Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **4** (2009), 139 – 153

COMPUTING OPTIMAL CONTROL WITH A QUASILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. This paper presents the numerical solution of a constrained optimal control problem (COCP) for quasilinear parabolic equations. The COCP is converted to unconstrained optimization problem (UOCP) by applying the exterior penalty function method. Necessary optimality conditions for the considered problem are established. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. Numerical results are reported.

1 Introduction

Optimal control problems for partial differential equations are currently of much interest. A large amount of the theoretical concept which governed by quasilinear parabolic equations has been investigated in the field of optimal control problems [1], [2] and [18]. These problems have dealt with the processes of hydro- and gas dynamics, heat physics, filtration, the physics of plasma and others [12] and [15]. From the mathematical point of view, the definition and refinement of the unknown parameters of the model present the problem of identification and optimal control of partial differential equations. The importance of investigating the identification and optimal control problems was developed in [6] and [9]. This paper presents the numerical solution of a constrained optimal control problem (COCP) for quasilinear parabolic equations. The COCP is converted to unconstrained optimization problem (UOCP) by applying the exterior penalty function method. Necessary optimality conditions for the considered problem are established. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. Numerical results are reported.

Let D be a bounded domain of the N-dimensional Euclidean space E_N. Let $V = \{v : v = (v_1, v_2, ..., v_N) \in E_N, \|v\|_{E_N} \leq R\}$, where $R > 0$ is a given number. Let l,T be given positive numbers and let $\Omega = \{(x,t) : x \in D = (0,l), t \in (0,T)\}$.

2000 Mathematics Subject Classification: 49J20; 49K20; 49M29; 49M30.

Keywords: Optimal control; Parabolic Equation; Penalty function methods; Existence theory; Necessary optimality conditions.

Now, we need to introduce some functional spaces as follows [10]:
0) E_N is the N-dimensional Euclidean space with

$$\langle v_1, v_2\rangle_{E_N} = \sum_{i=1}^{N}(v_1)_i(v_2)_i, \|v\|_{E_N} = \sqrt{\langle v, v\rangle_{E_N}}.$$

1) $L_2(D)$ is a Banach space which consisting of all measurable functions on D with the norm

$$\|z\|_{L_2(D)} = [\int_D \ |z|^2 \ dx \ ]^{\frac{1}{2}}.$$

2) $L_2(0, l)$ is a Hilbert space which consisting of all measurable functions on $(0, l)$ with

$$\langle z_1, z_2\rangle_{L_2(0,l)} = \int_0^l z_1(x)z_2(x)dx, \|z\|_{L_2(0,l)} = \sqrt{\langle z, z\rangle_{L_2(0,l)}}.$$

3) $L_2(0, T)$ is a Hilbert space which consisting of all measurable functions on $(0, T)$ with

$$\langle z_1, z_2\rangle_{L_2(0,T)} = \int_0^T z_1(t)z_2(t)dt, \|z\|_{L_2(0,T)} = \sqrt{\langle z, z\rangle_{L_2(0,T)}}.$$

4) $L_2(\Omega)$ is a Hilbert space which consisting of all measurable functions on Ω with

$$\langle z_1, z_2\rangle_{L_2(\Omega)} = \int_0^l\int_0^T z_1(x,t)z_2(x,t)dxdt, \|z\|_{L_2(\Omega)} = \sqrt{\langle z, z\rangle_{L_2(\Omega)}}.$$

5) $W_2^{1,0}(\Omega) = \{z \in L_2(\Omega) \ \ and \ \ \frac{\partial z}{\partial x} \in L_2(\Omega)\}$ is a Hilbert space with

$$\langle z_1, z_2\rangle_{W_2^{1,0}(\Omega)} = \int_\Omega \ [\ \ z_1z_2 + \frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}]dxdt$$

$$\|z\|_{W_2^{1,0}(\Omega)} = [\|z\|^2_{L_2(\Omega)} + \|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

6) $W_2^{1,1}(\Omega) = \{z \in L_2(\Omega) \ \ and \ \ \frac{\partial z}{\partial x} \in L_2(\Omega), \frac{\partial z}{\partial t} \in L_2(\Omega)\}$ is a Hilbert space with

$$\langle z_1, z_2\rangle_{W_2^{1,1}(\Omega)} = \int_\Omega \ [\ \ z_1z_2 + \frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x} + \frac{\partial z_1}{\partial t}\frac{\partial z_2}{\partial t}]dxdt$$

$$\|z\|_{W_2^{1,1}(\Omega)} = [\|z\|^2_{L_2(\Omega)} + \|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)} + \|\frac{\partial z}{\partial t}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

7) $V_2(\Omega)$ is a Banach space which consisting of the elements of the space $W_2^{1,0}(\Omega)$ with the norm

$$\|z\|_{V_2(\Omega)} = vraimax_{0\le t\le T}\|z(x,t)\|_{L_2(D)} + (\int_\Omega |\frac{\partial z}{\partial x}|^2)^{\frac{1}{2}}.$$

8) $V_2^{1,0}(\Omega)$ is a subspace of $V_2(\Omega)$, the elements of which have in sections $D_t = \{(x, \tau) : x \in D, \tau = t\}$ traces from $L_2(D)$ at all $t \in [0, T]$, continuously changing from $t \in [0, T]$ in the norm $L_2(D)$.

