Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **4** (2009), 111 – 117

SUBORDINATION AND SUPERORDINATION FOR CERTAIN ANALYTIC FUNCTIONS CONTAINING FRACTIONAL INTEGRAL

Rabha W. Ibrahim, Maslina Darus and Shaher Momani

Abstract. The purpose of the present article is to derive some subordination and superordination results for certain normalized analytic functions involving fractional integral operator. Moreover, this result is applied to find a relation between univalent solutions for fractional differential equation.

1 Introduction and Preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and for any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + \dots$. Let $\mathcal{A}$ be the class of all normalized analytic functions in U, such that $f(z)$ satisfies $f(0) = 0$ and $f'(0) = 1$.

Let F and G be analytic in the open unit disk U. The function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Alternatively, given two functions F and G, which are analytic in U, the function F is said to be subordinate to G in U if there exists a function h, analytic in U with

$$h(0) = 0 \quad and \quad |h(z)| < 1 \ \ for\ all\ z \in U$$

such that

$$F(z) = G(h(z)) \ \ for\ all\ z \in U.$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z), zp'(z))$ then p is called a solution of the differential superordination. An analytic function

2000 Mathematics Subject Classification: 34G10; 26A33; 30C45.
Keywords: Fractional calculus; Univalent solution; Subordination; Superordination.

q is called subordinant of the solution of the differential superordination if $q \prec p$. Subordination and superordination are studied by many authors for different classes of analytic functions (see[3]). To derive our results, we have to recall the following definitions and lemmas.

In [8], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1. *The fractional derivative of order α is defined, for a function $f(z)$ by*

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \leq \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 2. *The fractional integral of order α is defined, for a function $f(z)$, by*

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3. *[7] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where $E(f) := \{\zeta \in \partial U : lim_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.*

Lemma 4. *(see[6]). Let $q(z)$ be univalent in the open unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*
1. $Q(z)$ is starlike univalent in U, and
2. $\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.
If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 5. *(see[2]).Let q be convex univalent in the open unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*
1. $zq'(z)\varphi(q(z))$ is starlike univalent in U, and
2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.
If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Our work is organized as follows: In section 2, we will derive subordination and superordination results for normalized analytic functions involving fractional integral in the open unit disk U

$$q_1(z) \prec \frac{zI_z^\alpha f'(z)}{I_z^\alpha f(z)} \prec q_2(z),\ z \in U.$$

In section 3, we study the existence of univalent solution for the fractional differential equation

$$D_z^\alpha[\frac{u(z)}{z} I_z^\alpha f(z)] = h(z),\ \ 0 < \alpha \leq 1, \tag{1.1}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$, $h : U \to \mathbb{C}$, and $f : U \to \mathbb{C} - \{0\}$ are analytic functions in U. The existence is obtained by applying Schauder fixed point theorem.

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets) then it is said to be *completely continuous* on M. A mapping $A : X \to X$ is said to be a contraction if there exists a real number κ, $0 \leq \kappa < 1$ such that $\|Ax - Ay\| \leq \kappa\|x - y\|$ for all $x, y \in X$.

Theorem 6. *Arzela-Ascoli [4] Let E be a compact metric space and $\mathcal{C}(E)$ be the Banach space of real or complex valued continuous functions normed by*

$$\|f\| := sup_{t\in E}|f(t)|.$$

If $A = \{f_n\}$ is a sequence in $\mathcal{C}(E)$ such that f_n is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Theorem 7. *(Schauder) [1] Let X be a Banach space, $M \subset x$ a nonempty closed bounded convex subset and $P : M \to M$ is compact. Then P has a fixed point.*

