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EXISTENCE AND NONEXISTENCE RESULTS FOR SECOND-ORDER NEUMANN BOUNDARY VALUE PROBLEM

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Abstract. In this paper some existence and nonexistence results for positive solutions are obtained for second-order boundary value problem

$$-u'' + Mu = f(t, u), \quad t \in (0, 1)$$

with Neumann boundary conditions

$$u'(0) = u'(1) = 0,$$

where $M > 0,\ f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$. By making use of fixed point index theory in cones, some new results are obtained.

1 Introduction

In this paper, we are concerned with the second-order two-point Neumann boundary value problem

$$-u'' + Mu = f(t, u), \quad t \in (0, 1), \tag{1.1}$$

$$u'(0) = u'(1) = 0, \tag{1.2}$$

where $M > 0$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$.

In the last two decades, there has been much attention focused on questions of positive solutions for diverse nonlinear ordinary differential equation, difference equation, and functional differential equation boundary value problems, see [1]–[12], and the references therein. Recently, Neumann boundary value problems have deserved the attention of many researchers (see [9]–[5]). The goal of this paper is to study the existence and nonexistence results for second-order Neumann boundary value problem (1.1) and (1.2) under the new conditions by utilizing the fixed point index theory in cones.

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The paper is divided into six sections. In Section 2, we provide some preliminaries and various lemmas, which play key roles in this paper. In Section 3, we give the existence theorems of the *sublinear* Neumann boundary value problem. In Section 4, we establish the existence theorems of the *superlinear* Neumann boundary value problem. In Section 5, we obtain the existence of multiple positive solutions. In Section 6, we give the nonexistence of positive solution.

2 Preliminaries and lemmas

In Banach space $C[0,1]$ in which the norm is defined by $\|u\| = \max_{0\le t\le 1} |u(t)|$ for any $u \in C[0,1]$. We set $P = \{u \in C[0,1] | u(t) \ge 0,\ t \in [0,1]\}$ be a cone in $C[0,1]$. We denote by $B_r = \{u \in C[0,1] | \|u\| < r\} (r > 0)$ the open ball of radius r.

The function u is said to be a positive solution of BVP (1.1), ((1.2) if $u \in C[0,1] \cap C^2(0,1)$ satisfies (1.1), (1.2) and $u(t) > 0$ for $t \in (0,1)$.

Let $G(t,s)$ be the Green function of the problem (1.1), (1.2)with $f(t,u) \equiv 0$ (see [10], [11]), that is,

$$G(t,s) = \begin{cases} \dfrac{\mathrm{ch}(m(1-t))\mathrm{ch}(ms)}{m\mathrm{sh}m}, & 0 \le s \le t \le 1, \\ \dfrac{\mathrm{ch}(m(1-s))\mathrm{ch}(mt)}{m\mathrm{sh}m}, & 0 \le t \le s \le 1, \end{cases}$$

where $m = \sqrt{M}$, $\mathrm{ch}x = \dfrac{e^x + e^{-x}}{2}$, $\mathrm{sh}x = \dfrac{e^x - e^{-x}}{2}$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and $G(t,s) \ge 0$ for $0 \le t, s \le 1$. After direct computations we get

$$0 < \frac{1}{m\mathrm{sh}m} = \alpha \le G(t,s) \le \beta = \frac{\mathrm{ch}^2 m}{m\mathrm{sh}m},\ \forall\ 0 \le t,s \le 1. \tag{2.1}$$

Let

$$(Au)(t) = \int_0^1 G(t,s) f(s,u(s))ds, \quad t \in [0,1]. \tag{2.2}$$

We can verify that the nonzero fixed points of the operator A are positive solutions of the problem (1.1), (1.2).

Define

$$K = \{u \in P | u(t) \ge \gamma \|u\|,\ t \in [0,1]\},$$

where $0 < \gamma = \frac{\alpha}{\beta} < 1$. Then K is subcone of P.