2 Problem Formulation

We consider the heat exchange process described by the following quasilinear parabolic equation:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x}) + B(u,v)u(x,t) = f(x,t), (x,t) \in \Omega \tag{2.1}$$

with initial and boundary conditions

$$u(x,0) = \phi(x), x \in (0,l) \tag{2.2}$$

$$\lambda(u,v)\frac{\partial u}{\partial x}|_{x=0} = g_0(t), \lambda(u,v)\frac{\partial u}{\partial x}|_{x=l} = g_1(t), 0 \le t \le T \tag{2.3}$$

where $\phi(x) \in L_2(D)$, $g_0(t), g_1(t) \in L_2(0,T)$ and $f(x,t) \in L_2(\Omega)$ is a given function. Besides, functions $\lambda(u,v), B(u,v)$ are continuous on $(u,v) \in [r_1, r_2]$ x E_N, have continuous derivatives in u, $\forall (u,v) \in [r_1,r_2]$ x E_N and the derivatives $\frac{\partial\lambda(u,v)}{\partial u}, \frac{\partial B(u,v)}{\partial u}$ are bounded. Here r_1, r_2 are given numbers.

On the set V, under the conditions (2.1)-(2.3) and additional restrictions

$$\nu_0 \le \lambda(u,v) \le \mu_0, \qquad \nu_1 \le B(u,v) \le \mu_1 \qquad r_1 \le u(x,t) \le r_2 \tag{2.4}$$

is required to minimize the function [4]

$$J_\alpha(v) = \beta_0 \int_0^T [u(0,t) - f_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - f_1(t)]^2 dt + \alpha \|v - \omega\|^2_{E_N} \tag{2.5}$$

where $f_0(t)$, $f_1(t) \in L_2(0,T)$ are given functions, $\alpha \ge 0, \nu_i, \mu_i > 0, i = 0,1$, $\beta_0 \ge 0, \beta_1 \ge 0$, $\beta_0 + \beta_1 \ne 0$ are given numbers, $\omega \in E_N$ is also given : $\omega = (\omega_1, \omega_2, ..., \omega_N)$.

Definition 1. *The problem of finding a function* $u = u(x,t) \in V_2^{1,0}(\Omega)$ *from conditions (2.1)-(2.4) at given* $v \in V$ *is called the reduced problem.*

Definition 2. *The solution of the reduced problem (2.1)-(2.4) corresponding to the* $v \in V$ *is a function* $u(x,t) \in V_2^{1,0}(\Omega)$ *and satisfies the integral identity*

$$\int_0^l \int_0^T [u\frac{\partial \eta}{\partial t} - \lambda(u,v)\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x} - B(u,v)u(x,t)\eta + f(x,t)\eta]dxdt = \tag{2.6}$$
$$- \int_0^l \phi(x)\eta(x,0)dx + \int_0^T \eta(0,t)g_0(t)dt - \int_0^T \eta(l,t)g_1(t)dt,$$

$\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ *and* $\eta(x,T) = 0.$

The solution of the reduced problem (2.1)-(2.4) explicitly depends on the control v, therefore we shall also use the notation $u = u(x, t; v)$.

On the basis of adopted assumptions and the results of [11] follows that for every $v \in V$ the solution of the problem (2.1)-(2.4) exists, unique and $|u_x| \leq C_0$, $\forall (x,t) \in \Omega$, $\forall v \in V$, where C_0 is a certain constant.

The inequality constrained problem (2.1)-(2.5) is converted to a problem without inequality constrains $\{UOCP\}$ by applying the exterior penalty method [17], yielding the following function $\Phi_{\alpha,k}(v, A_k)$:

$$\Phi_{\alpha,k}(v, A_k) \equiv \Phi(v) = f_\alpha(v) + P_k(v) \tag{2.7}$$

where

$$\begin{aligned} Z(u,v) &= [\max\{\nu_0 - \lambda(u,v); 0\}]^2 + [\max\{\lambda(u,v) - \mu_0; 0\}]^2 \\ Y(u,v) &= [\max\{\nu_1 - B(u,v); 0\}]^2 + [\max\{B(u,v) - \mu_1; 0\}]^2 \\ Q^1(u) &= [\max\{r_1 - u(x,t;v); 0\}]^2, Q^2(u) = [\max\{u(x,t;v) - r_2; 0\}]^2 \\ P_k(v) &= A_k \int_0^l \int_0^T [Z(u,v) + Y(u,v) + Q^1(u) + Q^2(u)] dxdt \end{aligned}$$

and A_k , k=1,2,... are positive numbers, $\lim_{k\to\infty} A_k = +\infty$.

3 Well-posedness of the control problem

Optimal control problems of the coefficients of differential equations do not always have solution [14].

