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FAMILIES OF QUASI-PSEUDO-METRICS GENERATED BY PROBABILISTIC QUASI-PSEUDO-METRIC SPACES

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Abstract. This paper contains a study of families of quasi-pseudo-metrics (the concept of a quasi-pseudo-metric was introduced by Wilson [22], Albert [1] and Kelly [9]) generated by probabilistic quasi-pseudo-metric-spaces which are generalization of probabilistic metric space (PM-space shortly) [2, 3, 4, 6]. The idea of PM-spaces was introduced by Menger [11, 12], Schweizer and Sklar [18] and Serstnev [19]. Families of pseudo-metrics generated by PM-spaces and those generalizing PM-spaces have been described by Stevens [20] and Nishiure [14].

1 Introduction

The concept of a probabilistic metric space is a generalization of a metric spaces. The origin of the theory data back to a paper published by Menger in 1942 [11]. A foundational paper on the subject was written by Schweizer and Sklar in [16, 17] and numerous articles follows thereafter. The latter two authors gave an excellent treatment of the subject in their book published in 1983 [18].

The concept of a quasi-metric space (where the condition of symmetry in dropped) was introduced in Wilson [22] and further developed in Kelly [9].

In the development of the theory of quasi-pseudo-metric spaces two streams can be distinguished. The core of the first is the concept of a convergent sequence (see [Kelly [9]). The second stream, a structure topological one, connected with Kelly as well, originated from the observation that every quasi-pseudo-metric on a given set does naturally generate a dual quasi-pseudo-metric on the same set. Thus a system of two mutually conjugates functions appeared. The dropped symmetry condition thus manifested itself in an external nature of such systems. Since each quasi-pseudo-metric generates a topology, hence of systems of two topologies can be associated with every quasi-pseudo-metric (Kelly [9]).

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The purpose of this study is to invalidate a natural generalization of probabilistic metric space and quasi-pseudo-metric space (Birsan [2, 3, 4], Grabiec [6]).

This paper contains a study of families of quasi-pseudo-metrics generated by Probabilistic-quasi-pseudo-metric-spaces which are generalization of probabilistic metric spaces (PM-spaces) ([2, 3, 4, 6]). The idea of PM-spaces goes back to Menger [11], [12]. The families of pseudo-metrics generated by PM-spaces and these generalizing PM-spaces have been described by Stevens [20] and Nishiura [14].

2 Preliminaries

A *distance distribution function* (d.d.f.) is a non-decreasing function $F : [0, +\infty] \to [0, 1]$, which is left-continuous on $(0, +\infty)$, and assumes the values $F(0) = 0$ and $F(+\infty) = 1$. The set of all d.d.f's, denoted by Δ^+, is equipped with modified Leśvy metric d_L (see pp. 45 of [18]). The metric space (Δ^+, d_L) is compact and hence complete. Further, Δ^+ is partially ordered by usual order for real-valued functions.

Let u_a be the element of Δ^+ defined by

$$u_a = \begin{cases} 1_{(a,\infty]}, & \text{for all } a \in [0, +\infty), \\ 1_{\{+\infty\}}, & \text{for } a = \{+\infty\}. \end{cases}$$

A triangle function $*$ is defined to be a binary operation on Δ^+ which is non-descreasing in each component, and if $(\Delta^+, *)$ is an Abelian monoid with the identity u_0.

Triangle functions considered in this paper will be assumed to be continuous with respect to the topology induced by metric d_L.

Definition 1. *Let $p_L : \Delta^+ \times \Delta^+ \to I$ be defined by the following formula:*

$$p_L(F, G) = \inf\{h \in (0, 1] : G(t) \le F(t + h) + h, \quad t \in (0, \frac{1}{h})\}. \tag{1}$$

Observe that, for all $F, G \in \Delta^+$, we have $G(t) \le F(t + 1) + 1$. Hence the set of (1) is nonempty.

Lemma 2. *If $p_L(F, G) = h > 0$, then, for every $t \in (0, \frac{1}{h})$, $G(t) \le F(t + h) + h$.*

Proof. For arbitrary $s > 0$ let $J_s = (0, \frac{1}{s})$. Then $J_{s_2} \subseteq J_{s_1}$ whenever $0 < s_1 < s_2 < 1$. Let $t \in J_h$. Since the interval J_h is open, there exist $t_1 < t$ and $s > 0$ such that $t_1 \in J_{h+s}$. As $p_L(F, G) = h$, we get $G(t_1) \le F(t_1 + h + s) + (h + s)$. Let $s \to 0$. Then $G(t_1) \le F(t + h^+) + h$ since F is nondecreasing.

Next, let $t_1 \to t$. Using the left-continuity of G, we obtain $G(t) \le F(t + h) + h$ for $t \in J_h$. This completes the proof. □

Theorem 3. *The function $p_L : \Delta^+ \times \Delta^+ \to I$ defined by (1) is a quasi-pseudo-metric on Δ^+. Recall that a quasi-pseudo-metric space is an ordered pair (X, p), where X is a nonempty set and the function $p : X^2 \to R^+$ satisfies the following conditions: for all $x, y, z \in X$,*

$$d(x, x) = 0$$
$$d(x, y) \le d(x, z) + d(z, y).$$

Proof. For each $F \in \Delta^+$ we have $p_L(F, F) = 0$. This is the direct consequence of Definition 1. In order to prove the "triangle inequality":

$$p_L(F, H) \le p_L(F, G) + p_L(G, H) \quad \text{for} \quad F, G, H \in \Delta^+,$$

Let $x = p_L(F, G) > 0$ and $y = p_L(G, H) > 0$. If $x + y \ge 1$, then (1) is satisfied. Thus let $x + y < 1$ and $t \in J_{x+y}$. Then $t + y \in J_x$. Using this fact and Lemma 2, we obtain $H(t) \le G(t + y) + y \le F(t + y + x) + y + y$. Thus the equality $H(t) \le F(t+(x+y))+(x+y)$ holds for $t \in J_{x+y}$. Consequently, we have $p_L(F, H) \le x + y = p_L(F, G) + p_L(G, H)$. □

