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ON IRREDUCIBLE PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

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Abstract. The paper is a survey type article in which we present some results on irreducible projective representations of finite groups.

Section 2 includes Curtis and Reiner's theorem ([8]) in which is proved that a finite group has at most a finite number of inequivalent irreducible projective representations in an algebraically closed field K. Theorem 15 ([16]) gives an alternative proof of the main theorem of Morris ([15]), where the structure of a generalized Clifford algebra was determined. Similarly, Theorem 16 ([16]) gives the structure theorem for a generalized Clifford algebra which arises in the study of the projective representations of the generalized symmetric group. Section 2 is also dedicated to the study of degrees of irreducible projective representations of a finite group G over an algebraically closed field K. In Theorem 20, H. N. NG proved a generalization of Schur's result and showed that the degree of an irreducible projective representation of a finite group G belonging to $c \in H^2(G; K^*)$, where K is an algebraically closed field such that char$K$ does not divide $|G|$, divides the index of a class of abelian normal subgroups of G, which depends only on the 2-cohomology class c. In Theorem 27, Quinlan proved ([19]) that the representations theory of generic central extensions for a finite group G yields information on the irreducible projective representations of G over various fields.

In Section 3 we give a necessary and sufficient condition for a nilpotent group G to have a class of faithful faithful irreducible projective representation ([18]).

Section 4 includes NG's result in the case of a metacyclic group G with a faithful irreducible projective representation π over an algebraically closed field with arbitrary characteristic, which proved that the degree of π is equal to the index of any cyclic normal subgroup N whose factor group G/N is also cyclic and also a necessary and sufficient conditions for a metacyclic group to have a faithful irreducible projective representation ([18]).

In Section 5 we remind Barannyk's results ([2], [5]) in which he obtained conditions for a finite p-group to have a class of faithful irreducible projective representations.

Section 6 contains the most important results of the adaptation to projective representations of Clifford's theory of inducing from normal subgroups ([21], [13]).

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1 Introduction

In this paper we remind some of the most important results about irreducible projective representations of finite groups.

The theory of projective representations of finite groups over the complex number field was founded and developed by I. Schur. In [22], he reduced the problem to determine all projective representations of a finite group G to the determination of all linear representations of a certain finite group extension of G, which is called a representation-group of G. In [23] and [24], Schur determined all irreducible projective representations of some particular groups and also proved that the number of inequivalent projective representations of G with the multiplier α equals the number of α-regular classes of G ([22]). Then R. Frucht in [10], [11] determined the irreducible projective representations of finite abelian groups.

In Section 2, we present Curtis and Reiner's theorem ([8]) in which is proved that a finite group has at most a finite number of inequivalent irreducible projective representations in an algebraically closed field K. Theorem 15 ([16]) gives an alternative proof of the main theorem of Morris ([15]), where the structure of a generalized Clifford algebra was determined. Similarly, Theorem 16 ([16]) gives the structure theorem for a generalized Clifford algebra which arises in the study of the projective representations of the generalized symmetric group. In the proofs of these two theorems, Morris developed the method introduced by Eckmann in [9], who has proved the two above theorems in the case $n = 2$.

The complex irreducible projective representations of a finite group G may be described in terms of the complex irreducible ordinary representations of a covering group $\widehat{G}$ for G, which takes the form of a central extension of Schur multiplier of G. Given a free presentation of G it can be constructed a central extension F of a certain infinite abelian group R by G, which behaves as a covering group for G with respect to all fields. We shall refer to groups such as F as generic central extensions for G. Quinlan proved in [19] that their representations theory yields information on the irreducible projective representations of G over various fields (Theorem 27).

Section 2 is also dedicated to the study of degrees of irreducible projective representations of a finite group G over an algebraically closed field K. The degree of an irreducible projective representations of a finite group G over complex field was first studied by Schur ([22]). He proved that the degree of such a projective representation of G divides $|G|$. The proof depends on his method of representation-groups and his well-known theorem: The degree of an irreducible complex character of a finite group R divides $[R : Z(R)]$, the index of $Z(R)$ in the group R. In Theorem 20, H. N. NG proved a generalization of Schur's result and showed that the degree of an irreducible projective representation of a finite group G belonging to $c \in H^2(G; K^*)$, where K is an algebraically closed field such that char$K$ does

group; p-group.

not divide $|G|$, divides the index of a class of abelian normal subgroups of G, which depends only on the 2-cohomology class c.

The characterization of abelian groups that have faithful irreducible projective representations over K was first studied by Frucht [10],[11]. He showed that an abelian group H has a faithful irreducible projective representation over K if and only if H can be written as a direct product of two isomorphic subgroups. Yamazaki ([26]) showed that the above result holds when K is replaced by a field that contains a primitive $(\exp H)$-th root of unity. In Section 3 we give a necessary and sufficient condition for a nilpotent group G to have a class of faithful faithful irreducible projective representation ([18]).

Section 4 includes NG's result in the case of a metacyclic group G with a faithful irreducible projective representation π over an algebraically closed field with arbitrary characteristic, which proved that the degree of π is equal to the index of any cyclic normal subgroup N whose factor group G/N is also cyclic ([17]) and also a necessary and sufficient condition for a metacyclic group to have a faithful irreducible projective representation ([18]).

In Section 5 we remind Barannyk's results ([2], [5]) in which he obtained conditions for a finite p-group to have a class of faithful irreducible projective representations.

