

## SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY RUSCHEWEYH DERIVATIVE II

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**Abstract.** New class  $\Sigma_\lambda(\alpha, \beta, \gamma)$  of univalent meromorphic functions defined by Ruscheweyh derivative in the punctured unit disk  $U^*$  is introduced. We study several Hadamard product properties. Some results connected to inclusion relations, neighborhoods of the elements of class  $\Sigma_\lambda(\alpha, \beta, \gamma)$  and integral operators are obtained.

### 1 Introduction

Let  $\Sigma$  denote the class of meromorphic functions in the punctured unit disk  $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$  of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N} = \{1, 2, 3, \dots\}). \quad (1)$$

A function  $f \in \Sigma$  is meromorphic starlike function of order  $\beta$  ( $0 \leq \beta < 1$ ) if

$$- \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \beta, \quad (z \in U = U^* \cup \{0\}). \quad (2)$$

The class of all such functions is denoted by  $\Sigma^*(\beta)$ . A function  $f \in \Sigma$  is meromorphic convex function of order  $\beta$  ( $0 \leq \beta < 1$ ) if

$$- \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \beta, \quad (z \in U = U^* \cup \{0\}). \quad (3)$$

**Definition 1.** The Ruscheweyh derivative of order  $\lambda$  is denoted by  $D^\lambda f$  and is defined as following: If

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,$$

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then

$$D^\lambda f(z) = \frac{1}{z(1-z)^{\lambda+1}} * f(z) = z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n, \quad \lambda > -1, \quad z \in U^*, \quad (4)$$

where

$$D_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n+1)}{(n+1)!}. \quad (5)$$

**Definition 2.** The Hadamard product (or convolution) of two functions  $f(z)$  given by (1) and

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \quad (6)$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

**Definition 3.** Let  $f(z) \in \Sigma$  be given by (1). The new class  $\Sigma_\lambda(\alpha, \beta, \gamma)$  is defined by

$$\begin{aligned} \Sigma_\lambda(\alpha, \beta, \gamma) = & \left\{ f \in \Sigma : -\operatorname{Re} \left\{ \frac{z(D^\lambda f(z))' + \gamma z^2(D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + \gamma z(D^\lambda f(z))'} \right\} \right. \\ & \geq \alpha \left| \frac{z(D^\lambda f(z))' + \gamma z^2(D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + \gamma z(D^\lambda f(z))'} + 1 \right| + \beta, \quad z \in U, \quad 0 \leq \beta < 1, \\ & \left. \lambda > -1, \quad 0 \leq \gamma < \frac{1}{2}, \quad \alpha \geq 0 \right\}. \end{aligned} \quad (7)$$

## 2 Main Results

We will need the following lemma whose details can be found in [1].

**Lemma 4.** The function  $f(z)$  defined by (1) is in the class  $\Sigma_\lambda(\alpha, \beta, \gamma)$  if and only if

$$\sum_{n=1}^{\infty} (n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)] D_n(\lambda) a_n \leq (1 - \beta)(1 - 2\gamma) \quad (8)$$

where  $0 \leq \beta < 1, \alpha \geq 0, 0 \leq \gamma < \frac{1}{2}, \lambda > -1, D_n(\lambda)$  given by (5).

**Theorem 5.** Let the function  $f(z)$  defined by (1) and the function  $g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$  ( $b_n \geq 0, n \in \mathbb{N}$ ) be in the class  $\Sigma_\lambda(\alpha, \beta, \gamma)$ . Then the function  $k(z)$  defined

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by  $k(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n$  is in the class  $\Sigma_{\lambda}(\alpha, \sigma, \gamma)$ , where  $(\alpha \geq 0, 0 \leq \gamma < \frac{1}{2}, 0 \leq \beta < 1, 0 \leq \sigma < 1, n \in \mathbb{N}$  and  $n \geq 1)$  and  $\sigma$  is given by

$$\sigma = 1 - \frac{(1 - \beta)^2(1 - 2\gamma)(n + 1)(\alpha + 1)}{(1 - \beta)^2(1 - 2\gamma) + (n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]^2}.$$