Lemma 3. *Let $\delta v = (\delta v_1, \delta v_2, \cdots, \delta v_N)$ be an increment of control on element $v \in V$ such that $v + \delta v \in V$. Let $\delta u(x,t) = u(x,t; v + \delta) - u(x,t; v)$ and $u = u(x,t;v)$. At above adopted assumptions, then $\delta u(x,t)$ satisfies the following estimation*

$$\|\delta u(x,t)\|_{V_2^{1,0}(\Omega)} \leq C_1[\|\delta\lambda \frac{\partial u(x,t)}{\partial x}\|^2_{L_2(\Omega)} + \|\delta B u(x,t)\|^2_{L_2(\Omega)}]^{\frac{1}{2}} \tag{3.1}$$

where $\delta\lambda = \lambda(u, v+\delta v) - \lambda(u,v)$, $\delta B = B(u, v+\delta v) - B(u,v)$ and $C_1 \geq 0$ is constant not depending on δv.

Proof. The proof is quite similar to the one given in Farag [5]. □

Corollary 4. *From the above adopted assumptions the right part of estimation (3.1) converges to zero at $\|\delta v\|_{E_N} \to 0$. Therefore $\|\delta u\|_{V_2^{1,0}(\Omega)} \to 0$ as $\|\delta v\|_{E_N} \to 0$. Hence from trace theorem [13] we get*

$$\|\delta u(0,t)\|_{L_2(0,T)} \to 0, \|\delta u(l,t)\|_{L_2(0,T)} \to 0, at\ \|\delta v\|_{E_N} \to 0 \tag{3.2}$$

Lemma 5. *The function* $J_0(v) = \beta_0 \int_0^T [u(0,t) - f_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - f_1(t)]^2 dt$ *is continuous on* V.

Proof. Let $\delta v = (\delta v_1, \delta v_2, \cdots, \delta v_N)$ be an increment of control on element $v \in V$ such that $v + \delta v \in V$. For the increment of the function $J_0(v)$ the following holds

$$\delta J_0(v) = J_0(v+\delta v) - J_0(v) = 2\beta_0 \int_0^T [u(0,t) - f_0(t)]\delta u(0,t)dt$$
$$+2\beta_1 \int_0^T [u(l,t) - f_1(t)]\delta u(l,t)dt + \beta_0 \int_0^T [\delta u(0,t)]^2 dt + \beta_1 \int_0^T [\delta u(l,t)]^2 dt. \quad (3.3)$$

Applying the Cauchy Bunyakoviskii inequality, we obtain

$$|\delta J_0(v)| \le 2\|\beta_0[u(0,t) - f_0(t)\|_{L_2(0,T)}\|\delta u(0,t)\|_{L_2(0,T)} + \beta_0\|\delta u(0,t)\|^2_{L_2(0,T)}$$
$$+2\beta_1\|u(l,t) - f_1(t)\|_{L_2(0,T)}\|\delta u(l,t)\|_{L_2(0,T)} + \beta_1\|\delta u(l,t)\|^2_{L_2(0,T)}. \quad (3.4)$$

From corollary 4 and (3.4), the continuity of the function $J_0(v)$ on V follows. Then the lemma 5 is proved. □

Theorem 6. *The problem (2.1)-(2.5) at any* $\alpha \ge 0$ *has at least one solution.*

Proof. The set V is closed and bounded in E_N. From the affirmation of lemma 5 follows the continuity of function $J_0(v)$ on V, the function $J_\alpha(v) = J_0(v) + \alpha\|v - \omega\|^2_{E_N}$ will be continuous on V also. Then from the Weierstrass's Theorem [9] follows that the problem (2.1)-(2.5) has at least one solution. Theorem 6 is proved. □

Theorem 7. *The problem (2.1)-(2.5) at* $\alpha > 0$, *at almost all* $\omega \in E_N$ *has a unique solution.*

Proof. From lemma 5 follows the continuity of function J_0 and $J_\alpha(v), \alpha > 0$ will be continuous on V also. Besides,E_N is a uniformly convex space, then from a theorem in [7] follows the existence of a dense subset K of the space E_N such that for any $\omega \in K$ at $\alpha > 0$ the problem (2.1)-(2.5) has a unique solution. Consequently for almost all $\omega \in E_N$ and $\omega > 0$ the problem (2.1)-(2.5) has a unique solution. Theorem 7 is proved. □

4 The adjoint problem

The Lagrangian function $L(x,t,u,v,\Theta)$ for the OCP problem is defined as

$$L(x,t,u,v,\Theta) = \beta_0 \int_0^T [u(0,t) - f_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - f_1(t)]^2 dt + \quad (4.1)$$
$$+\alpha\|v-\omega\|^2_{E_N} + A_k \int_0^l \int_0^T [Z(u,v) + Y(u,v) + Q^1(u) + Q^2(u)]dxdt$$
$$+ \int_0^l \int_0^T \Theta[\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x}) + B(u,v)u(x,t) + f(x,t)]dxdt.$$

Setting the variation in the Lagrangian equal to zero, the first order necessary condition for minimizing $L(x,t,u,v,\Theta)$, we obtain the adjoint problem [4]:

$$\Theta_t + (\lambda(u,v)\Theta_x)_x - \lambda_u(u,v)\Theta_x u_x = (B_u u + B)\Theta(x,t) + A_k[Z_u(u,v) + Y_u + Q_u^2 + Q_u^1], (x,t) \in \Omega \quad (4.2)$$

$$\Theta(x,T) = 0, x \in D \quad (4.3)$$

$$\lambda\Theta_x \mid_{x=0} = 2\beta_0[u(0,t) - f_0(t)]$$
$$\lambda\Theta_x \mid_{x=l} = -2\beta_1[u(l,t) - f_1(t)], t \in [0,T], \quad (4.4)$$

where $u = u(x,t)$ is the solution of problem (2.1)-(2.3) corresponding to $v \in V$.

