B(E)))$ defined by

$$\Gamma([\rho_{ij}]^n_{i,j=1})(a) = [\rho_{ij}(a)]^n_{i,j=1}$$

for all $a\in A$.

Let a and b be two elements in A and let $\xi_{1,...,}\xi_n$ be n elements in E. We have

$$\begin{aligned}
\Gamma(\rho_S)(a^*b)\left((\xi_i)^n_{i=1}\right) &= \left[\widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a^*b)\widetilde{V_\rho}\Big|_E\right]^n_{i,j=1}\left((\xi_i)^n_{i=1}\right)\\
&= \left[\widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a^*b)\widetilde{V_\rho}\right]^n_{i,j=1}\left((\xi_i)^n_{i=1}\right)\\
&= \left[\widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a^*)\widetilde{\Phi_\rho}(b)\widetilde{V_\rho}\right]^n_{i,j=1}\left((\xi_i)^n_{i=1}\right)\\
&= \left[\widetilde{V_\rho}^* \widetilde{\Phi_\rho}(a)^* S_{ij}\widetilde{\Phi_\rho}(b)\widetilde{V_\rho}\right]^n_{i,j=1}\left((\xi_i)^n_{i=1}\right)\\
&= (M_a)^* S M_b\left((\xi_i)^n_{i=1}\right),
\end{aligned}$$

where

$$M_a = \begin{bmatrix} \widetilde{\Phi_\rho}(a)\widetilde{V_\rho} & 0 & ... & 0\\ 0 & \widetilde{\Phi_\rho}(a)\widetilde{V_\rho} & ... & 0\\ \cdot & \cdot & ... & \cdot\\ 0 & 0 & ... & \widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\end{bmatrix}$$

Let $a_1,\dots,a_m$ be m elements in A, let $T_1,\dots,T_m$ be m elements in $M_n(L_B(E))$ and let $\xi_1,...,\xi_n$ be n elements in E. We have

$$
\begin{aligned}
&\left\langle \sum_{k,l=1}^{n} \left(T_l^* \Gamma(\rho_S)(a_l^* a_k) T_k\right) \left((\xi_i)_{i=1}^n\right), \left((\xi_i)_{i=1}^n\right) \right\rangle \\
&= \sum_{k,l=1}^{n} \left\langle \left(\left(M_{a_l}\right)^* S M_{a_k} T_k\right) \left((\xi_i)_{i=1}^n\right), T_l \left((\xi_i)_{i=1}^n\right) \right\rangle \\
&= \sum_{k,l=1}^{n} \left\langle \left(\left(M_{a_l}\right)^* S M_{a_k} T_k\right) \left((\xi_i)_{i=1}^n\right), T_l \left((\xi_i)_{i=1}^n\right) \right\rangle_{B^{**}} \\
&= \sum_{k,l=1}^{n} \left\langle \left(\left(M_{a_l}\right)^* S M_{a_k} \widetilde{T_k}\right) \left((\xi_i)_{i=1}^n\right), \widetilde{T}_l \left((\xi_i)_{i=1}^n\right) \right\rangle_{B^{**}} \\
&= \left\langle S \left(\sum_{k=1}^{n} M_{a_k} \widetilde{T_k}\right) \left((\xi_i)_{i=1}^n\right), \left(\sum_{l=1}^{n} M_{a_l} \widetilde{T}_l\right) \left((\xi_i)_{i=1}^n\right) \right\rangle_{B^{**}} \geq 0
\end{aligned}
$$

since $S = [S_{ij}]_{i,j=1}^n$ is an element in $C(\rho)$ such that $\langle S\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \geq 0$ for all $\eta_1, ..., \eta_n$ in E_ρ. This implies that $\Gamma(\rho_S) \in CP_\infty(A, M_n(L_B(E)))$ and the lemma is proved. $\square$

Remark 2. *1. By [4, the proof of Theorem 2.2], $\langle T^\rho\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \geq 0$ for all $\eta_1, ..., \eta_n$ in E_ρ, and so $\rho_{T^\rho} \in CP_\infty^n(A, L_B(E))$. Moreover, $\rho_{T^\rho} = \rho$.*

2. If S_1 and S_2 are two elements in $C(\rho)$ such that $\langle S_k\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \geq 0$ for all $\eta_1, ..., \eta_n$ in E_ρ, and for all $k = 1, 2$, then $\rho_{S_1+S_2} = \rho_{S_1} + \rho_{S_2}$.