*Proof.* We must find the largest  $\sigma$  such that

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \sigma)]D_n(\lambda)}{(1 - \sigma)(1 - 2\gamma)} a_n b_n \leq 1, \quad D_n(\lambda) \text{ is given by (5).}$$

Since  $f(z)$  and  $g(z)$  are in  $\Sigma_{\lambda}(\alpha, \beta, \gamma)$ , then

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)} b_n \leq 1.$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)} \sqrt{a_n b_n} \leq 1.$$

We want only to show that

$$\begin{aligned} & \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \sigma)]D_n(\lambda)}{(1 - \sigma)(1 - 2\gamma)} a_n b_n \\ & \leq \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)} \sqrt{a_n b_n}. \end{aligned}$$

This equivalently to

$$\sqrt{a_n b_n} \leq \frac{(1 - \sigma)[n(\alpha + 1) + (\alpha + \beta)]}{(1 - \beta)[n(\alpha + 1) + (\alpha + \sigma)]}.$$

Then

$$\sigma \leq 1 - \frac{(1 - \beta)^2(1 - 2\gamma)(n + 1)(\alpha + 1)}{(1 - \beta)^2(1 - 2\gamma) + (n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]^2}.$$

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**Theorem 6.** Let the function  $f(z)$  defined by (1) and  $g(z)$  given by  $g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$  be in the class  $\Sigma_{\lambda}(\alpha, \beta, \gamma)$ , then the function  $k(z)$  defined by  $k(z) = z^{-1} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) z^n$  is in the class  $\Sigma_{\lambda}(\alpha, \sigma, \gamma)$ , where  $0 \leq \gamma < \frac{1}{2}$ ,  $\lambda > -1$ ,  $0 \leq \sigma < 1$ ,  $0 \leq \beta < 1$ ,  $\alpha \geq 0$  and

$$\sigma = 1 - \frac{8(1-\beta)^2(1-2\gamma)(1+\alpha)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2 + 4(1-\beta)^2(1-2\gamma)}.$$

*Proof.* We must find the largest  $\sigma$  such that

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} (a_n^2 + b_n^2) \leq 1. \quad (9)$$

Since  $f(z)$  and  $g(z)$  are in  $\Sigma_{\lambda}(\alpha, \beta, \gamma)$ , we get

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \right\}^2 \leq 1 \quad (10)$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} b_n \right\}^2 \leq 1. \quad (11)$$

Combining the inequalities (10) and (11), gives

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} \right\}^2 (a_n^2 + b_n^2) \leq 1.$$

But,  $k(z) \in \Sigma_{\lambda}(\alpha, \beta, \gamma)$  if and only if

$$\sum_{n=1}^{\infty} \left\{ \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \right\} (a_n^2 + b_n^2) \leq 1. \quad (12)$$

The inequality (12) would obviously imply (9) if

$$\begin{aligned} & \frac{(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \\ & \leq \frac{[(n\gamma - \gamma + 1)(n(\alpha + 1) + (\alpha + \beta))]D_n(\lambda)^2}{2[(1-\beta)(1-2\gamma)]^2} \\ & = \frac{q^2}{2}. \end{aligned}$$

Hence

$$\frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \sigma)]D_n(\lambda)}{(1-\sigma)(1-2\gamma)} \leq \frac{q^2}{2}$$

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or

$$\frac{(1-\sigma)}{(1+\alpha)} \geq \frac{2(n+1)(n\gamma-\gamma+1)D_n(\lambda)}{(1-2\gamma)q^2+2(n\gamma-\gamma+1)D_n(\lambda)}.$$

The right hand is decreasing function of  $n$  and its maximum if  $n = 1$ .

Now

$$\begin{aligned} & \frac{(1-\sigma)}{(1+\alpha)} \\ & \geq \frac{2(n+1)(n\gamma-\gamma+1)(1-\beta)^2(1-2\gamma)D_n(\lambda)}{[(n\gamma-\gamma+1)(n(\alpha+1)+(\alpha+\beta))D_n(\lambda)]^2+2(n\gamma-\gamma+1)(1-\beta)^2(1-2\gamma)D_n(\lambda)} \end{aligned} \quad (13)$$

Simplify (13) we obtain

$$\frac{(1-\sigma)}{(1+\alpha)} \geq \frac{8(1-\beta)^2(1-2\gamma)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2+4(1-\beta)^2(1-2\gamma)}$$

or

$$\sigma \leq 1 - \frac{8(1-\beta)^2(1-2\gamma)(1+\alpha)}{(\lambda+1)(\lambda+2)(2\alpha+\beta+1)^2+4(1-\beta)^2(1-2\gamma)}.$$

This completes the proof of theorem.  $\square$

We get the inclusive properties of the class  $\Sigma_\lambda(\alpha, \beta, \gamma)$ .