Definition 8. *The solution of the adjoint problem (4.2)-(4.4) corresponding to the $v \in V$ is a function $\Theta(x,t) \in V_2^{1,0}(\Omega)$ and the following integral identity is then satisfied to*

$$\int_0^l \int_0^T [\Theta\frac{\partial\gamma}{\partial t} + \lambda(u,v)\frac{\partial\Theta}{\partial x}\frac{\partial\gamma}{\partial x} + \lambda_u(u,v)\frac{\partial\Theta}{\partial x}\frac{\partial u}{\partial x}\gamma + (B_u u + B)\Theta(x,t)\gamma(x,t)]dxdt$$
$$= -A_k \int_0^l \int_0^T [Z_u(u,v) + Y_u + Q_u^2 + Q_u^1]\gamma(x,t)dxdt \quad (4.5)$$

where $\forall\gamma = \gamma(x,t) \in W_2^{1,1}(\Omega)$ *and* $\gamma(x,0) = 0$.

On the basis of adopted assumptions and the results of [10] follows that for every $v \in V$ the solution of the conjugated boundary value problem (4.2)-(4.4) is exists, unique and $|\Theta_x| \leq C_2$ almost at all $(x,t) \in \Omega$, $\forall v \in V$, where C_2 is a certain constant.

5 Gradient of the Cost functional

The sufficient differentiability conditions of function (2.7) and its gradient formulae will be obtained by defining the Hamiltonian function [3] $H(u,\Theta,v)$ as:

$$H(u,\Theta,v) \equiv -\int_0^l \int_0^T [\lambda(u,v)\Theta_x u_x + B(u,v)u(x,t)\Theta(x,t)]dxdt$$
$$-A_k \int_0^l \int_0^T [Z(u,v) + Y(u,v)]dxdt - \alpha\|v-\omega\|^2_{E_N} \quad (5.1)$$

Theorem 9. *It is assumed that the following conditions are fulfilled:*
(i) The functions $\lambda(u,v), B(u,v)$ satisfy the Lipshitz condition for v.
(ii) The first derivatives of the functions $\lambda(u,v), B(u,v)$ with respect to v are continuous functions and for any $v \in V$ such that $\|v\|_{E_N} \leq R$, the functions $\lambda_v(u,v), B_v(u,v)$

belong to $L_\infty(\Omega)$.
(iii) Operators $\int_0^l\int_0^T \lambda_v(u,v)dxdt$ *and* $\int_0^l\int_0^T B_v(u,v)dxdt$ *are bounded in* E_N.

Therefore, the function $\Phi_{\alpha,k}(v)\equiv\Phi(v)$ *is differentiable and its gradient is given by the expression :*

$$\frac{\partial\Phi(v)}{\partial v}=-\frac{\partial H}{\partial v}\equiv(-\frac{\partial H}{\partial v_1},-\frac{\partial H}{\partial v_2},...,-\frac{\partial H}{\partial v_N}). \tag{5.2}$$

Proof. Suppose that $v\equiv(v_1,v_2,...,v_N)$, $\delta v\equiv(\delta v_1,\delta v_2,...,\delta v_N)$,$\delta v\in E_N$,$v+\delta v\in V$ and denoting $\delta u\equiv u(x,t;v+\delta v)-u(x,t;v)$. The increment of the function $\Phi(v)$ can be expressed as:

$$\begin{gathered}\delta\Phi(v)=\Phi(v+\delta v)-\Phi(v)=\\ =2\beta_0\int_0^T[u(0,t)-f_0(t)]\delta u(0,t)dt+2\beta_1\int_0^T[u(l,t)-f_1(t)]\delta u(l,t)dt+\\ +A_k\int_0^l\int_0^T[Z_u(u,v+\delta v)+Y_u(u,v+\delta v)+Q_u^1(u)+Q_u^2(u)]\delta u(x,t)dxdt+\\ +A_k\int_0^l\int_0^T[Z(u,v+\delta v)-Z(u,v)+Y(u,v+\delta v)-Y(u,v)]dxdt\\ +2\alpha\langle v-\omega,\delta v\rangle_{E_N}+R_1(\delta v)\end{gathered} \tag{5.3}$$

where

$$R_1(\delta v)=\beta_0\int_0^T[\delta u(0,t)]^2dt+\beta_1\int_0^T[\delta u(l,t)]^2dt+\alpha\|\delta v\|_{E_N}^2. \tag{5.4}$$

Using the estimation (3.2), then the inequality $|R_1(\delta v)|\le C_3\|\delta v\|_{E_N}$ could be verified where C_3 is a constant not dependent on δv.