The definition of the quasi-pseudo-metric p_L immediately yields the following observations:

Remark 4. *For every $F \in \Delta^+$ and every $t > 0$, the following hold (recall that $u_0 = 1_{(0,\infty]} \in \Delta^+$):*

$$\begin{aligned} p_L(F, u_0) &= \inf\{h \in (0, 1] : u_0(t) \le F(t + h) + h, \quad t \in J_h\} \\ &= \inf\{h \in (0, 1] : F(h+) > 1 - k\}, \\ F(t) > 1 - t \quad &\textit{iff} \quad p_L(F, u_0) < t. \end{aligned}$$

Lemma 5. *If $F, G \in \Delta^+$ and $F \le G$, then $p_L(G, u_0) \le p_L(F, u_0)$.*

Proof. This is an immediate consequence of Remark 4. □

Lemma 6. *If $\emptyset \ne A \subset \Delta^+$, then $G \in \Delta^+$ where*

$$G(t) = \sup\{F(t) : F \in A\}.$$

Proof. This follows from the information about lower semicontinuous functions. □

Definition 7. *Let $q_L : \Delta^+ \times \Delta^+ \to I$ be given by the formula:*

$$q_L(F, G) = p_L(G, F) \quad \text{for all} \quad F, G \in \Delta^+.$$

The function q_L is also a quasi-pseudo-metric on Δ^+. The functions p_L and q_L are called conjugate and the structure on Δ^+ generated by p_L is denoted by (Δ^+, p_L, q_L).

Theorem 8. *Given a structure* (Δ^+, p_L, q_L)*, the function* $d_L : \Delta^+ \times \Delta^+ \to I$ *defined by:*

$$d_L(F,G) = \max(p_L(F,G), q_L(F,G)) \quad \textit{for} \quad F, G \in \Delta^+$$

is a metric on the set Δ^+*.*

Proof. It suffices to show that the following condition holds:

$$d_L(F,G) = 0 \quad \text{iff} \quad F = G.$$

Let $t_0 \in (0, +\infty)$ and $F(t_0) < G(t_0)$. Since F and G are left-continuous, there exists $0 < t' < t_0$ such that $F(t') < G(t')$. Now, take $h < t_0 - t'$. By (1) and the fact that G is nondecreasing, we obtain the inequality:

$$G(t') \leq G(t_0 - h) \leq F(t_0 - h + h) + h.$$

If $h \to 0$, then we get $G(t_0-) = G(t_0) \leq F(t_0)$, which is a contradiction. Taking into account that $F(0) = G(0)$ and $F(+\infty) = G(+\infty) = 1$, we eventually get the equality $F(t) = G(t)$ for any $t \in [0, +\infty]$. □

Remark 9. *Note that the metric given by Theorem 2 is equivalent to the metric defined by Schweizer and Sklar ([18], Definition 4.2.1).*

Now, we state some facts related to the convergence in (Δ^+, d_L) and the weak convergence in the set Δ^+.

Definition 10. *A sequence* $\{F_n\}$*, where* $F_n \in \Delta^+$*, is said to be weakly convergent to* $F \in \Delta^+$ *(denoted by* $F_n \xrightarrow{w} F$*) if and only if the sequence* $\{F_n(t)\}$ *is convergent to* $F(t)$ *for every point* t *of continuity of* F*.*

Let us recall the well-known fact that the convergence in every point of continuity of the function F fails to be equivalent to the convergence in any point of $(0, +\infty)$. Indeed, consider the sequence $\{S_{(a-1/n,a)}\}$, where $a > 1$, and the function $S_{(a-1/n,a)}$ in Δ^+ is defined as follows:

$$S_{(a-\frac{1}{n},a)}(t) = \begin{cases} 0 & \text{if } 0 \leq t < a - \frac{1}{n}, \\ \frac{t-(a-\frac{1}{n})}{a-(a-\frac{1}{n})} & \text{if } t \in [a - \frac{1}{n}, a), \\ 1 & \text{if } t \in [a, +\infty]. \end{cases}$$

Notice that $S_{(a-1/n,a)} \xrightarrow{\mathrm{w}} u_a$, while, for every $n \in N$, we have

$$S_{(a-1/n,a)}(a) = 1 \neq 0 = u_a(a).$$

Theorem 11. *Let* $\{F_n\}_{n \in \mathbf{N}}$ *be a sequence of the functions of* Δ^+ *and let* $F \in \Delta^+$*. Then* $F_n \xrightarrow{w} F$ *if and only if* $d_L(F_n, F) \to 0$*.*

Proof. Assume that $d_L(F_n, F) \to 0$ and let $t_0 > 0$ be a point of continuity of F. It follows that for sufficiently small $h > 0$, the interval $(t_0 - h, t_0 + h)$ is contained in the interval $(0, \frac{1}{h})$ and the following hold:

$$F(t_0) - h \leq F_n(t_0 + h) \quad \text{and} \quad F_n(t_0) \leq F(t_0 + h) + h$$

for sufficiently large $n \in \mathbf{N}$ and for $t \in (0, \frac{1}{h})$. Thus, by the monotonicity of F_n and F we obtain:

$$F(t_0 - 2h) - f \leq F_n(t_0 - h) \leq F_n(t_0) \leq F_n(t_0 + h) \leq F(t_0 + 2h) + h.$$

Since h is sufficiently small and F is continuous at t_0, it follows that $F_n(t_0) \to F(t_0)$.