Lemma 1. *Suppose that $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Then $A : K \to K$ is a completely continuous operator.*

Proof. Let $u \in K$. Since $G(t,s) \geq 0$, $(t,s) \in [0,1] \times [0,1]$, by the definition, we have $(Au)(t) \geq 0$, $t \in [0,1]$. On the other hand, by ((2.1)) we have

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \alpha \int_0^1 f(s,u(s))ds, \tag{2.3}$$

$$\|Au\| = \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \leq \beta \int_0^1 f(s,u(s))ds, \tag{2.4}$$

for every $t \in [0,1]$, by ((2.3)) and (2.4) we have

$$(Au)(t) \geq \gamma \|Au\|.$$

Thus, we assert that $A : K \to K$. The completely continuity of A follows from the Arzera-Ascoli theorem. □

We also need the following lemmas(see [6]).

Lemma 2. *Let E be Banach space, K be a cone in E, and $\Omega(K)$ be a bounded open set in K with $\theta \in \Omega(K)$. Suppose that $A : \overline{\Omega(K)} \to K$ is a completely continuous operator. If*

$$\mu Au \neq u, \quad \forall\ u \in \partial\Omega(K),\ 0 < \mu \leq 1,$$

then the fixed point index $i(A, \Omega(K), K) = 1$.

Lemma 3. *Let E be Banach space, K be a cone in E, and $\Omega(K)$ be a bounded open set in K. Suppose that $A : \overline{\Omega(K)} \to K$ is a completely continuous operator. Suppose that the following two conditions are satisfied:*

(i) $\inf_{u\in\partial\Omega(P)} \|Au\| > 0.$

(ii) $\mu Au \neq u,\ \forall\ u \in \partial\Omega(P),\ \mu \geq 1,$

then the fixed point index $i(A, \Omega(P), P) = 0$.

3 Existence results in sublinear case

Theorem 4. *Suppose that $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and*

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > M, \tag{3.1}$$

$$\limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < M. \tag{3.2}$$

Then the Neumann boundary value problem (1.1), (1.2) has at least one positive solution.

Proof. It follows from (3.1) that there exists $r_1 > 0$ such that

$$f(t,u) \geq Mu, \quad \forall\, t \in [0,1],\ 0 \leq u \leq r_1. \tag{3.3}$$

If $u \in \partial B_{r_1} \cap K$, we have $\gamma r_1 = \gamma\|u\| \leq u(t) \leq r_1,\ 0 \leq t \leq 1$. It follows from (3.3) that

$$\begin{aligned}
\inf_{u\in\partial B_{r_1}\cap K} \|Au\| = & \inf_{u\in\partial B_{r_1}\cap K} \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \\
\geq & \ \alpha \inf_{u\in\partial B_{r_1}\cap K} \int_0^1 f(s,u(s))ds \\
\geq & \ \alpha M \inf_{u\in\partial B_{r_1}\cap K} \int_0^1 u(s)ds \\
\geq & \ \alpha M \int_0^1 \gamma r_1 ds \\
\geq & \ \alpha M \gamma r_1 \\
> & \ 0.
\end{aligned}$$

We may suppose that A has no fixed point on $\partial B_{r_1} \cap K$ (Otherwise, the proof is finished). Next, we show that

$$\mu Au \neq u,\ \forall\, u \in \partial B_{r_1} \cap K,\ \mu \geq 1. \tag{3.4}$$

If otherwise, then there exist $u_1 \in \partial B_{r_1} \cap K$ and $\mu_1 \geq 1$ such that $\mu_1 A u_1 = u_1$. Hence $\mu_1 > 1$. By the definition of A, $u_1(t)$ satisfies the differential equation

$$\begin{cases} -u_1'' + Mu_1 = \mu_1 f(t,u_1), & 0 < t < 1, \\ u_1'(0) = u_1'(1) = 0 \end{cases}$$

Integrating this equation from 0 to 1 and from (3.3) we get

$$M\int_0^1 u_1(t)dt = \mu_1 \int_0^1 f(t,u_1)dt \geq \mu_1 M \int_0^1 u_1(t)dt.$$