**Theorem 7.** Let  $\alpha \geq 0$ ,  $0 \leq \beta < 1$ ,  $0 \leq \gamma < \frac{1}{2}$ ,  $\sigma \geq 0$ , then

$$\Sigma_\lambda(\alpha, \beta, \gamma) \subseteq \Sigma_\lambda(\alpha, \sigma, 0)$$

where

$$\sigma \leq 1 - \frac{(n+1)(\alpha+1)(1-\beta)(1-2\gamma)}{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\gamma)}, \quad n \in \mathbb{N}, \quad n \geq 1.$$

*Proof.* Let  $f(z) \in \Sigma_\lambda(\alpha, \beta, \gamma)$ , then from Lemma 4, we have

$$\sum_{n=1}^{\infty} \frac{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n \leq 1. \quad (14)$$

We want to find the value  $\sigma$  such that

$$\sum_{n=1}^{\infty} \frac{[n(1+\alpha)+(\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)} a_n \leq 1. \quad (15)$$

The inequality (14) would obviously imply (15) if

$$\frac{[n(1+\alpha)+(\alpha+\sigma)]D_n(\lambda)}{(1-\sigma)} \leq \frac{(n\gamma-\gamma+1)[n(1+\alpha)+(\alpha+\beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} = q.$$

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Therefore,

$$\frac{[n(1 + \alpha) + (\alpha + \sigma)]D_n(\lambda)}{(1 - \sigma)} \leq q. \quad (16)$$

Hence

$$\frac{(1 - \sigma)}{(1 + \alpha)} \geq \frac{(n + 1)D_n(\lambda)}{(q + D_n(\lambda))} (n \geq 1, n \in \mathbb{N}). \quad (17)$$

The right hand side of (17) decreases as  $n$  increases and so is maximum for  $n = 1$ . So (17) is satisfied provided

$$\frac{(1 - \sigma)}{(1 + \alpha)} \geq \frac{(n + 1)(1 - \beta)(1 - 2\gamma)}{(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \beta)] + (1 - \beta)(1 - 2\gamma)} = y.$$

Obviously  $y < 1$  and

$$\sigma \leq 1 - \frac{(n + 1)(\alpha + 1)(1 - \beta)(1 - 2\gamma)}{(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \beta)] + (1 - \beta)(1 - 2\gamma)}.$$

This completes the proof of the theorem.  $\square$

### 3 $(n, \delta)$ -Neighborhoods on $\Sigma_\lambda^\sigma(\alpha, \beta, \gamma)$

The next, we determine the inclusion relation involving  $(n, \delta)$  - neighborhoods. Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruscheweyh [4], but for meromorphic function studied by Liu and Srivastava [3], we define the  $(n, \delta)$ -neighborhood of a function  $f(z) \in \Sigma$  by

$$N_{n,\delta}(f) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \quad (18)$$

For the identity function  $e(z) = z$ , we have

$$N_{n,\delta}(e) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|b_n| \leq \delta \right\}.$$

**Definition 8.** A function  $f \in \Sigma$  is said to be in the class  $\Sigma_\lambda^\sigma(\alpha, \beta, \gamma)$  if there exists a function  $g \in \Sigma_\lambda(\alpha, \beta, \gamma)$  such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \sigma, \quad (z \in U, 0 \leq \sigma < 1).$$

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**Theorem 9.** If  $g \in \Sigma_\lambda(\alpha, \beta, \gamma)$  and

$$\sigma = 1 - \frac{\delta(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2)}{(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2) - 2(1 - \beta)(1 - 2\gamma)} \quad (19)$$

then  $N_{n,\delta}(g) \subset \Sigma_\lambda^\sigma(\alpha, \beta, \gamma)$ .