Conversely, assume that $F_n \xrightarrow{w} F$. Let $h \in (0, 1]$. Since the set of continuity points of F is dense in $[0, +\infty]$, there exists a finite set $A = \{a_0, a_1, \ldots, a_p\}$ of continuity points of F such that: $a_0 = 0, a_p \leq \frac{1}{h}$, $a_{m-1} < a_m \leq a_{m+1} + h$ for $m = 1, 2, \ldots, p$. Since A is finite, for sufficiently large $n \in \mathbf{N}$, we obtain $|F_n(a_m) - F(a_m)| \leq h$ for all a_m. Let $t_0 \in (0, \frac{1}{n})$. Then $t_0 \in [a_{m-1}, a_m]$ for some m. Therefore we have $F(t_0) \leq F(a_m) \leq F_n(a_m) + h \leq F_n(t_0 + h) + h$, i.e. condition (13) is satisfied. By interchanging the role of F_n and F we obtain that $F_n(t_0) \leq F(t_0 + h) + h$, which implies that $d_L(F_n, F) \to 0$. This completes the proof. □

From the Helly's theorem, it follows that, from every sequence in Δ^+, one can select a subsequence which is weakly convergent. This fact and Theorem 11 yield the following result:

Theorem 12. *The metric space (Δ^+, d_L) is compact, and hence complete.*

3 t–Norms and Their Properties

Now, we shall give some definitions and properties of t-norms (Menger [11], [12], Schweizer, Sklar [18]) defined on the unit interval $I = [0, 1]$. A t-norm $T : I^2 \to I$ is an Abelian semigroup with unit, and the t-norm T is nondecreasing with respect to each variable.

Definition 13. *Let T be a t-norm.*

(1) T is called a continuous t-norm if the function T is continuous with respect to the product topology on the set $I \times I$.

(2) The function T is said to be left-continuous if, for every $x, y \in (0, 1]$, the following condition holds:

$$T(x, y) = \sup\{T(u, v) : 0 < u < x,\ 0 < v < y\}.$$

(3) The function T is said to be right-continuous if, for every $x, y \in [0,1)$, the following condition holds:

$$T(x,y) = \inf\{T(u,v) : x < u < 1,\ y < v < 1\}.$$

Note that the continuity of a t-norm T implies both left and right-continuity of it.

Definition 14. *Let T be a t-norm. For each $n \in N$ and $x \in I$, let*

$$x^0 = 1,\ x^1 = x \quad \text{and} \quad x^{n+1} = T(x^n, x),\ \text{for all } n \geq 1.$$

Then the function T is called an Archimedean t-norm if, for every $x, y \in (0,1)$, there is an $n \in N$ such that

$$x^n < y, \quad \text{that is}, \quad x^n \leq y \quad \text{and} \quad x^n \neq y. \tag{2}$$

Note that $([0,1], T)$ is a semigroup, we have

$$T(x^n, x^m) = x^{n+m} \ \text{ for all } \ n, m \in \mathbf{N}.$$

From an immediate consequence of the above definition, we have the following:

Lemma 15. *A continuous t-norms is Archimedean if and only if*

$$T(x,x) < x \ \text{ for all } \ x \in (0,1).$$

Proof. Let $a \in (0,1)$ be fixed and $y_n = a^n$. Since

$$y_{n+1} = a^{n+1} = T(a^n, a) \leq T(a^n, 1) = a^n = y_n,$$

the sequence $\{y_n\}$ is non-increasing and bounded and so there exists $y = \lim_{n\to\infty} y_n$. Since $a^{2n} = T(a^n, a^n)$ and T is continuous, we deduce that $y = T(y,y)$.

If $T(x,x) < x$ for all $x \in (0,1)$, then $y \in \{0,1\}$ and, since $a^n \leq a < 1$, we have $y = 0$.

Conversely, if there exists $a \in (0,1)$ such that $T(a,a) = a$, then $a^{2n} = a$ for all $n \in \mathbf{N}$ and hence the sequence $\{a^n\}$ does not converge to 0. Therefore, $T(x,x) < x$ for all $x \in (0,1)$. This completes the proof. □

Lemma 16. *Let T is a continuous t-norm and strictly increasing in $(0,1]^2$ then it is Archimedean.*

Proof. By the strict monotonicity of T, for any $x \in (0,1)$, we have $T(x,x) < x$. □

Definition 17. *Let T be a t-norm. Then T is said to be positive if $T(x,y) > 0$ for all $x, y \in (0,1]$.*

Note that every t-norm satisfying the assumption of Lemma 16 is positive.

We shall now establish the notation related to a few most important t-norms defined by:

$$M(x,y) = \mathrm{Min}(x,y) = x \wedge y \tag{3}$$

for all $x, y \in I$. The function M is continuous and positive, but is not Archimedean (in fact, it fails to satisfy the strict monotonicity condition).

$$\Pi(x,y) = x \cdot y \tag{4}$$

for all $x, y \in I$. The function Π is strictly increasing and continuous and hence it is a positive archimedean t-norm.

$$W(x,y) = \mathrm{Max}(x+y-1, 0) \tag{5}$$

for all $x, y \in I$. The function W is continuous and Archimedean, but it is not positive and hence it fails to be a strictly increasing t-norm.

$$Z(x,y) = \begin{cases} x & \text{if } x \in I \text{ and } y = 1, \\ y & \text{if } x = 1 \text{ and } y \in I, \\ 0 & \text{if } x, y \in [0,1). \end{cases} \tag{6}$$

The function Z is Archimedean and right-continuous, but it fails to be left-continuous.

For any t-norm T, we have

$$Z \leq T \leq M \ \text{ in particular}$$
$$Z < W < \Pi < M.$$

4 Triangle Functions and Their Properties

In this section, we shall now present some properties of the triangle functions on Δ^+ (Šerstnev [19], Schweizer, Sklar [18]).

The ordered pair $(\Delta^+, *)$ is an Abelian semigroup with the unit $u_0 \in \Delta^+$ and the operation $* : \Delta^+ \times \Delta^+ \to \Delta^+$ is a nondecreasing function. We note that $u_\infty \in \Delta^+$ is a zero of Δ^+. Indeed, we obtain

$$u_\infty \leq u_\infty * F \leq u_\infty * u_0 = u_\infty \text{ for all } F \in \Delta^+.$$

Definition 18. *Let $T(\Delta^+, *)$ denote the family of all triangle functions on the set Δ^+. Then the relation $\leq$ defined by*

$$*_1 \leq *_2 \text{ iff } F *_1 G \leq F *_2 G \text{ for all } F, G \in \Delta^+ \text{ partially orders the family } T(\Delta^+, *). \tag{7}$$

Now, we are going to define the next relation in the $T(\Delta^+, *)$. *It will be denoted by* $\gg$ *and is defined as follows:*

$$*_1 \gg *_2 \text{ iff for all } F, G, P, Q \in \Delta^+ \quad [(F *_2 P) *_1 (G *_2 R)] \geq [(F *_G) *_2 (P * R)]. \quad (8)$$

By putting $G = P = u_0$ *we obtain* $F *_1 R \geq F *_2 R$ *for* $F, R \in \Delta^+$ *and hence* $*_1 \geq *_2$. *Then follows that* $*_1 \gg *_2 \Rightarrow *_1 \geq *_2$.