Since $M\int_0^1 u_1(t)dt \geq M\gamma r_1 > 0$, we see that $\mu_1 \leq 1$, which is a contradiction. Hence (3.4) is true and we have from Lemma 3 that

$$i(A, B_{r_1} \cap K, K) = 0. \tag{3.5}$$

It follows from (3.2) that there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$f(t,u) \leq \sigma M u, \quad \forall\, t \in [0,1],\ u \geq r_2.$$

Set $C = \max_{0\le t\le 1,\ 0\le u\le r_2} |f(t,u) - \sigma M u| + 1$, it is clear that

$$f(t,u) \le \sigma M u + C,\ \forall\ t \in [0,1],\ \ u \ge 0. \tag{3.6}$$

Let

$$W = \{u \in K | u = \mu A u,\ 0 < \mu \le 1\}.$$

In the following, we prove that W is bounded.

For any $u \in W$, we have $u = \mu A u$, then $u(t)$ satisfies the differential equation

$$\begin{cases} -u'' + Mu = \mu f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases}$$

Integrating this equation from 0 to 1 and from (3.6) we have

$$\begin{aligned} M\int_0^1 u(t)dt = & \ \mu \int_0^1 f(t,u(t))dt \\ \le & \int_0^1 f(t,u(t))dt \\ \le & \ \sigma M \int_0^1 u(t)dt + C. \end{aligned}$$

Consequently, we obtain that

$$\int_0^1 u(t)dt \le \frac{C}{(1-\sigma)M}. \tag{3.7}$$

By definition of K, $\int_0^1 u(t)dt \ge \gamma \|u\|$, from which and (3.7) we get that

$$\|u\| \le \frac{1}{\gamma}\int_0^1 u(t)dt \le \frac{C}{(1-\sigma)M\gamma}.$$

So W is bounded.

Select $r_3 > \max\{r_2, \sup W\}$. Then from the homotopy invariance property of fixed point index we have

$$i(A, B_{r_3} \cap K, K) = i(\theta, B_{r_3} \cap K, K) = 1. \tag{3.8}$$

By (3.5) and (3.8), we have that

$$i(A,\ (B_{r_3}\cap K)\backslash(\overline{B}_{r_1}\cap K),\ K) = i(A,\ B_{r_3}\cap K,\ K) - i(A,\ B_{r_1}\cap K,\ K) = 1.$$

Then A has at least one fixed point on $(B_{r_3}\cap K)\backslash(\overline{B}_{r_1}\cap K)$. This means that the sublinear Neumann boundary value problem (1.1), (1.2) has at least one positive solution. □

From Theorem 4 we immediately obtain the following

Corollary 5. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and*

$$\liminf_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t,u)}{u} = +\infty,$$

$$\limsup_{u \to +\infty} \max_{t \in [0,1]} \frac{f(t,u)}{u} = 0.$$

Then the Neumann boundary value problem (1.1), (1.2) *has at least one positive solution.*

Corollary 6. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, denote*

$$f_0 = \liminf_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t,u)}{u}, \quad f^\infty = \limsup_{u \to +\infty} \max_{t \in [0,1]} \frac{f(t,u)}{u}$$

In addition, assume that $0 \le f^\infty < f_0 \le +\infty$,

$$\lambda \in \left(\frac{M}{f_0}, \frac{M}{f^\infty} \right). \tag{3.9}$$

Then the eigenvalue problem

$$\begin{cases} -u'' + Mu = \lambda f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least one positive solution.

Proof. By (3.9), we know that

$$\liminf_{u \to 0^+} \min_{t \in [0,1]} \frac{\lambda f(t,u)}{u} > M, \quad \limsup_{u \to +\infty} \max_{t \in [0,1]} \frac{\lambda f(t,u)}{u} < M.$$

So Corollary 6 holds from Theorem 4. □

4 Existence results in superlinear case

In this section, we give the existence theorems of positive solutions for the superlinear Neumann boundary value problem.