From results in ([8], p. 995-996), we put $\gamma=\delta u(x,t)$ in identity (4.5) and put $\eta=\Theta(x,t)$ in (2.6) and substract the obtained relations, then we have

$$\begin{gathered}2\beta_0\int_0^T[u(0,t)-f_0(t)]\delta u(0,t)dt+2\beta_1\int_0^T[u(l,t)-f_1(t)]\delta u(l,t)dt+\\ +A_k\int_0^l\int_0^T[Z_u(u,v+\delta v)+Y_u(u,v+\delta v)+Q_u^1(u)+Q_u^2(u)]\delta u(x,t)dxdt=\\ =\int_0^l\int_0^T[\delta\lambda u_x\Theta_x+u(x,t)\Theta(x,t)\delta B]dxdt+R_2(\delta v)\end{gathered} \tag{5.5}$$

where

$$\begin{gathered}R_2(\delta v)=\int_0^l\int_0^T\{[\lambda(u+\delta u,v+\delta v)-\lambda(u,v)]\tfrac{\partial\delta u}{\partial x}\tfrac{\partial\Theta}{\partial x}+\\ +[\tfrac{\partial\lambda(u+\theta_1\delta u,v+\delta v)}{\partial u}-\tfrac{\partial\lambda(u,v)}{\partial u}]\tfrac{\partial u}{\partial x}\tfrac{\partial\Theta}{\partial x}\delta u+\\ +[B(u+\delta u,v+\delta v)-B(u,v)]\Theta\tfrac{\partial\delta u}{\partial x}\\ +[\tfrac{\partial B(u+\theta_2\delta u,v+\delta v)}{\partial u}-\tfrac{\partial B(u,v)}{\partial u}]\Theta(x,t)u(x,t)\delta u(x,t)\}dxdt\end{gathered} \tag{5.6}$$

and $\theta_i\in(0,1), i=1,2$ are positive numbers.

In virtue of assumption (i), $R_2(\delta v)$ is estimated as $|R_2(\delta v)|\le C_4\|\delta v\|_{E_N}$, where C_4 is a constant and independent of δv. Using the above assumptions, we can estimate the following expansions as:

$$Z(u, v + \delta v) - Z(u, v) = \langle Z_v(u, v), \delta v\rangle_{E_N} + O(\|\delta v\|_{E_N})$$

$$Y(u, v + \delta v) - Y(u, v) = \langle Y_v(u, v), \delta v\rangle_{E_N} + O(\|\delta v\|_{E_N})$$

$$\delta\lambda = \langle \lambda_v(u, v), \delta v\rangle_{E_N} + O(\|\delta v\|_{E_N}),\ \delta B = \langle B_v(u, v), \delta v\rangle_{E_N} + O(\|\delta v\|_{E_N}).$$

By substituting the last three expansions in (5.3) and (5.5), we obtain

$$\delta\Phi(v) = \int_0^l \int_0^T \langle \lambda_v(u, v) u_x \Theta_x + B_v(u, v) u(x, t)\Theta(x, t)$$
$$+A_k(Z_v(u, v) + Y_v(u, v)), \delta v\rangle_{E_N} dxdt + 2\alpha\langle v - \omega, \delta v\rangle_{E_N} + R_3(\delta v) \tag{5.7}$$

where $R_3(\delta v) = R_1(\delta v) + R_2(\delta v) + O(\|\delta v\|_{E_N})$.

From the formula of $R_3(\delta v)$, we have

$$|R_3(\delta v)| \leq C_5\|\delta v\|_{E_N} \tag{5.8}$$

where C_5 is a constant and independent of δv.

From (5.7), (5.8) and using the function $H(u, \Theta, v)$, we have

$$\delta\Phi(v) = \langle -\frac{\partial H(u, \Theta, v)}{\partial v}, \delta v\rangle_{E_N} + O(\|\delta v\|_{E_N}) \tag{5.9}$$

which shows the differentiability of the function $\Phi(v)$ and also gives the gradient formulae of the function $\Phi(v)$. This completes the proof of the theorem. □

6 Necessary optimality conditions

Above the differentiability of the function $\Phi(v)$ and its gradient formulae are proved. Now, we are able to pass to prove the necessary conditions for optimization for the optimal control problem (2.1)-(2.3),(2.7).

Theorem 10. *Let all conditions of theorem 9 be fulfilled. In order that $v^* \in V$ be a solution of UOCP problem it is necessary that*

$$H(u^*, \Theta^*, v^*) = \max_{v\in V} H(u^*, \Theta^*, v) \tag{6.1}$$

where $u^*(x, t), \Theta^*(x, t)$ are, respectively, solutions of the basic problem (2.1)-(2.3) and the adjoint problem (4.2)-(4.4) at $v^* \in V$.