Theorem 19. *Let* T *be a left-continuous t-norm. Then the function* $T : \Delta^+ \times \Delta^+ \to \Delta^+$ *defined by*

$$\mathbf{T}(F, G)(t) = T(F(t), G(t)) \tag{9}$$

for any $t \in [0, +\infty]$ *is a triangle function on the set* Δ^+.

Theorem 20. *For every triangle function* $*$, *the following inequality holds:*

$$* \leq \mathbf{M},$$

where M *is the t-norm of Definition 17.*

Proof. For every F, $G \in \Delta^+$, we have by definition of $(\Delta^+, *)$, $F * G \leq F * u_0 = F$ and, by symmetry, also $F * G \leq G$. Thus, for every $t \in [0, +\infty]$, we have

$$(F * G)(t) \leq M(F(t), G(t)) = M(F, G)(t). \tag{10}$$

□

Theorem 21. *If* T *is a left-continuous t-norm, then the function* $*_T : \Delta^+ \times \Delta^+ \to \Delta^+$ *defined by*

$$F *_T G(t) = \sup\{T(F(u), G(s)) : u + s = t, \ u, s > 0\} \tag{11}$$

is a triangle function on Δ^+.

Proof. The function $F *_T G \in \Delta^+$ is nondereasing and satisfies the condition $F *_T G(+\infty) = 1$ for all $F, G \in \Delta^+$. Thus it suffices to check that $F *_T G$ is left-continuous, i.e., for every $t \in (0, +\infty)$ and $h > 0$, there exists $0 < t_1 < t$ such that

$$F *_T G(t_1) > F *_T G(t) - h.$$

Let $t \in (0, +\infty)$. Then there exist $u, s > 0$ such that $u + s = t$ and

$$T(F(u), G(s)) > F *_T G(t) - \frac{h}{2}. \tag{12}$$

By the left-continuity of F, G and the t-norm T, it follows that there are numbers $0 \leq u_1 < u$ and $0 \leq s_1 \leq s$ such that

$$T(F(u_1), G(s_1)) > T(F(u), G(s)) - \frac{h}{2}. \tag{13}$$

Now, put $t_1 = u_1 + s_1$. Then $t_1 < t$ and, by (11), we obtain

$$F *_T G(t) \geq T(F(u_1), G(s_1)). \tag{14}$$

This completes the proof. □

Theorem 22. *Let T be a continuous t-norm. Then the triangular functions $*_T$ and T are uniformly continuous on (Δ^+, d_L).*

Proof. (see Theorem 7.2.8 [18]) Let us observe that the continuity of the t-norm T implies its uniform continuity on $I \times I$ with the product topology. Take an $h \in (0, 1)$. Then there exists $s > 0$ such that

$$T(\mathrm{Min}(z + s, 1), w) < T(z, w) + \frac{h}{4}$$

and

$$T(z, \mathrm{Min}(w + s, 1)) < T(z, w) + \frac{h}{4} \tag{15}$$

for all $z, w \in I$. Let $u < 1/s$ and $v < 1/s$ be such that $u + v < 2/h$. Next, by (11), for every $F, G \in \Delta^+$ and $t \in (0, 2/h)$, there exist $u, v > 0$ such that $u + v = t$ and

$$F *_T G(t) < T(F(u), G(v)) + \frac{h}{4}.$$

Now, let $F_1 \in \Delta^+$ be such that $d_L(F, F_1) < s$, which means that

$$F(u) \leq F_1(u + s) + s$$

for all $u \in (0, \frac{1}{s})$. Since $u + v = t < 2/h$, we have $u < 2/h$. Therefore, we obtain

$$\begin{aligned} F *_T G(t) &< T(\mathrm{Min}(F_1(u + s) + s, 1), G(v)) + \frac{h}{2} \\ &< T(F_1(u + s), G(v)) + \frac{h}{2} \end{aligned}$$

and

$$\begin{aligned} F *_T G(t) &< F_1 *_T G(u + s + v) + \frac{h}{2} \\ &\leq F_1 *_T G(u + v + \frac{h}{2}) + \frac{h}{2} \\ &= F_1 *_T G(t + \frac{h}{2}) + \frac{h}{2}. \end{aligned}$$

Thus, by (1), we have

$$p_L(F_1 *_T G, G) \leq \frac{h}{2}, \quad q_L(F *_T G, F_1 *_T G) \leq \frac{h}{2}$$

and so we have

$$d_L(F_1 *_T G, F *_T G) \leq \frac{h}{2}.$$

If $d_L(G, G_1) < s$, then we have

$$d_L(F_1 *_T G_1, F_1 *_T G) \leq \frac{h}{2}$$

and so let $F, F_1, G, G_1 \in \Delta^+$ satisfy the conditions $d_L(F, F_1) < s$ and $d_L(G, G_1) < s$. Then we have

$$\begin{aligned}
&d_L(F_1 *_T G_1, F *_T G) \\
&\leq d_L(F_1 *_T G_1, F_1 *_T G) + d_L(F_1 *_T G, F *_T G) \\
&\leq \frac{h}{2} + \frac{h}{2} = h.
\end{aligned}$$

It follows that the triangle function $*_T$ is uniformly continuous in the space (Δ^+, d_L). The second part is a simple restatement of the first one. This completes the proof. □

Remark 23. *There exist triangle functions which are not continuous on* (Δ^+, d_L). *Among them, there is the function* $*_Z$ *of* (11) *and* (6). *Indeed, this can be seen by the following example.*

Let $F_n(t) = 1 - e^{-\frac{t}{n}}$, where $n \in N$. Then

$$F_n \xrightarrow{\text{w}} u_0$$

while the sequence $\{F_n *_Z F_n\}$ fails to be weakly convergent to $u_0 *_Z u_0$ because $F_n *_Z F_n = u_\infty$ for all $n \in N$. We note that this example actually shows much more: the triangle function $*_Z$ is not continuous on (Δ^+, d_L). In particular, it is not continuous at the point (u_0, u_0).