Theorem 7. *Suppose that* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and*

$$\liminf_{u \to +\infty} \min_{t \in [0,1]} \frac{f(t,u)}{u} > M, \tag{4.1}$$

$$\limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u} < M. \tag{4.2}$$

Then the Neumann boundary value problem (1.1), (1.2) *has at least one positive solution.*

Proof. It follows from (4.1) that there exists $\varepsilon > 0$ such that $f(t,u) \geq (M+\varepsilon)u$ when u is sufficiently large. Hence there exists $b_1 \geq 0$ such that

$$f(t,u) \geq (M+\varepsilon)u - b_1, \quad \forall\, t \in [0,1],\ 0 \leq u < +\infty. \tag{4.3}$$

Take

$$R > \max\left\{1, \frac{b_1}{\gamma\varepsilon}\right\}. \tag{4.4}$$

If $u \in \partial B_R \cap K$, we have $\gamma R = \gamma\|u\| \leq u(t) \leq R,\ 0 \leq t \leq 1$. It follows from (4.4) that

$$\begin{aligned}
\inf_{u\in\partial B_R\cap K} \|Au\| = & \inf_{u\in\partial B_R\cap K}\max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \\
\geq & \ \alpha \inf_{u\in\partial B_R\cap K} \int_0^1 f(s,u(s))ds \\
\geq & \ \alpha(M+\varepsilon)\int_0^1 u(s)ds - \alpha b_1 \\
\geq & \ \alpha(M+\varepsilon)\gamma R - \alpha b_1 \\
> & \ \alpha\varepsilon\gamma R - \alpha b_1 \\
> & \ 0.
\end{aligned}$$

Next, we show that

$$\mu Au \neq u,\ \forall\ u \in \partial B_R \cap K,\ \mu \geq 1. \tag{4.5}$$

If otherwise, then there exist $u_2 \in \partial B_R \cap K$ and $\mu_2 \geq 1$ such that $\mu_2 A u_2 = u_2$. Hence $\mu_2 > 1$. By the definition of A, $u_2(t)$ satisfies the differential equation

$$\begin{cases} -u_2'' + Mu_2 = \mu_2 f(t,u_2), & 0 < t < 1, \\ u_2'(0) = u_2'(1) = 0. \end{cases}$$

Integrating this equation from 0 to 1 and from (4.3) we have

$$\begin{aligned}
M\int_0^1 u_2(t)dt = & \ \mu_2 \int_0^1 f(t,u_2(t))dt \\
\geq & \int_0^1 f(t,u_2(t))dt \\
\geq & \ (M+\varepsilon)\int_0^1 u_2(t)dt - b_1.
\end{aligned}$$

Consequently, we obtain that

$$\int_0^1 u_2(t)dt \leq \frac{b_1}{\varepsilon}. \tag{4.6}$$

By definition of K, $\int_0^1 u_2(t)dt \geq \gamma\|u\| = \gamma R$, from which and (4.6) we get that

$$R \leq \frac{b_1}{\gamma\varepsilon}, \tag{4.7}$$

which contradicts (4.4). Hence (4.5) is true and by Lemma 3, we have

$$i(A, B_R \cap K, K) = 0. \tag{4.8}$$

It follows from (4.2) that there exists $0 < r < 1$ such that

$$f(t,u) \leq Mu, \ \forall\ t \in [0,1], \ 0 \leq u \leq r. \tag{4.9}$$

We may suppose that A has no fixed point on $\partial B_r \cap K$ (otherwise, the proof is finished). In the following we show that

$$\mu Au \neq u, \ \forall\ u \in \partial B_r \cap K, \ 0 \leq \mu \leq 1. \tag{4.10}$$