*Proof.* Let  $f \in N_{n,\delta}(g)$ , then we get from (18) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}).$$

Since  $g \in \Sigma_\lambda(\alpha, \beta, \gamma)$ , we have from Lemma 4

$$\sum_{n=1}^{\infty} b_n \leq \frac{2(1 - \beta)(1 - 2\gamma)}{(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2)}$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \leq \frac{\delta(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2)}{(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2) - 2(1 - \beta)(1 - 2\gamma)} = 1 - \sigma.$$

Thus, by Definition 8,  $f \in \Sigma_\lambda^\sigma(\alpha, \beta, \gamma)$  for  $\sigma$  given by (19).  $\square$

## 4 Integral Operators

Next, we consider integral transforms of functions in the class  $\Sigma_\lambda(\alpha, \beta, \gamma)$ .

**Theorem 10.** Let the function  $f(z)$  given by (1) be in  $\Sigma_\lambda(\alpha, \beta, \gamma)$ . Then the integral operator

$$F(z) = c \int_0^1 u^c f(uz) du, \quad (0 < u \leq 1, 0 < c < \infty) \quad (20)$$

is in  $\Sigma_\lambda(\alpha, \delta, \gamma)$ , where

$$\delta = \frac{(c + 2)(2\alpha + \beta + 1) - c(1 - \beta)(2\alpha + 1)}{(c + 2)(2\alpha + \beta + 1) + c(1 - \beta)}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2(1 - \beta)(1 - 2\gamma)}{(2\alpha + \beta + 1)(\lambda + 1)(\lambda + 2)} z.$$

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*Proof.* Let  $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n$  in  $\Sigma_{\lambda}(\alpha, \beta, \gamma)$ . Then

$$\begin{aligned} F(z) &= c \int_0^1 u^c f(uz) du \\ &= c \int_0^1 \left( \frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} a_n u^{n+c} z^n \right) du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \delta)]D_n(\lambda)}{(c + n + 1)(1 - \delta)(1 - 2\gamma)} a_n \leq 1. \quad (21)$$

Since  $f \in \Sigma_{\lambda}(\alpha, \beta, \gamma)$ , we have

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)} a_n \leq 1.$$

Note that (21) is satisfied if

$$\frac{c(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \delta)]D_n(\lambda)}{(c + n + 1)(1 - \delta)(1 - 2\gamma)} \leq \frac{(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \beta)]D_n(\lambda)}{(1 - \beta)(1 - 2\gamma)}.$$

Rewriting the inequality, we have

$$\begin{aligned} &c(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \delta)]D_n(\lambda)(1 - \beta)(1 - 2\gamma) \\ &\leq (c + n + 1)(1 - \delta)(1 - 2\gamma)(n\gamma - \gamma + 1)[n(1 + \alpha) + (\alpha + \beta)]D_n(\lambda). \end{aligned}$$

Solving for  $\delta$ , we have

$$\delta \leq \frac{(c + n + 1)[n(1 + \alpha) + (\alpha + \beta)] - c(1 - \beta)[n(1 + \alpha) + \alpha]}{(c + n + 1)[n(1 + \alpha) + (\alpha + \beta)] + c(1 - \beta)} = F(n).$$

A computation shows that

$$\begin{aligned} &F(n + 1) - F(n) = \\ &= [((c + n + 2)((n + 1)(1 + \alpha) + (\alpha + \beta)) - c(1 - \beta)((n + 1)(1 + \alpha) + \alpha)) \times \\ &\quad ((c + n + 1)(n(1 + \alpha) + (\alpha + \beta)) + c(1 - \beta)) - ((c + n + 1)(n(1 + \alpha) + (\alpha + \beta)) \\ &\quad - c(1 - \beta)(n(1 + \alpha) + \alpha))((c + n + 2)((n + 1)(1 + \alpha) + (\alpha + \beta)) + c(1 - \beta))] / \\ &= [((c + n + 1)(n(1 + \alpha) + (\alpha + \beta)) + c(1 - \beta)) \times \\ &\quad ((c + n + 2)((n + 1)(1 + \alpha) + (\alpha + \beta)) + c(1 - \beta))] > 0 \end{aligned}$$

for all  $n$ . This means that  $F(n)$  is increasing and  $F(n) \geq F(1)$ . Using this, the result follows.  $\square$