We finish this section by showing a few properties of the relation defined in (8) in the context of triangle functions (22).

Lemma 24. *If* T_1 *and* T_2 *are continuous t-norms, then triangle functions* T_1, T_2 *given by* (9),

$$\mathbf{T}_1 \gg \mathbf{T}_2 \quad \textit{if and only if} \quad *_{T_1} \gg *_{T_2}.$$

Lemma 25. *If* T *is a continuous t-norm and* $*_T$ *is the triangle function of* (9), *then*

$$\mathbf{T} \gg *_T, \tag{16}$$

$$\mathbf{M} \gg * \quad \textit{for all triangle functions} \quad * . \tag{17}$$

5 Properties of $PqpM$–Spaces

First, we give the definition of $PqpM$-spaces and some properties of $PqpM$-spaces and others.

Definition 26. *([2, 3, 4, 6]) By a PqpM-space we mean an ordered triple $(X, P, *)$, where X is a nonempty set, the operation $*$ is triangle function and $P : X^2 \to \Delta^+$ satisfies the following conditions (by P_{xy} we denote the value of P at $(x, y) \in X^2$): for all $x, y, z \in X$,*

$$P_{xx} = u_0, \tag{18}$$
$$P_{xy} * P_{yz} \leq P_{xz}. \tag{19}$$

If P satisfies also the additional condition:

$$P_{xy} \neq u_0 \quad \text{if} \quad x \neq y, \tag{20}$$

then $(X, P, *)$ is called a probabilistic quasi-metric space (denoted by PqM-space). Moreover, if P satisfies the condition of symmetry:

$$P_{xy} = P_{yx}, \tag{21}$$

then $(X, P, *)$ is called a probabilistic metric space (denoted by PM-space).

Definition 27. *[6] Let $(X, P, *)$ be a PqpM-space and let $Q : X^2 \to \Delta^+$ be defined by the following condition:*

$$Q_{xy} = P_{yx}$$

for all $x, y \in X$. Then the ordered triple $(X, Q, *)$ is also a PqpM-space. We say that the function P is a conjugate Pqp-metric of the function Q. By $(X, P, Q, *)$ we denote the structure generated by the Pqp-metric P on X.

Now, we shall characterize the relationships between Pqp-metrics and probabilistic pseudo-metrics.

Lemma 28. *Let $(X, P, Q, *)$ be a structure defined by a Pqp-metric P and let*

$$*_1 \gg * \tag{22}$$

Then the ordered triple $(X, F^{*_1}, *)$ is a probabilistic pseudo-metric space (denoted by PPM-space) whenever the function $F^{*_1} : X^2 \to \Delta^+$ is defined in the following way:

$$F^{*_1}_{xy} = P_{xy} *_1 Q_{xy} \tag{23}$$

for all $x, y \in X$. If, additionally, P satisfies the condition:

$$P_{xy} \neq u_0 \quad \text{or} \quad Q_{xy} \neq u_0 \tag{24}$$

for $x \neq y$, then $(X, F^{*_1}, *)$ is a PM-space.

Proof. For any $x, y \in X$, we have

$$F_{xy}^{*_1} \in \Delta^+ \quad \text{and} \quad F_{xy}^{*_1} = F_{yx}^{*_1}.$$

By (18), we obtain

$$F_{xx}^{*_1} = P_{xx} *_1 Q_{xx} = u_0 *_1 u_0 = u_0.$$

Next, by (19) and (22) and the monotonicity of triangle function, we obtain

$$\begin{aligned} F_{xy}^{*_1} &= P_{xy} *_1 Q_{xy} \\ &\geq (P_{xz} * P_{xz}) *_1 (Q_{xz} * Q_{zy}) \\ &\geq (P_{xz} *_1 Q_{xz}) * (P_{zy} *_1 Q_{zy}) \\ &= F_{xz}^{*_1} * F_{zy}^{*_1}. \end{aligned}$$

The proof of the second part of the theorem is a direct consequence of the fact that the conditions (24) and (23) both imply the statement that

$$F_{xz}^{*_1} = P_{xy} *_1 Q_{xy} = u_0 \quad \text{if and only if} \quad P_{xy} = Q_{xy} = u_0.$$

It follows that, whenever $x \neq y$, $P_{xy} \neq u_0$ or $Q_{xy} \neq u_0$ and hence $P_{xy} *_1 Q_{xy} \neq u_0$. This completes the proof. □

Remark 29. *For an arbitrary triangle function (22), we know, by Lemma 25, that $M \gg *$. Using (23), we have*

$$F_{P \vee Q} = F^M(x, y) \geq F^{*_1}(x, y) \quad \textit{for all} \quad x, y \in X. \tag{25}$$

for all $x, y \in X$.

The function F^M will be called the natural probabilistic pseudo-metric generated by the Pqp-metric P. It is the "greatest" among all the probabilistic pseudo-metrics generated by P.

Definition 30. *Let X be a nonempty set and $P : X^2 \to D^+$, where $D^+ = \{F \in \Delta^+; \lim_{t\to\infty} F(t) = 1\}$ and T is t-norm. The triple (X, P, T) is called a quasi-pseudo-Menger space if it satisfies the following axioms:*

$$P_{xx} = u_0 \tag{26}$$

$$P_{xy}(u + v) \geq T(P_{xz}(u), P_{zy}(v)) \quad \textit{for all } x, y, z \in X \textit{ and } u, v \in R. \tag{27}$$

If P satisfies also the additional condition:

$$P_{xy} \neq u_0 if x \neq y \tag{28}$$

then (X, P, T) is a quasi-Menger space.