If otherwise, there exist $u_3 \in \partial B_r \cap K$ and $0 \leq \mu_3 \leq 1$ such that $\mu_3 Au_3 = u_3$. Thus $0 \leq \mu_3 < 1$. By the definition of A, $u_3(t)$ satisfies the differential equation

$$\begin{cases} -u_3'' + Mu_3 = \mu_3 f(t, u_3), & 0 < t < 1, \\ u_3'(0) = u_3'(1) = 0 \end{cases}$$

Integrating this equation from 0 to 1 and from (4.9) we get

$$M\int_0^1 u_3(t)dt = \mu_3 \int_0^1 f(t,u_3)dt \leq \mu_3 M \int_0^1 u_3(t)dt.$$

Since $M\int_0^1 u_3(t)dt \geq M\gamma r > 0$, we see that $\mu_3 \geq 1$, which is a contradiction. Hence (4.10) is true and we have from Lemma 2 that

$$i(A, B_r \cap K, K) = 1. \tag{4.11}$$

By (4.8) and (4.11) we have

$$i(A, (B_R \cap K)\backslash(\overline{B_r} \cap K), K) = i(A, B_R \cap K, K) - i(A, B_r \cap K, K) = -1.$$

Then A has at least one fixed point on $(B_R \cap K)\backslash(\overline{B}_r \cap K)$. This means that the superlinear Neumann boundary value problem (1.1), (1.2) has at least one positive solution. □

From Theorem 7 we immediately obtain the following

Corollary 8. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and*

$$\liminf_{u \to +\infty} \min_{t \in [0,1]} \frac{f(t,u)}{u} = +\infty,$$

$$\limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u} = 0.$$

Then the Neumann boundary value problem (1.1), (1.2) has at least one positive solution.

Corollary 9. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, denote*

$$f^0 = \limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u}, \quad f_\infty = \liminf_{u \to +\infty} \min_{t \in [0,1]} \frac{f(t,u)}{u}$$

In addition, assume that $0 \le f^0 < f_\infty \le +\infty$*,*

$$\lambda \in \Big(\frac{M}{f_\infty}, \frac{M}{f^0} \Big). \tag{4.12}$$

Then the eigenvalue problem

$$\begin{cases} -u'' + Mu = \lambda f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least one positive solution.

Proof. By (4.12), we know that

$$\liminf_{u \to +\infty} \min_{t \in [0,1]} \frac{\lambda f(t,u)}{u} > M, \quad \limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{\lambda f(t,u)}{u} < M.$$

So Corollary 8 holds from Theorem 7. □

5 Existence results of twin positive solutions

In this section we need the following well-know lemma (see [6]).

Lemma 10. *Let* E *be a Banach space, and* P *be a cone in* E*, and* $\Omega(P)$ *be a bounded open set in* P*. Suppose that* $A : \overline{\Omega(P)} \to P$ *is a completely continuous operator.*

(i) If $\|Au\| > \|u\|$*,* $u \in \partial\Omega(P)$*, then the fixed point index* $i(A, \Omega(P), P) = 0$*.*

(ii) If $\theta \in \Omega(P)$ *and* $\|Au\| < \|u\|$*,* $u \in \partial\Omega(P)$*, then the fixed point index* $i(A, \Omega(P), P) = 1$*.*

Theorem 11. *Suppose that $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ is continuous. In addition, assume that*

$$\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{f(t,u)}{u} < M, \tag{5.1}$$

$$\limsup_{u\to +\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u} < M. \tag{5.2}$$

If there exists $r_0 > 0$ such that

$$f(t,u) > \xi r_0, \quad \forall\ t\in[0,1],\ \ u\in[\gamma r_0, r_0], \tag{5.3}$$

where $\gamma\in(0,1)$, $\xi=\alpha^{-1}$, then the Neumann boundary value problem (1.1), (1.2) has at least two positive solutions.