3. If S is an element in $C(\rho)$ such that $\langle S\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \geq 0$ for all $\eta_1, ..., \eta_n$ in E_ρ, and α is a positive number, then $\rho_{\alpha S} = \alpha \rho_S$.

4. If S_1 and S_2 are two elements in $C(\rho)$ such that

$$0 \leq \langle S_1\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \leq \langle S_2\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}}$$

for all $\eta_1, ..., \eta_n$ in E_ρ, then $\rho_{S_1} \leq \rho_{S_2}$.

Let $\rho = [\rho_{ij}]_{i,j=1}^n \in CP_\infty^n(A, L_B(E))$. We denote by $[0, \rho]$ the set of all completely n-positive linear maps $\theta = [\theta_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ such that $\theta \leq \rho$ (that is, $\rho - \theta \in CP_\infty^n(A, L_B(E))$) and by $[0, T^\rho]$ the set of all elements $S = [S_{ij}]_{i,j=1}^n$ in $C(\rho)$ such that

$$\langle (T^\rho - S)\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n \rangle_{B^{**}} \geq 0$$

for all $\eta_1, ..., \eta_n$ in E_ρ.

The following theorem gives a characterization of the order relation on the set of completely n-positive linear maps between C^*-algebras in terms of the representation associated to each completely n-positive linear map by [3, Theorem 2.2].

Theorem 3. *The map $S \to \rho_S$ from $[0, T^\rho]$ to $[0, \rho]$ is an affine order isomorphism.*

Proof. By Lemma 1 and Remark 2, we get that the map $S \to \rho_S$ from $[0, T_\rho]$ to $[0, \rho]$ is well-defined, affine and preserves the order relation.

We show that this map is injective. Let $S = [S_{ij}]_{i,j=1}^n \in [0, T^\rho]$ such that $\rho_S = 0$. Then by definition of ρ_S, $\left.\widetilde{V_\rho}^* S_{ij} \widetilde{\Phi_\rho}(a) \widetilde{V_\rho}\right|_E = 0$ for all $a \in A$ and $i, j \in \{1, \dots, n\}$.

Let $i, j \in \{1, \dots, n\}$. Then we have

$$\begin{aligned}
\langle S_{ij}\Phi_\rho(a)V_\rho\xi, \Phi_\rho(b)V_\rho\eta\rangle_{B^{**}} &= \left\langle S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\xi, \widetilde{\Phi_\rho}(b)\widetilde{V_\rho}\eta\right\rangle_{B^{**}} \\
&= \left\langle \widetilde{V_\rho}^*\widetilde{\Phi_\rho}(b)^* S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\xi, \eta\right\rangle_{B^{**}} \\
&= \left\langle \widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(b)^*\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\xi, \eta\right\rangle_{B^{**}} \\
&= \left\langle \widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(b^*a)\widetilde{V_\rho}\xi, \eta\right\rangle_{B^{**}} = 0
\end{aligned}$$

for all $a, b \in A$ and for all $\xi, \eta \in E$. From this fact and taking into account that $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ and $S_{ij}|_E$ is a bounded module morphism from E to $E^\#$ [2, Remark 2.2], we deduce that $S_{ij}|_E = 0$ and then, by [9, pp. 442-443], $S_{ij} = 0$, since $S_{ij}|_E$ is a bounded module morphism from E to $E^\#$ and since $S_{ij}|_E = 0$. Therefore $S = 0$.