Moreover, if P satisfies the condition of symmetry $P_{xy} = P_{yx}$, then (X, P, T) is called a Menger-space (see [11, 12]).

Definition 31. *Let (X, p) be a quasi-pseudo-metric-space and $G \in D^+$ be distinct from u_0. Define a function $G_p : X^2 \to D^+$ by*

$$G_p(x, y) = G\left(\frac{t}{p(x, y)}\right) \quad \text{for all } t \in R^+ \tag{29}$$

and $G(\frac{t}{0}) = G(\infty) = 1$, for $t > 0, G(\frac{0}{0} = G(0) = 0$. Then (X, G_p) is called a P-simple space generated by (X, p) and G.

Theorem 32. *Every P-simple space (X, G_p) is a quasi-pseudo-Menger space respect to the t-norm M.*

Proof. For all $x, y, z \in X$, by the triangle condition for the quasi-pseudo-metric p, we have

$$p(x, y) \geq p(x, y) + p(y, z).$$

Assume,that all at $p(x, z), p(x, y)$ and $p(y, z)$ are distinct from zero. For any $t_1, t_2 > 0$, we obtain

$$\frac{t_1 + t_2}{p(x, z)} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)} \tag{30}$$

and hence we infer that

$$\max\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)} \geq \min\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\}. \tag{31}$$

This inequality and the monotonicity of G imply that

$$G_p(x, z)(t_1 + t_2) \geq \min(G_p(x, y)(t_1), G_p(y, z)(t_2)),$$

for $t_1, t_2 \geq 0$. This completes the proof. □

6 The family of Pqp–metrics on a get X

Definition 33. *Let $P[X, *]$ denote the family of all Pqp-metrics defined on a set X with respect to a triangle function $*$. Define on X a relation $\prec$ in the following way:*

$$P_1 \prec P_2 \quad \text{iff} \quad P_1(x, y) \geq P_2(x, y) \quad \text{for all} \quad x, y \in X. \tag{32}$$

We note that $\prec$ is a partial order on the family $P[X, *]$. We distinguish elements P_0 and P_∞ in it:

$$P_0(x, y) = u_0 \quad \text{for all} \quad x, y \in X, \tag{33}$$

$$P_\infty(x, y) = u_0, \quad \text{and} \quad p_\infty(x, y) = u_\infty \quad \text{for} \quad x \neq y. \tag{34}$$

We note that $P_0 \prec P \prec P_\infty$ for every $P \in P[X, *]$.

Now, we give the definition of certain binary operation $\oplus$ on $P[X,*]$. Let for all $P_1, P_2 \in P[X,*]$:

$$P_1 \oplus P_2(x,y) = P_1(x,y) * P_2(x,y), \quad x,y \in X. \tag{35}$$

We note that $P_1 \oplus P_2 \in P[X,*]$. Indeed, we prove the condition (18) directly: $P_1 \oplus P_2(x,x) = P_1(x,x) * P_2(x,x) = u_0$.

The condition (19) follows from $F * u_0 = F$ when applied to P_1 and P_2:

$$\begin{aligned} P_1 \oplus P_2(x,y) &= P_1(x,y) \oplus P_2(x,y) \\ &\geq (P_1(x,y) * P_1(z,y)) * (P_2(x,z) * P_2(z,y)) \\ &= (P_1(x,z) * P_2(x,z)) * (P_1(z,y) * P_2(z,y)) \\ &= (P_1 \oplus P_2(x,y)) * (P_1(z,y) \oplus P_2(z,y)). \end{aligned}$$

This shows that $P_1 \oplus P_2$ is a Pqp-metric. Notice also that for each $P \in P[X,*]$ the following property holds:

$$P_0 \oplus P = P. \tag{36}$$

Indeed, $P_0 \oplus P(x,y) = u_0 * P_{xy} = P(x,y)$.

The operation $\oplus$ is also commutative and associative. This is a consequence of the form of (22). Thus we have the following corollary:

Lemma 34. *The ordered triple $(P[X,*], \oplus, p_0)$ is an Abelian semi-group with respect to the operation $*$, and has the neutral element P_0.*

The following gives a relationship between the relation $\prec$ and the operation $\oplus$.

Lemma 35. *Let $(P[X,*], \oplus, P_0)$ be as in Lemma 35. Then, for all $P, P_1, P_2 \in P[X,*]$, the following hold:*

$$P_0 \prec P, \tag{37}$$

$$P_1 \oplus P \prec P_2 \oplus P \quad \textit{whenever} \quad P_1 \prec P_2. \tag{38}$$

Proof. That the first property holds true follows from the Definition 33. The relation $P_1 \prec P_2$ means, by (32), that $P_1(x,y) \geq P_2(x,y)$, $x,y \in X$. Since 22 is a monotone function, we get $P_1(x,y) * P(x,y) \geq P_2(x,y) * P(x,y)$. This shows the validity of the second condition.

Let us define in $P[X,*]$ get another operation, denoted by $\vee$. For any $P_1, P_2 \in P[X,*]$, let

$$P_1 \vee P_2 = \min(P_1, P_2) = M(P_1, P_2). \tag{39}$$

By Lemma 5 it follows that $M \gg *$ for all $*$. Thus we have $P_1 \vee P_2 \in P[X,*]$. □

The following accounts for some properties of the operation v.

Lemma 36. *The ordered pair* $(P[X,*],\vee)$ *is a* $\vee$*-semi-lattice (see Grätzer [4]) satisfying the following conditions: for all* $P, P_1, P_2 \in P[X,*]$,

$$P_1 \prec P_2 \quad \textit{iff} \quad P_1 \vee P_2 = P_2, \tag{40}$$

$$(P \oplus P_1) \vee (P \oplus P_2) \prec P \oplus (P_1 \vee P_2). \tag{41}$$

Proof. $P \vee P = M(P,P) = P$, hence $\vee$ satisfies the indempotency. It is also commutative. This yields the first part of the Lemma. Next, observe that if $P_1 \prec P_2$, then $P_1(x,y) \geq P_2(x,y)$, $x,y \in X$. Thus $M(P_1,P_2) = P_2$. We have shown the first property. For a proof of the second one notice that $P_1 \prec P_1 \vee P_2$ and $P_2 \prec P_1 \vee P_2$. By (38) we get $P \oplus P_1 \prec P \oplus (P_1 \vee P_2)$ and $P \oplus P_1 \prec P \oplus (P_1 \vee P_2)$. Since $(P[X,*],\vee)$ is a $\vee$-semilattice, the condition (41) follows. This completes the proof. □

7 Families of quasi-pseudo-metrics generated by $PqpM$-metrics

We shall now give some classification of $PqpM$-spaces with respect to the so-called "triangle condition".