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**Theorem 11.** *If  $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$ , then the integral operator*

$$F(z) = c \int_0^1 u^c f(uz) du, \quad (0 < u \leq 1, 0 < c < \infty)$$

*is in  $\Sigma_\lambda(\alpha, \frac{1+\beta c}{2+c}, \gamma)$ . The result is sharp for*

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)} z^n.$$

*Proof.* By Definition of  $F$ , we get

$$F(z) = c \int_0^1 u^c f(uz) du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n.$$

By Lemma 4, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1 - \frac{1+\beta c}{2+c})(1-2\gamma)(c+n+1)} a_n \leq 1. \quad (22)$$

Since if  $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$ , then (22) satisfies if

$$\frac{c}{(1 - \frac{1+\beta c}{2+c})(1-2\gamma)(c+n+1)} \leq \frac{1}{(1-\beta)(1-2\gamma)}$$

or equivalently, when

$$Y(n, c, \beta) = \frac{c(1-\beta)}{(1 - \frac{1+\beta c}{2+c})(c+n+1)} \leq 1$$

since  $Y(n, c, \beta)$  is a decreasing function of  $n$  ( $n \geq 1$ ), then the proof is completed. The result is sharp for

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)} z^n.$$

□

**Theorem 12.** *Let the function  $f(z)$  given by (1) be in  $\Sigma_\lambda(\alpha, \beta, \gamma)$ ,*

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = z^{-1} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} a_n z^n, \quad c > 0. \quad (23)$$

*Then  $F(z)$  is in  $\Sigma_\lambda(\alpha, \beta, \gamma)$  for  $|z| \leq r(\alpha, \beta, c, \delta)$ , where*

$$r(\alpha, \beta, c, \delta) = \inf_n \left( \frac{c(1-\delta)[n(\alpha + 1) + (\alpha + \beta)]}{(1-\beta)(c+n+1)[n(\alpha + 1) + (\alpha + \delta)]} \right)^{\frac{1}{n+1}},$$

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$n = 1, 2, 3, \dots$ .

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)} z^n, \quad n = 1, 2, 3, \dots$$

*Proof.* Let

$$w = -\frac{z(D^\lambda f(z))' + \gamma z^2(D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + z(D^\lambda f(z))'}$$

Then it is sufficient to show that

$$|w - 1| < |w + 1 - 2\delta|.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \delta)]D_n(\lambda)(c + n + 1)}{c(1-\delta)(1-2\gamma)} a_n |z|^{n+1} \leq 1 \quad (24)$$

since  $f \in \Sigma_\lambda(\alpha, \beta, \gamma)$ , then by Lemma 4, we have

$$\sum_{n=1}^{\infty} (n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)a_n \leq (1-\beta)(1-2\gamma).$$

The equation (24) is satisfied if

$$\begin{aligned} & \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \delta)]D_n(\lambda)(c + n + 1)}{c(1-\delta)(1-2\gamma)} a_n |z|^{n+1} \\ & \leq \frac{(n\gamma - \gamma + 1)[n(\alpha + 1) + (\alpha + \beta)]D_n(\lambda)}{(1-\beta)(1-2\gamma)} a_n. \end{aligned}$$

Solving for  $|z|$ , we obtain the result.  $\square$

**Corollary 13.** Let the function  $f(z)$  given by (1) be in  $\Sigma_\lambda(0, \beta, \gamma)$  and  $F(z)$  given by (23). Then  $F(z)$  is in  $\Sigma_\lambda(0, \beta, \gamma)$  for  $|z| < r(0, \beta, \gamma, c, \delta)$ , where

$$r(0, \beta, \gamma, c, \delta) = \inf_n \left( \frac{c(1-\delta)(n+\beta)}{(1-\beta)(c+n+1)(n+\delta)} \right)^{\frac{1}{n+1}}, \quad n = 1, 2, 3, \dots$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\gamma)}{(n\gamma - \gamma + 1)(n+\beta)D_n(\lambda)} z^n, \quad n = 1, 2, 3, \dots$$

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