We prove now that the map $S \to \rho_S$ is surjective. Let $\sigma = [\sigma_{kl}]_{k,l=1}^n$ be an element in $[0, \rho]$. Then $\widetilde{\rho} - \widetilde{\sigma} \in CP_\infty^n(A, L_B(E))$, where $\widetilde{\rho} = \frac{1}{n}\sum_{j=1}^n \rho_{jj}$ and $\widetilde{\sigma} = \frac{1}{n}\sum_{j=1}^n \sigma_{jj}$. Let $(\Phi_\rho, E_\rho, V_\rho, T^\rho)$ and $(\Phi_\theta, E_\theta, V_\theta, T^\theta)$ be the constructions associated with ρ respectively θ by [3, Theorem 2.2]. Then $(\Phi_\rho, E_\rho, V_\rho)$ and $(\Phi_\theta, E_\theta, V_\theta)$ are the KSGNS constructions associated with $\widetilde{\rho}$ respectively $\widetilde{\sigma}$ [5, Theorem 5.6]. Since $\widetilde{\rho} - \widetilde{\sigma} \in CP_\infty^n(A, L_B(E))$, there is a bounded module morphism $W : E_\rho \to E_\theta$ such that

$$W(\Phi_\rho(a)V_\rho\xi) = \Phi_\theta(a)V_\theta\xi$$

for all $a \in A$ and for all $\xi \in E$. Moreover, $W\,\Phi_\rho(a) = \Phi_\theta(a)W$ for all $a \in A$, and $WV_\rho = V_\theta$ [2, the proof of Theorem 2.6]. Let $S = \left[\widetilde{W}^* T_{ij}^\theta \widetilde{W}\right]_{i,j=1}^n$. Clearly, $S \in M_n(\widetilde{\Phi_\rho}(A)')$. From

$$\begin{aligned}
\left.\widetilde{V_\rho}^* S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\right|_E &= \left.\widetilde{V_\rho}^*\widetilde{W}^* T_{ij}^\theta \widetilde{W}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\right|_E \\
&= \left.\widetilde{V_\theta}^* T_{ij}^\theta \widetilde{\Phi_\theta}(a)\widetilde{V_\theta}\right|_E
\end{aligned}$$

for all $i, j \in \{1, ..., n\}$ and taking into account that $\left[\left.\widetilde{V_\theta}^* T_{ij}^\theta \widetilde{\Phi_\theta}(a)\widetilde{V_\theta}\right|_E\right]_{i,j=1}^n \in M_n(L_B(E))$, we deduce that $\left[\widetilde{V_\rho}^{*} S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E\right]_{i,j=1}^n \in M_n(L_B(E))$. Therefore, $S \in C(\rho)$. Moreover,

$$\langle S\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n\rangle_{B^{**}} \geq 0$$

for all $\eta_1, ..., \eta_n \in E_\rho$, since

$$\begin{aligned}
&\left\langle S\left(\left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right), \left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{W}^{*}T_{ij}^{\theta}\widetilde{W}\widetilde{\Phi_\rho}\left(a_j\right)\widetilde{V_\rho}\right]_{i,j=1}^n \left(\left(\xi_k\right)_{k=1}^n\right), \left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{V_\rho}^{*}\widetilde{\Phi_\rho}\left(a_i\right)^{*}\widetilde{W}^{*}T_{ij}^{\theta}\widetilde{W}\widetilde{\Phi_\rho}\left(a_j\right)\widetilde{V_\rho}\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{V_\theta}^{*}\widetilde{\Phi_\theta}\left(a_i\right)^{*}T_{ij}^{\theta}\widetilde{\Phi_\theta}\left(a_j\right)\widetilde{V_\theta}\right]_{i,j=1}^n \left(\xi_k\right)_{k=1}^n, \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[T_{ij}^{\theta}\right]_{i,j=1}^n \left(\left(\Phi_\theta\left(a_k\right)V_\theta\xi_k\right)_{k=1}^n\right), \left(\Phi_\theta\left(a_k\right)V_\theta\xi_k\right)_{k=1}^n\right\rangle_{B^{**}} \geq 0
\end{aligned}$$

for all $a_1, ..., a_n \in A$ and for all $\xi_1, ..., \xi_n \in E$, and since $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ. From

$$\begin{aligned}
\left(\rho_S\right)_{ij}(a) &= \widetilde{V_\rho}^{*}S_{ij}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E\\
&= \widetilde{V_\rho}^{*}\widetilde{W}^{*}T_{ij}^{\theta}\widetilde{W}\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}\Big|_E\\
&= \widetilde{V_\theta}^{*}T_{ij}^{\theta}\widetilde{\Phi_\theta}(a)\widetilde{V_\theta}\Big|_E = \sigma_{ij}(a)
\end{aligned}$$