Proof. Suppose that $v^* \equiv (v_1^*, v_2^*, \cdots, v_N^*)$ is an optimal control. Suppose the contrary, i.e. will be found such a control $\overline{v} = v^* + \zeta\delta v \in V$ and number $\beta_3 > 0$ for which

$$H(u^*, \Theta^*, \overline{v}) - H(u^*, \Theta^*, v) \geq \beta_3 > 0, \tag{6.2}$$

where $\zeta > 0$ is a constant, $\overline{v} \equiv (\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_N) \equiv (v_1^* + \zeta\delta v_1, v_2^* + \zeta\delta v_2, \cdots, v_N^* + \zeta\delta v_N)$, $\delta v = (\delta v_1, \delta v_2, \cdots, \delta v_N)$.

If in (6.2) to take into account the formulae (5.2), then we obtain

$$\zeta\langle \frac{\partial\Phi(\hat{v})}{\partial v}, \delta v\rangle_{E_N} \leq -\beta_3 < 0, \tag{6.3}$$

where $\hat{v} = \zeta\theta_0\delta v = \zeta\theta_0(\delta v_1, \delta v_2, \cdots, \delta v_N)$,$\theta_0 \in (0, 1)$ are positive numbers.

Hence and from formula of finite increment, we have

$$\begin{aligned}\Phi(\overline{v}) - \Phi(v^*) &= \zeta\langle \tfrac{\partial\Phi(v^o)}{\partial v}, \delta v\rangle_{E_N} = \zeta\langle \tfrac{\partial\Phi(\hat{v})}{\partial v}, \delta v\rangle_{E_N} + \zeta\langle \tfrac{\partial\Phi(v^o)}{\partial v} - \tfrac{\partial\Phi(\hat{v})}{\partial v}, \delta v\rangle_{E_N}\\ &\leq -\beta_3 + \zeta\langle \tfrac{\partial\Phi(v^o)}{\partial v} - \tfrac{\partial\Phi(\hat{v})}{\partial v}, \delta v\rangle_{E_N} \leq -\beta_3 + \zeta O(\|\delta v\|_{E_N}),\end{aligned} \tag{6.4}$$

where $v^o = \zeta\theta_1\delta v = \zeta\theta_1(\delta v_1, \delta v_2, \cdots, \delta v_N)$,$\theta_1 \in (0, 1)$ are positive numbers.

Let $0 < \zeta_1 < \zeta$ is such a number that $-\beta_3 + \zeta_1 O(\|\delta v\|_{E_N}) < 0$. Put $\overline{\overline{v}} = v^* + \zeta_1\delta v$. Reasoning as in the proof of (6.4), we obtain

$$\Phi(\overline{\overline{v}}) - \Phi(v^*) \leq -\beta_3 + \zeta_1 O(\|\delta v\|_{E_N}) < 0. \tag{6.5}$$

This contradicts to the optimality of control v^*. Hence, we obtain the validity of relation (6.1). The theorem is proved. □

7 Numerical Results

The outlined of the algorithm for solving UOCP problem are as follows:

1- Given $k = 0, \sigma > 0, It = 0, A_k > 0, \epsilon > 0$ and $v^{(It)} \in V$.

2- At each iteration It, do

Solve (2.1)-(2.3), then find $u(., v^{(It)})$.

Solve the adjoint problem for (4.2)-(4.4), then find $\Theta(., v^{(It)})$.

Find optimal control $v_*^{(It+1)}$ using conjugate gradient method [16].

End do.

3- If $|\Phi(v^{(It+1)}) - \Phi(v^{(It)})| < \epsilon$, then Stop, else, go to Step 4.

4- Set $v^{(It+1)} = v^{(It)}, It = It + 1, k = k + 1, A_{k+1} = A_k * \sigma$ and go to Step 2.

The following theorem represents the main contribution of the convergence theory for the control sequence, generated by the above algorithm for UOCP problem.

Theorem 11. *Let $\{v^{(It)}\}$ be a sequence of minimizers to the UOCP problem which generated by the above algorithm for any increasing sequence of values A_k. Then $\{v^{(It)}\}$ converge to the optimum solution v_* of the constrained problem COCP as $A_k \to \infty$.*

Proof. The proof is similar to that of Theorem 7.4 [16]. □

The problem (2.1)-(2.5) is considered as one of the identification problems on definition of unknown coefficients of parabolic quasilinear equation type. The numerical results were carried out for the following examples:

1. $u = x + t, \lambda = tan^{-1}(u), B = u^2(1 - u^2), x \in [0, 0.9], t \in [0, 0.001]$
2. $u = x + t, \lambda = e^{-u} sinu, B = ln(1 + u), x \in [0, 0.7], t \in [0, 0.001]$
3. $u = x + t, \lambda = ln(\frac{1}{1-u}), B = e^{sinu}, x \in [0, 0.9], t \in [0, 0.001]$