Definition 37. *Let* X *be a nonempty set. Let* $P : X^2 \to \Delta^+$ *satisfy the condition* (18) *and let, for all* $x,y,z \in X$*, the following implication hold:*

$$\text{If } P_{xy}(t_2) = 1 \text{ and } P_{yz}(t_2) = 1, \text{ then} \tag{42}$$

$$P_{xy}(t_1 + t_2) = 1 \text{ for all } t_1, t_2 > 0. \tag{43}$$

Then the ordered pair (X,P) *in called a statistical quasi-pseudo-metric space. We write* $SpqM$*-space.*

Topics related to the "triangle condition" belong to the mast important ones in the theory of PM-spaces. We mention here the mast important papers in a chronological order (see Menger [11], Wald [21], Schweizer and Sklar [16, 17], Muštari and Serstnev [13], Brown [5], Istrătescu [8], Radu [15].

Definition 38. *Let* T *be* t*-norm ones a function* $P : X^2 \to \Delta^+$ *is assumed to satisfy the condition* (18) *and, for all* $x,y,z \in X$*, let*

$$P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)), \quad t_1, t_2 > 0. \tag{44}$$

Then (X,P,T) *is called a quasi-pseudo-Menger space.*

Condition (44) is called a Menger condition and comes from a paper by Schweizer and Sklar ([13, 14]). It is modification of an inequality of Menger ([7, 8]).

Lemma 39. *Each quasi-pseudo-Menger space is an* $SqpM$*-space.*

Proof. Assume $P_{xy}(t_1) = 1$ and $P_{yz}(t_2) = 1$ for any $t_1, t_2 > 0$. By (M.2), we have

$$P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) = T(1,1) = 1.$$

Let X be a nonempty set and let $P : X^2 \to \Delta^+$ satisfy the condition (18). For each $a \in [0,1)$ define $p_a : X \to \mathbf{R}$ by

$$p_a(x,y) = \inf\{t > 0 : P_{xy}(t) > a \quad \text{for} \quad x, y \in X\}. \tag{45}$$

Since P_{xy} is nondecreasing and let-continuous, the following equivalence holds for $x, y \in X$ and $a \in [0,1)$:

$$p_a(x,y) < t \quad \text{iff} \quad P_{xy}(t) > a. \tag{46}$$

The family $D(X, P, a)$ of all functions p_a has the following properties which are the consequences of (46):

$$p_a(x,y) \geq 0, \tag{47}$$
$$p_a(x,x) = 0 \quad \text{for} \quad x, y \in X \quad \text{and} \quad a \in [0,1). \tag{48}$$

Under the additional assumption that P satisfies the following condition: for all $a \in [0,1)$,

$$P_{xy}(t_1) > a \quad \text{and} \quad P_{yz}(t_2) > a \Rightarrow P_{xz}(t_1 + t_2) > a \tag{49}$$
$$\text{for all} \quad x, y, z \in X \quad \text{and} \quad t_1, t_2 > 0, \tag{50}$$

then for every $a \in [0,1)$ the function p_a satisfies

$$p_a(x,z) \leq p_a(x,y) + p_a(y,z) \quad \text{for} \quad x, y, z \in X. \tag{51}$$

This completes the proof. □

As a consequence of this fact we conclude the following:

Lemma 40. *The family $D(X, P, a)$ of all the functions p_a with $a \in [0,1)$ is a family of quasi-pseudo-metrics if and only if the function P satisfies* (5.3.5). *For any $a \in (0,1)$, p_a is a quasi-metric if and only if $p_{xy}(0+) < a$ for all $x \neq y$ in X.*

Proof. For the first assertion, it suffices to show the triangle condition (51). Given an arbitrary $s > 0$, put $t_1 = p_a(x,y) + \frac{s}{2}$ and $t_2 = p_a(y,z) + \frac{s}{2}$. By (46) we then have $P_{xy}(t_1) > a$ and $P_{yz}(t_2) > a$. By (49) this yields the inequality $P_{xz}(t_1 + t_2) > a$ which is equivalent to $p_a(x,z) < t_1 + t_2 = p_a(x,y) + p_a(y,z) + s$. Since s is arbitrary, we obtain the required inequality (51).

The second assertion follows from the fact that $p_a(x,y) = 0$ if and only if $P_{xy}(t) > a$ for all $t > 0$, i.e., when $P_{xy}(0+) \geq a$. The proof is complete. □

Remark 41. *Observe that if $P : X^2 \to \Delta^+$ satisfies the conditions (18) and (49), then (X, P) is a statistical quasi-pseudo-metric space.*

Indeed, let $P_{xy}(t_1) = 1$ and $P_{yz}(t_2) = 1$. Then it follows by (49) that $P_{xz}(t_1 + t_2) > a$ for all $a \in [0, 1)$. Thus $P_{xz}(t_1 + t_2) = 1$. Thus $P_{xz}(t_1 + t_2) = 1$. This shows that the condition (37) of Definition 37 holds true.