for all $i, j \in \{1, 2, ..., n\}$ and for all $a \in A$, we deduce that $\sigma = \rho_S$. To proof the surjectivity of the map $S \to \rho_S$ it remained to show that

$$\langle (T^\rho - S)\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n\rangle_{B^{**}} \geq 0$$

for all $\eta_1, ..., \eta_n$ in E_ρ. We have

$$\begin{aligned}
&\left\langle S\left(\left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right), \left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{V_\rho}^{*}\widetilde{\Phi_\rho}\left(a_i\right)^{*}S_{ij}\widetilde{\Phi_\rho}\left(a_j\right)\widetilde{V_\rho}\right]_{i,j=1}^n \left(\left(\xi_k\right)_{k=1}^n\right), \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{V_\rho}^{*}S_{ij}\widetilde{\Phi_\rho}\left(a_i^{*}a_j\right)\widetilde{V_\rho}\right]_{i,j=1}^n \left(\left(\xi_k\right)_{k=1}^n\right), \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \sigma\left(\left[a_i^{*}a_j\right]_{i,j=1}^n\right)\left(\left(\xi_k\right)_{k=1}^n\right), \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&\leq \left\langle \rho\left(\left[a_i^{*}a_j\right]_{i,j=1}^n\right)\left(\left(\xi_k\right)_{k=1}^n\right), \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[\widetilde{V_\rho}^{*}\widetilde{\Phi_\rho}\left(a_i\right)^{*}T_{ij}^{\rho}\widetilde{\Phi_\rho}\left(a_j\right)\widetilde{V_\rho}\right]_{i,j=1}^n \left(\left(\xi_k\right)_{k=1}^n\right), \left(\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}\\
&= \left\langle \left[T_{ij}^{\rho}\right]_{i,j=1}^n \left(\left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right), \left(\Phi_\rho\left(a_k\right)V_\rho\xi_k\right)_{k=1}^n\right\rangle_{B^{**}}
\end{aligned}$$

for all $a_1, ..., a_n$ in A and for all $\xi_1, ..., \xi_n \in E$. From this fact and taking into account that $\{\Phi_\rho(a) V_\rho \xi; a \in A \text{ and } \xi \in E\}$ spans a dense subspace in E_ρ, we deduce that

$$\langle S\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n\rangle_{B^{**}} \leq \langle T^\rho\left((\eta_i)_{i=1}^n\right), (\eta_i)_{i=1}^n\rangle_{B^{**}}$$

for all $\eta_1, ..., \eta_n$ in E_ρ, and thus the theorem is proved. □

Definition 4. *Let A and B be two C^*-algebras and let E be a Hilbert C^*-module over B. A completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ is said to be pure if for every completely n-positive linear map $\theta = [\theta_{ij}]_{i,j=1}^n \in [0, \rho]$, there is a positive number α such that $\theta = \alpha\rho$.*

Corollary 5. *Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $CP_\infty^n(A, L_B(E))$. Then ρ is pure if and only if $[0, T^\rho] = \{\alpha T^\rho; 0 \leq \alpha \leq 1\}$.*

Let A be a unital C^*-algebra, let B be a C^*-algebra and let E be a Hilbert B-module. We denote by $CP_\infty^n(A, L_B(E), T^0)$, where $T^0 = [\widetilde{V_\rho}^* T_{ij}^\rho \widetilde{V_\rho}]_{i,j=1}^n$, the set of all completely n-positive linear maps $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ such that $\rho(I_n) = T^0$, where I_n is an $n \times n$ matrix with all the entries equal to 1_A, the unity of A.

Proposition 6. *Let $\rho = [\rho_{ij}]_{i,j=1}^n \in CP_\infty^n(A, L_B(E), T^0)$. If the map $S = [S_{ij}]_{i,j=1}^n \longrightarrow [\widetilde{V_\rho}^* S_{ij} \widetilde{V_\rho}]_{i,j=1}^n$ from $C(\rho)$ to $M_n(L_{B^{**}}(\widetilde{E}))$ is injective, then ρ is an extreme point in the set $CP_\infty^n(A, L_B(E), T^0)$.*