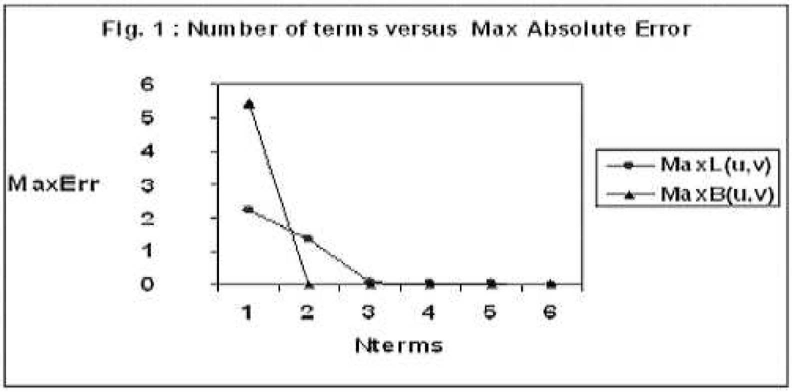
The input data of parameters and functions in our optimal control problem are given as follows:

Table 1

Parameters	ν_0	ν_1	μ_0	μ_1	r_0	r_1	β_0	β_1	α	ϵ
Example 1	0	0.7334	0	0.1527	0	0.901	0.9	0.9	0.9	10^{-6}
Example 2	0	0.3199	0	0.5312	0	0.701	0.9	0.9	0.9	10^{-6}
Example 3	0	2.3126	1	2.1901	0	0.901	0.9	0.9	0.9	10^{-6}

Table 2

Functions	Example 1	Example 2	Example 3
$\phi(x)$	x	x	x
$g_0 = f_0$	$tan^{-1}t$	$e^{-t} sint$	$ln(\frac{1}{1-t})$
$g_1 = f_1$	$tan^{-1}(0.9 + t)$	$e^{-(0.7+t)} sin(0.7 + t)$	$ln(\frac{1}{0.1-t})$
$f(x, t)$	$\frac{(x+t)^2}{1+(x+t)^2} + \frac{(x+t)^3}{[1-(x+t)^2]^{-1}}$	$1 + ln(1 + x + t)^{(x+t)}$ $+e^{-(x+t)} sin(x + t)$ $-e^{-(x+t)} cos(x + t)$	$2 + (x + t)e^{sin(x+t)}$ $-(x + t)$



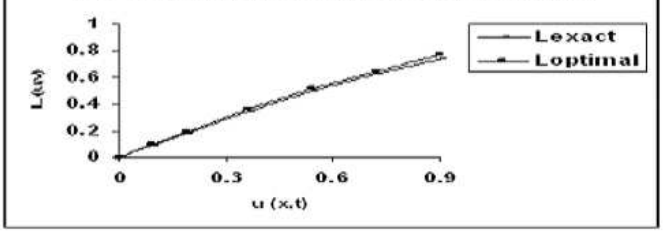


Fig. 2: Coefficients of QPDEs (Exact & Optimal)

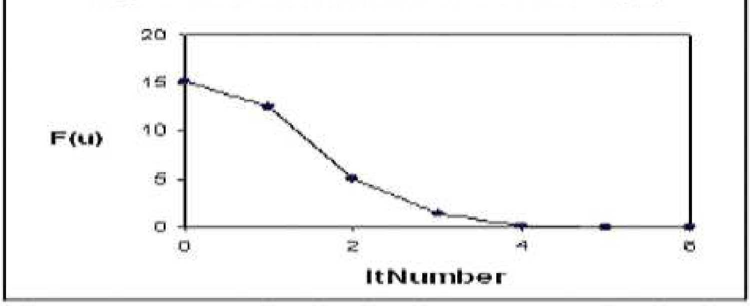


Fig. 3 : Iteration Number versus F(u)

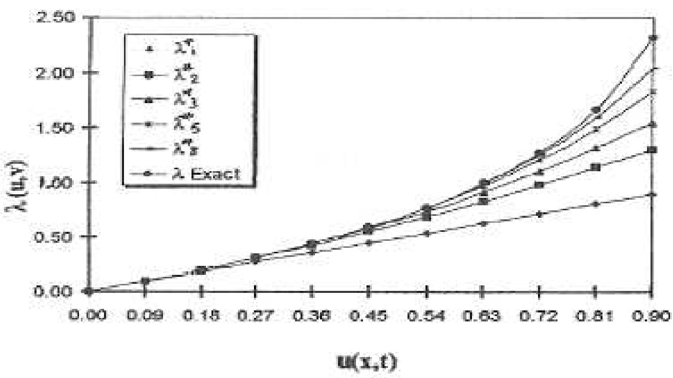


Fig. 4: The coefficient $\lambda(u,v) = \ln\left(\frac{1}{1-u}\right)$ **(Example 3)**

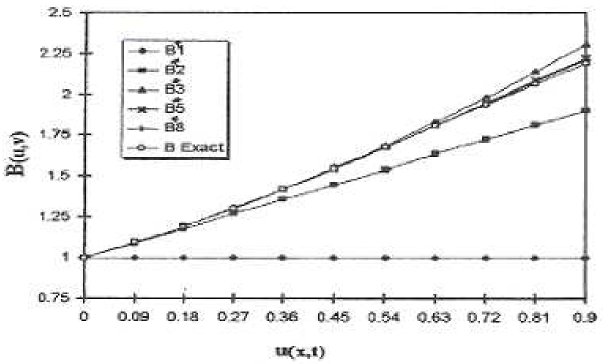


Fig. 5: The coefficient $B(u,v)=e^{\sin u}$ (Example 3)