Proof. It follows from (5.1) and (5.2) that there exists $0 < r_4 < r_0$ such that $f(t,u) \le Mu$ for $0 \le u \le r_4$ and there exist $0 < \sigma < 1$ and $r_5 > r_0$ such that $f(t,u) \le \sigma M u$ for $u \ge r_5$. We may suppose that A has no fixed point on $\partial B_{r_4}\cap K$ and $\partial B_{r_5}\cap K$. Otherwise, the proof is completed.

We have from the proof in Theorem 7 and the permanence property of fixed point index that $i(A, B_{r_4}\cap K, K) = 1$. It follows from the proof in Theorem 4 that $i(A, B_{r_5}\cap K, K) = 1$.

For every $u\in B_{r_0}\cap K$, we have $\gamma r_0 = \gamma\|u\| \le u(t) \le r_0,\ 0\le t\le 1$. It follows from (5.3) that

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s)f(s,u(s))ds\\ &\ge \alpha\int_0^1 f(s,u(s))ds\\ &> \alpha\xi r_0\\ &= r_0,\quad t\in[0,1].\end{aligned}$$

Then $\|Au\| > \|u\|$, for any $u\in\partial B_{r_0}\cap K$. Hence we have from Lemma 10 that $i(A, B_{r_0\cap K}, K) = 0$.

Therefore,

$$i(A,(B_{r_0}\cap K)\backslash(B_{r_4}\cap K),K) = i(A,B_{r_0}\cap K,K) - i(A,B_{r_4}\cap K,K) = -1,$$

$$i(A,(B_{r_5}\cap K)\backslash(B_{r_0}\cap K),K) = i(A,B_{r_5}\cap K,K) - i(A,B_{r_0}\cap K,K) = 1.$$

Then A has at least two fixed points on $(B_{r_0}\cap K)\backslash(B_{r_4}\cap K)$ and $(B_{r_5}\cap K)\backslash(B_{r_0}\cap K)$. This means that the Neumann boundary value problem (1.1), (1.2) has at least two positive solutions. □

Theorem 12. *Suppose that* $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ *is continuous. In addition, assume that*

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > M, \tag{5.4}$$

$$\liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} > M. \tag{5.5}$$

If there exists $r_0' > 0$ *such that*

$$f(t,u) < \xi' r_0', \quad \forall\ t\in[0,1],\ u\in[\gamma r_0', r_0'], \tag{5.6}$$

where $\gamma\in(0,1)$, $\xi' = \beta^{-1}$, *then the Neumann boundary value problem* (1.1), (1.2) *has at least two positive solutions.*

Proof. It follows from (5.4) and (5.5) that there exists $0 < r_4' < r_0'$ such that $f(t,u) \geq Mu$ for $0 \leq u \leq r_4'$ and there exist $r_5' > r_0'$ and $\varepsilon > 0$ such that $f(t,u) \geq (M+\varepsilon)u$ for $u \geq r_5'$. We may suppose that A has no fixed point on $\partial B_{r_4'} \cap K$ and $\partial B_{r_5'} \cap K$. Otherwise, the proof is completed.

We have from the proof in Theorem 4 and the permanence property of fixed point index that $i(A, B_{r_4'} \cap K, K) = 0$. It follows from the proof in Theorem 7 that $i(A, B_{r_5'} \cap K, K) = 0$.

For every $u \in B_{r_0'} \cap K$, we have $\gamma r_0' = \gamma\|u\| \leq u(t) \leq r_0'$, $0 \leq t \leq 1$. It follows that

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s) f(s,u(s))ds \\ &\leq \int_0^1 \beta f(s,u(s))ds \\ &< \beta\xi' r_0' \\ &= r_0', \quad t\in[0,1]. \end{aligned}$$

Then $\|Au\| < \|u\|$, for any $u \in \partial B_{r_0'} \cap K$. Hence we have from Lemma 10 that $i(A, B_{r_0'} \cap K, K) = 1$.