2 Subordination and superordination results

By using Lemma 4, we first prove the following subordination

Theorem 8. *Let $f \in A$ and $q(z)$ be univalent in U. Assume that $zq'(z)$ is starlike univalent in U and*

$$\Re\{2 + \frac{zq''(z)}{q'(z)}\} > 0,\ \ z \in U.$$

If the subordination

$$[\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}] \prec q(z) + zq'(z)$$

holds then

$$\frac{zI_z^{\alpha}f'(z)}{I_z^{\alpha}f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Setting

$$p(z) := \frac{zI_z^{\alpha}f'(z)}{I_z^{\alpha}f(z)},$$

$$\theta(\omega) := \omega \ \ and \ \ \phi(\omega) := 1,$$

it can easily be observed that $\theta(z), \phi(z)$ are analytic in $\mathbb{C}$. Also, we let

$$Q(z) := zq'(z)\phi(z) = zq'(z),$$

$$h(z) := \theta(q(z)) + Q(z) = q(z) + zq'(z).$$

By the assumptions of the theorem we find that $Q(z)$ is starlike univalent in U and that

$$\Re\{\frac{zh'(z)}{Q(z)}\} = \Re\{2 + \frac{zq''(z)}{q'(z)}\} > 0.$$

Now we must show that

$$p(z) + zp'(z) \prec q(z) + zq'(z).$$

A computation shows that

$$\begin{aligned} p(z) + zp'(z) &= [\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}] \\ &\prec q(z) + zq'(z) \end{aligned}$$

Thus we have, $p(z) \prec q(z)$ and q is the best dominant. □

By using Lemma 5, we prove the following superordination.

Theorem 9. *Let $f \in A$ and $q(z)$ be convex univalent in U. Let the following assumptions hold: $zq'(z)$ is starlike univalent in U, $\frac{zI_z^{\alpha}f'(z)}{I_z^{\alpha}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$[\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}]$$

is univalent in U. If the subordination

$$q(z) + zq'(z) \prec [\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}]$$

holds then

$$q(z) \prec \frac{zI_z^{\alpha}f'(z)}{I_z^{\alpha}f(z)},$$

and $q(z)$ is the best subordinant.

Proof. Setting

$$p(z) := \frac{zI_z^\alpha f'(z)}{I_z^\alpha f(z)},$$

$$\vartheta(\omega) := \omega, \ \ and \ \ \varphi(\omega) := 1,$$

it can be easily observed that both $\vartheta(\omega)$ and $\varphi(\omega)$ are analytic in $\mathbb{C}$. Now,

$$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} = 1 > 0.$$

Then a computation shows that

$$\begin{aligned} q(z) + zq'(z) &= [\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}] \\ &= p(z) + zp'(z). \end{aligned}$$

Thus by applying Lemma 5, our proof of the theorem is complete. □

Combining the results of differential subordination and superordination, we state the following (sandwich result).

Theorem 10. *Let $f \in \mathcal{A}$, $q_1(z)$ be convex univalent in U, $q_2(z)$ be univalent in U, $zq_i'(z)$, $i = 1, 2$ be starlike univalent in U, $\frac{zI_z^\alpha f'(z)}{I_z^\alpha f(z)} \in \mathcal{H}[0,1] \cap Q$ and $[\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}]$ be univalent in U. If the subordination*

$$q_1(z) + zq_1'(z) \prec [\frac{zf'(z)}{f(z)}][\frac{z}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}] \prec q_2(z) + zq_2'(z),$$

holds then

$$q_1(z) \prec \frac{zI_z^\alpha f'(z)}{I_z^\alpha f(z)} \prec q_2(z)$$

and $q_1(z)$, $q_2(z)$ are the best subordinant and the best dominant, respectively.

3 Existence of univalent solution

In this section, we establish the existence of univalent solution for the equation (1.1). Let $\mathcal{B} := \mathcal{C}[U, \mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup. norm

$$\|u\| := sup_{z\in U}|u(z)|.$$

Lemma 11. *If the function $f \in \mathcal{A}$, then the initial value problem (1.1) is equivalent to the nonlinear Volterra integral equation*

$$u(z)=\frac{zI_z^\alpha h(z)}{I_z^\alpha f(z)};\ \ \alpha>0,\ \ I_z^\alpha f(z)\neq 0. \tag{3.1}$$

In other words, every solution of the Volterra equation (3.1) is also a solution of the initial value problem (1.1). The proof comes from the properties of the fractional operators (see [5]).