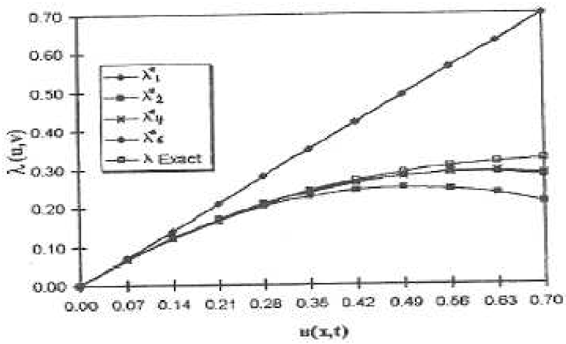


Fig. 6: The coefficient $\lambda(u,v)=e^{-u}\sin u$ (Example 2)

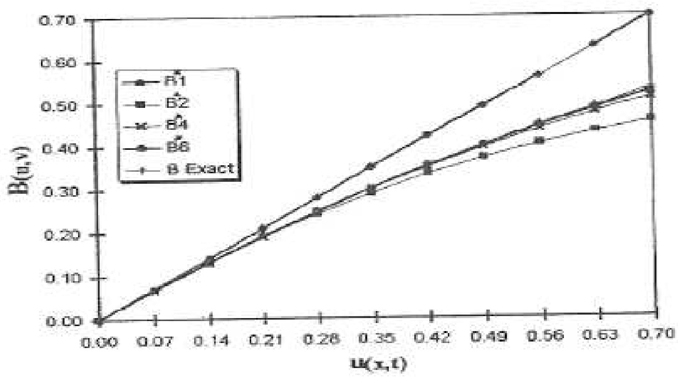


Fig. 7: The coefficient $B(u,v)=\ln(1+u)$ (Example 2)

The numerical study has given the following results:

1. Knowing the computed optimal control values v^* obtained by the above numerical algorithm, we can calculate the approximate values of the unknown coefficients $\lambda(u,v), B(u,v)$ each of which can be represented in a series as $\sum_{k=1}^{Nc} v_k u^k$, for example $\lambda(u,v) = \sum_{k=1}^{Nc/2} v_{2k-1}u^{2k-1}, B(u,v) = \sum_{k=1}^{Nc/2} v_{2k}u^{2k}$, where Nc is the number of controls.

2. For example 3,the maximum absolute errors for $\lambda(u,v), B(u,v)$ versus the number of control (Nterms) are displayed in Fig. 1. The curves denoted by Maxλ(u,v) and MaxB(u,v) are the maximum absolute errors $\Gamma_\lambda = max|\lambda_{Exact} - \lambda^*_{Nc/2}|$ and $\Gamma_B = max|B_{Exact} - B^*_{Nc/2}|$. It is clear that the maximum absolute errors decrease by increasing the number of control (Nterms).

3. For example 1, in Fig. 2, the curves denoted by Lexact and Loptimal are the exact values and approximate values of $\lambda(u,v)$ with the optimal control v^* versus the values of $u(x,t)$.

4. For example 2, the curve of the computing values of $\Phi(v)$ denoted by $F(u)$ in Fig. 3 versus the iteration numbers (It) are displayed.

5. For examples 2 and 3, in figures Fig. 4-Fig. 7 the curves denoted by $\lambda_1^*, \lambda_2^*, \lambda_3^*, \cdots$ and $B_1^*, B_2^*, B_3^*, \cdots$ are the approximate values of $\lambda(u,v)$ and $B(u,v)$ with v^*, while $\lambda_{Exact}, B_{Exact}$ are the exact values of $\lambda(u,v), B(u,v)$. Obviously, by increasing Nc, the coefficients $\lambda(u,v)$ and $B(u,v)$ are agree with the exact value.

6. For example 1, in Table 3, we report the number of function evaluations NEF needed to compute the solution with an accuracy on the modified function $\Phi(v)$ of the order 10^{-6}. The above algorithm takes 6 iterations for decreasing $\Phi(v)$ to the value $0.8731989E-06$.

Table 3

It	$\Phi(v)$	$f_\alpha(v)$	$P_k(v)$	A_k	NEF
0	15.2245300	15.2245300	0.0000000	0.0000000	1
1	12.5949100	12.5939900	9.240004E-04	0.500000E+01	169
2	5.0931420	5.0926800	4.620002E-04	0.100000E+02	506
3	1.4406350	1.4404040	2.310001E-04	0.200000E+02	674
4	6.352364E-02	6.340814E-02	1.155001E-04	0.400000E+02	842
5	7.121307E-04	6.543807E-04	5.775003E-05	0.800000E+02	1010
6	8.731989E-05	8.371051E-05	3.609377E-06	0.160000E+03	1176

Acknowledgement. The author gratefully acknowledges the referee, who made useful suggestions and remarks which helped me improve the paper.

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