Therefore,

$$i(A, (B_{r_0'} \cap K)\backslash(B_{r_4'} \cap K), K) = i(A, B_{r_0'} \cap K, K) - i(A, B_{r_4'} \cap K, K) = 1,$$

$$i(A, (B_{r_5'} \cap K)\backslash(B_{r_0'} \cap K), K) = i(A, B_{r_5'} \cap K, K) - i(A, B_{r_0'} \cap K, K) = -1.$$

Then A has at least two fixed points on $(B_{r_0'} \cap K)\backslash(B_{r_4'} \cap K)$ and $(B_{r_5'} \cap K)\backslash(B_{r_0'} \cap K)$. This means that the Neumann boundary value problem (1.1), (1.2) has at least two positive solutions. □

6 Nonexistence results

In this section we are concerned with the nonexistence of positive solutions.

Theorem 13. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and*

$$\inf_{u\to 0^+} \min_{0\le t\le 1} \frac{f(t,u)}{u} > M. \tag{6.1}$$

Then the Neumann boundary value problem (1.1), (1.2) *has no positive solution* $u_0 \in K$.

Proof. If Neumann BVP (1.1), (1.2) has a nonzero solution $u_0 \in K$, then u_0 satisfies

$$\begin{cases} -u_0'' + Mu_0 = f(t,u_0), & 0 < t < 1, \\ u_0'(0) = u_0'(1) = 0. \end{cases} \tag{6.2}$$

It follows from (6.1) that there exists $\varepsilon' > 0$ such that

$$f(t,u_0(t)) \ge (M+\varepsilon')u_0(t),\ \ t \in [0,1]. \tag{6.3}$$

Integrating Eq. (6.2) from 0 to 1 and from (6.3) we get

$$M\int_0^1 u_0(t)dt = \int_0^1 f(t,u_0(t))dt \ge (M+\varepsilon')\int_0^1 u_0(t)dt.$$

Since $\int_0^1 u_0(t)dt > 0$, we conclude that $M \ge M+\varepsilon'$, which is a contradiction. Therefore Neumann BVP ((1.1), (1.2) has no positive solution. □

Theorem 14. *Suppose* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and*

$$\inf_{u\to 0^+} \min_{0\le t\le 1} \frac{f(t,u)}{u} < M. \tag{6.4}$$

Then the Neumann boundary value problem (1.1), (1.2) *has no positive solution* $\widetilde{u}_0 \in K$.

Proof. If Neumann BVP (1.1), (1.2) has a nonzero solution $\widetilde{u}_0 \in K$, then $\widetilde{u}_0$ satisfies

$$\begin{cases} -\widetilde{u}_0'' + M\widetilde{u}_0 = f(t,\widetilde{u}_0), & 0 < t < 1, \\ \widetilde{u}_0'(0) = \widetilde{u}_0'(1) = 0. \end{cases} \tag{6.5}$$

It follows from (6.4) that there exists $0 < \varepsilon'' < M$ such that

$$f(t,\widetilde{u}_0(t)) \le (M-\varepsilon'')\widetilde{u}_0(t),\ \ t \in [0,1]. \tag{6.6}$$

Integrating Eq. (6.5) from 0 to 1 and from (6.6) we get

$$M\int_0^1 \widetilde{u}_0(t)dt = \int_0^1 f(t,\widetilde{u}_0(t))dt \leq (M-\varepsilon'')\int_0^1 \widetilde{u}_0(t)dt.$$

Since $\int_0^1 \widetilde{u}_0(t)dt > 0$, we conclude that $M \leq M-\varepsilon''$, which is a contradiction. Therefore Neumann BVP (1.1), (1.2) has no positive solution. □

Remark 15. *From similar arguments and techniques, the results presented in this paper could be obtained for the following second-order Neumann boundary value problem:*

$$\begin{cases} u'' + Mu = f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Remark 16. *If we call* $g(t,u) = f(t,u) - Mu$ *then, the role of* M *is superfluous. That is, we can consider, without loss of the generality in the paper, that* $M = 0$ *and the positivity of* f *means that* $\frac{g(t,u)}{u}$ *is bounded from below for* $t \in [0,1]$ *and* $u > 0$.

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