Proof. Let $\theta = [\theta_{ij}]_{i,j=1}^n, \sigma = [\sigma_{ij}]_{i,j=1}^n \in CP_\infty^n(A, L_B(E), T^0)$ and $\alpha \in (0,1)$ such that $\alpha\theta + (1-\alpha)\sigma = \rho$. Since $\alpha\theta \in [0, \rho]$, by Theorem 1, there is an element $S_1 = [S_{ij}^1]_{i,j=1}^n \in [0, T^\rho]$ such that $\alpha\theta = \rho_{S_1}$. Then

$$\begin{aligned}
\widetilde{V_\rho}^*(S_{ij}^1 - \alpha T_{ij}^\rho)\widetilde{V_\rho}\Big|_E &= \widetilde{V_\rho}^* S_{ij}^1 \widetilde{V_\rho}\Big|_E - \alpha \widetilde{V_\rho}^* T_{ij}^\rho \widetilde{V_\rho}\Big|_E \\
&= (\rho_{S_1})_{ij}(1_A) - \alpha\rho_{ij}(1_A) \\
&= \alpha\theta_{ij}(1_A) - \alpha\rho_{ij}(1_A) \\
&= \alpha T_{ij}^0 - \alpha T_{ij}^0 = 0
\end{aligned}$$

From this fact and taking into account that $\widetilde{V_\rho}^*(S_{ij}^1 - \alpha T_{ij}^\rho)\widetilde{V_\rho}\Big|_E$ is an element in $L_B(E)$, we conclude that $\widetilde{V_\rho}^*(S_{ij}^1 - \alpha T_{ij}^\rho)\widetilde{V_\rho} = 0$ for all $i, j \in \{1, 2, \ldots n\}$ and since the map $S = [S_{ij}]_{i,j=1}^n \longrightarrow [\widetilde{V_\rho}^* S_{ij} \widetilde{V_\rho}]_{i,j=1}^n$ is injective, $S_1 = \alpha T^\rho$. Thus we showed that $\theta = \rho$. In the same way we prove that $\sigma = \rho$ and so ρ is an extreme point in $CP_\infty^n(A, L_B(E), T^0)$. □

References

[1] J. Heo, *Completely multi-positive linear maps and representation on Hilbert C^*-modules*, J. Operator Theory **41** (1999), 3-22. MR1675235(2000a:46103). Zbl 0994.46019.

[2] M. Joiţa, *A Radon-Nikodym theorem for completely multi- positive linear maps and its applications*, Topological algebras and applications, Contemp. Math. **427** (2007), 235–245. MR2326359. Zbl pre05180876.

[3] M. Joiţa, T.-L. Costache, M. Zamfir, *Representations associated with completely n-positive linear maps between C^*-algebras*, Stud. Cercet. Stiint., Ser. Mat. **16** (2006), Supplement Proceedings of ICMI 45, Bacău, Sept. 18-20, 2006, 111-122. MR2318951.

[4] M. Joiţa, T.-L. Costache, M. Zamfir, *On the order structure on the set of completely multi-positive linear maps on C^*-algebras* (submitted).

[5] E. C. Lance, *Hilbert C^*- modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge, 1995. MR1325694(96k:46100). Zbl 0822.46080.

[6] H. Lin, *Bounded module maps and pure completely positive maps*, J. Operator Theory **26** (1991), 121-139. MR1214924(92h:35103). Zbl 0791.46032.

[7] W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443-468. MR0355613(50 #8087). Zbl 0239.46062.

[8] C. Y. Suen, *An $n \times n$ matrix of linear maps of a C^*-algebra*, Proc. Amer. Math. Soc. **112** (1991), no.3, 709-712. MR1069296(92a:46069). Zbl 0755.46024.

[9] S. K. Tsui, *Completely positive module maps and completely positive extreme maps*, Proc. Amer. Math. Soc. **124** (1996), 437-445. MR1301050(96d:46074). Zbl 0846.46037.

Maria Joiţa
Department of Mathematics,
Faculty of Chemistry,
University of Bucharest,
Romania.
e-mail: mjoita@fmi.unibuc.ro

Tania- Luminita Costache
Faculty of Applied Sciences,
University "Politehnica",
Bucharest,
Romania.
e-mail: lumycos@yahoo.com

Mariana Zamfir
Department of Mathematics and Informatics
Technical University of Civil Engineering Bucharest
Romania.
e-mail: zacos@k.ro