The following observation is a consequence of the preceding remark:

Corollary 42. *Let the function P satisfy the conditions (18) and (49) and let, for every $x, y \in X$, there exists a number $t_{xy} < \infty$ such that $P_{xy}(t_{xy}) = 1$. Then the function p_a is a quasi-pseudo-metric for every $a \in [0, 1]$. In particular, $p_1 : X^2 \to R$ is given by the following formula:*

$$p_1(x, y) = \inf\{t > 0 : P_{xy}(t) = 1 \quad \text{for} \quad x, y \in X\}. \tag{52}$$

Proof. Let $s > 0$. Let $t_1 = p_1(x, y) + \frac{s}{2}$ and $t_2 = p_1(y, z) + \frac{s}{2}$. Then $P_{xy}(t_1) = 1$ and $P_{yz}(t_2) = 1$, and thus, by (45), we have $P_{xz}(t_1 + t_2) = 1$. We now have $p_1(x, z) < t_1 + t_2 = p_1(x, y) + p_1(y, z) + s$. Finally, the condition (51) is satisfied on account of s being arbitrary. □

Remark 43. *Let $(X, P, *_M)$ be a quasi-pseudo-Menger space. Then the function P satisfies the condition (49). Indeed, let $P_{xy}(t_1) > a$ and $P_{yz}(t_2) > a$. By (M.2), we get $P_{xz}(t_1 + t_2) \geq \min(P_{xy}(t_1), P_{yz}(t_2)) > \min(a, a) = a$.*

The following is an immediate consequence of Lemma 40 and Remark 43:

Corollary 44. *If $(X, P, *_M)$ is a quasi-pseudo-Menger space, then the family $D(X, P, a)$ defined in (45) is a family of the quasi-pseudo-metrics on X for all $a \in [0, 1)$.*

Theorem 45. *Let (X, P, T) be a quasi-pseudo-Menger space. Let the function $d(x) = T(x, x)$ be strictly increasing and continuous on some interval $[a, b) \subset I$. Then, if $T(a, a) = a$, then the function p_a of (45) is a quasi-pseudo-metric in X. For $a > 0, p_a$ is a quasi-metric in X if and only if $P_{xy}(0+) < a$ whenever $x \neq y$.*

Proof. It suffices to show that the property (49) holds true for any $a \in [0, 1)$, which satisfies the assumption of the theorem.

Let $P_{xy}(t_1) > a$ and $P_{yz}(t_2) > a$. Since P_{xy} and P_{yz} are nondecreasing and left-continuous, there exists $s > 0$ such that $a+s < b, P_{xy}(t_1) > a+s$ and $P_{yz}(t_2) > a+s$. The properties of the function $d(x) = T(x, x)$ and the condition (44) yield the inequality $P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) \geq T(a + s, a + s) > a$. The assertion is now a consequence of Lemma 40. □

Theorem 46. *Let (X, P, T) be a quasi-pseudo-Menger space such that $T \geq \Pi$. Then the family $D(X, P, p_a)$ of all the functions $p_a : X^2 \to R$ given by*

$$p_a(x, y) = \inf\{t > 0 : P_{xy}(t) > a(t), \quad x, y \in X\}, \tag{53}$$

consists of quasi-pseudo-metrics, if all the functions $a : [0, +\infty] \to [0, 1]$ *are defined by the following formula:*

$$a(t) = \begin{cases} e^{-at}, & t \in [0, +\infty), \\ 0, & t = +\infty, \ where \ a \in (0, +\infty). \end{cases} \tag{54}$$

The functions p_a *are quasi-metrics if and only if* $P_{xy}(0+) < 1$ *whenever* $x \neq y$.

Proof. Observe that for every $a \in (0, +\infty)$ the functions are strictly decreasing. Let $t_1 = p_a(x, y) + \frac{s}{2}$ and $t_2 = p_a(y, z) + \frac{s}{2}$, $s > 0$. This means that by (46) the following inequalities hold:

$$P_{xy}(t_1) \geq a(p_a(x, y)) > a(t_1),$$
$$P_{yz}(t_2) \geq a(p_a(y, z)) > a(t_2).$$

By (44) and the inequality $T \gg \Pi$, we obtain

$$\begin{aligned} P_{xz}(t_1 + t_2) &\geq T(P_{xy}(t_1), P_{yz}(t_2)) \\ &\geq T(a(p_a(x, y), a(p_a(y, z))) \\ &\geq \Pi(a(p_a(x, y), a(p_a(y, z))) \\ &> \Pi(a(t_1), a(t_2)) = e^{-at_1} \cdot e^{-at_2} \\ &= e^{-a(t_1+t_2)} = a(t_1 + t_2). \end{aligned}$$

This means that $p_a(x, z) < t_1 + t_2 = p_a(x, y) + p_a(y, z) + s$ for any $s > 0$, so that the triangle condition holds. This completes the proof. □

Theorem 47. *Let* (X, P, T) *be a quasi-pseudo-Menger space with* $T \geq W$ (28). *Then the family* $D(X, P, p_a)$ *of all the functions* p_a *of* (53) *consists of quasi-pseudo-metrics, provided the functions* $a : [0, +\infty] \to [0, 1]$ *are defined by the following formula:*

$$a(t) = \begin{cases} 1 - \frac{t}{a}, & t \in [0, a], \\ 0, & t > a \ where \ a \in (0, +\infty). \end{cases} \tag{55}$$

Proof. Let $t_1 = p_a(x, y) + \frac{s}{2}$ and $t_2 = p_a(y, z) + \frac{s}{2}$, $s > 0$. By (46), we have

$$P_{xy}(t_1) \geq a(p_a(x, y)) > a(t_1) \quad \text{and} \quad P_{yz}(t_2) \geq a(p_a(y, z)) > a(t_2).$$

By (44) and the inequality $T \geq W$, we get

$$\begin{aligned} P_{xz}(t_1 + t_2) &\geq T(P_{xy}(t_1), P_{yz}(t_2)) \geq T(a(p_a(x, y)), a(p_a(y, z))) \\ &\geq W(a(p_a(x, y)), a(p_a(y, z))) > W(a(t_1), a(t_2)) \\ &= \mathrm{Max}\Big(1 - \frac{t_1}{a} + 1 - \frac{t_2}{a} - 1, 0\Big) \\ &= 1 - \frac{t_1 + t_2}{a} = a(t_1 + t_2). \end{aligned}$$

Therefore $p_a(x,z) < t_1 + t_2 = p_a(x,y) + p_a(y,z) + s$ for every $s > 0$, i.e., the tirangle inequality holds. □

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