Theorem 12. *If $f,h\in A$, then the equation (1.1) has at least one locally univalent solution.*

Proof. Define an operator $P:\mathcal{B}\to\mathcal{B}$ as follows

$$Pu(z):=\frac{zI_z^\alpha h(z)}{I_z^\alpha f(z)},\ \ for\,all\ z\in U. \tag{3.2}$$

We can observed that $|(Pu)(z)|<\frac{\|h\|}{\|f\|}:=r$ thus $P:B_r\to B_r$. Now we show that P is an equicontinuous mapping on $S:=\{u\in\mathcal{B}:\|u\|\le r\}$. For $z_1,z_2\in U$ such that $|z_1-z_2|<\epsilon,\ \epsilon>0$, then we obtain, for all $u\in S$,

$$\begin{aligned}|Pu(z_1)-Pu(z_2)|&=|\frac{z_1I_{z_1}^\alpha h(z_1)}{I_{z_1}^\alpha f(z_1)}-\frac{z_2I_{z_2}^\alpha h(z_2)}{I_{z_2}^\alpha f(z_2)}|\\&\le|\frac{z_1\|h\|}{\|f\|}-\frac{z_2\|h\|}{\|f\|}|\\&\le r|z_1-z_2|\\&<r\epsilon.\end{aligned}$$

Hence P is equicontinuous mapping on S. The Arzela-Ascoli theorem yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a solution of the initial value problem (1). Now for $z_1\neq z_2\in U$, we can verify that $Pu(z_1)\neq Pu(z_2)$. Hence P is univalent in U. □

In the next theorem , we use the results of subordination and superordination to establish the relation between solutions of the problem (1.1).

Theorem 13. *Let the assumption of Theorem 10 be hold. Then solutions of the problem (1.1) satisfy the subordination $q_1(z)\prec u(z)\prec q_2(z)$.*

Proof. Setting $h(z)=f'(z)$, we see the result of the theorem. □

Acknowledgement. The work presented here was supported by eScienceFund 04-01-02-SF0425, MOSTI, Malaysia.

References

[1] K. Balachandar and J. P. Dauer, *Elements of Control Theory*, Narosa Publishing House, 1999.

[2] T. Bulboaca, *Classes of first-order differential superordinations*, Demonstr.Math. **35** (2) (2002), 287-292. MR1907301(2003m:30054). Zbl 1010.30020.

[3] Nak Eun Cho, Oh Sang Kwon, S. Owa and H. M. Srivastava, *A class of integral operators preserving subordination and superordination for meromorphic functions*, Appl. Math. and Compu. **193** (1) (2007), 463-474. MR2385804(2009a:30054).

[4] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press. 1977. MR0479787(58 #2). Zbl 0448.46002.

[5] R. W. Ibrahim and M. Darus, *Subordination and superordination for univalent solutions for fractional differential equations*, J. Math. Anal. Appl., vol. **345**, no. 2, (2008), 871-879. MR2429186. Zbl 1147.30009.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinants: Theory and Applications*, Monographs and Texbooks in Pure and Applied Mathematics, vol **225**, Marcel Dekker, New York, 2000. MR1760285(2001e:30036). Zbl 0456.30022.

[7] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, **48** (10) (2003), 815-826. MR2014390(2004i:30015). Zbl 1039.30011.

[8] H. M. Srivastava and S. Owa, *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989. MR1199135(93f:30003). Zbl 0683.00012.

Rabha W. Ibrahim, Maslina Darus
School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600, Selangor Darul Ehsan,
Malaysia.

e-mail:rabhaibrahim@yahoo.com
e-mail:maslina@ukm.my

Shaher Momani
Department of Mathematics,
Mutah. University,
P.O. Box 7,
Al-Karak,
Jordan.

e-mail:shahermm@yahoo.com