Section 6 contains the most important results of the adaptation to projective representations of Clifford's theory of inducing from normal subgroups ([21], [13]).

2 Equivalent irreducible projective representations of finite groups and their degrees

Let G be a finite group, let K be a field and let K^* be the multiplicative subgroup of non-zero elements in K.

Definition 1. *([17]) A map $\alpha\colon G \times G \to K^*$ is called a multiplier (or a factor set or a 2-cocycle) on G if*

i) $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$;

ii) $\alpha(x, e) = \alpha(e, x) = 1$ for all $x \in G$, where e is the identity of G.

The set of all possible multipliers on G can be given an abelian group structure by defining the product of two multipliers as their pointwise product. The resulting group we denote by $Z^2(G; K^*)$.

The set of all multipliers α satisfying

$$\alpha(x, y) = \mu(xy)\mu(x)^{-1}\mu(y)^{-1}$$

for an arbitrary function $\mu\colon G \to K^*$ such that $\mu(e) = 1$, forms an invariant subgroup $B^2(G; K^*)$ of $Z^2(G; K^*)$. Thus we may form the quotient group $H^2(G; K^*) = Z^2(G; K^*)/B^2(G; K^*)$.

Definition 2. *([17]) Let $GL(n,K)$ be the group of all non-singular $n \times n$ matrices over K. A projective representation of G over K is a map $\pi\colon G \to GL(n,K)$ such that $\pi(x)\pi(y) = \alpha(x,y)\pi(xy)$ for all $x, y \in G$, where α is the associated multiplier.*

The integer n is the degree of π and it is denoted by $\deg\, \pi$.

Remark 3. *([17]) From the associativity of $GL(n,K)$ and Definition 2 results the condition i) in Definition 1.*

In the context of projective representations, $H^2(G;K^*)$ is known as the multiplier (or multiplicator) group of G. The elements of $H^2(G;K^*)$ are known as 2-cohomology classes.

Definition 4. *([17]) Let α be a multiplier on G and let c be the 2-cohomology class of α in $H^2(G;K^*)$, the Schur multiplier of G over K. We say that the projective representation π belongs to c.*

Definition 5. *([25]) Two projective representations π_1 and π_2 are equivalent (or projectively equivalent) if there are a non-singular matrix U and a map $\mu\colon G \to K^*$ such that $\mu(e) = 1$ and*

$$\pi_1(x) = \mu(x)U^{-1}\pi_2(x)U$$

for all $x \in G$.

Remark 6. *([25]) If α is the associated multiplier of π_1 and β is the associated multiplier of π_2, then the projective equivalence of π_1 and π_2 yields*

$$\mu(xy)\alpha(x,y) = \mu(x)\mu(y)\beta(x,y) \tag{2.1}$$

for all $x, y \in G$.

The multipliers that satisfy (2.1) are called equivalent.

Remark 7. *([1]) In fact, two multipliers α and β are equivalent (or cohomologous) if $\alpha\beta^{-1} \in B^2(G;K^*)$.*

Definition 8. *([17]) A projective representation π of G is called reducible if it is projectively equivalent to a projective representation ρ of the form*

$$\rho(x) = \begin{pmatrix} \rho_1(x) & * \\ 0 & \rho_2(x) \end{pmatrix}$$

for all $x \in G$, where ρ_1 and ρ_2 are projective representations of G.

An irreducible projective representation is one that is not reducible.

Definition 9. *([17]) Let I_n be the identity element of $GL(n,K)$. The set*

$$\ker \pi = \{x \in G; \pi(x) \in K^* I_n\}$$

is the kernel of π. When $\ker \pi = \{e\}$, we say that the projective representation π is faithful.

In what follows G will denote a finite group of order n and K an algebraically closed field of an arbitrary characteristic.

Definition 10. *([8]) A group G^* is called an extension of a group G by a group H if there is a homomorphism φ of G^* onto G with kernel H.*

Definition 11. *([8]) Let G^* be an extension of G with kernel N and suppose that N is contained in the center of G^*. Then $G^*/N \cong G$ and we may find a set of coset representatives $\{v_x, x \in G\}$ in one-to-one correspondence with the elements of G such that for all $x, y \in G$ we have*

$$v_x v_y = v_{xy} a(x, y),$$

where $a(x, y) \in N$. Now let $\pi\colon G^* \to GL(n, K)$ be an ordinary representation of G^* such that for all $x, y \in G$,

$$\pi(a(x, y)) = \alpha(x, y) I_n,$$

for some element $\alpha(x, y) \in K^*$. Then the map $\pi\colon G \to GL(n, K)$ given by $x \longmapsto \pi(v_x)$ defines a projective representation of G with the associated multiplier α. We say that a projective representation of G which is constructed in this way from an ordinary representation of G^* can be lifted to G^*.

Definition 12. *([8]) A representation-group G^* of G is a finite group G^* which is an extension of G with kernel contained in the center of G^* such that every projective representation of G can be lifted to G^*.*

Theorem 13. *([8]) A finite group G has at most a finite number of inequivalent irreducible projective representations in an algebraically closed field K.*

Proof. Let G^* be a representation-group and let $\pi\colon G \to GL(n, K)$ be an irreducible projective representation of G. By replacing π by an equivalent representation if necessary, we can find a representation $\pi^*\colon G^* \to GL(n, K)$ such that $\pi^*(v_x) = \pi(x)$, for all $x \in G$. Then π^* is an irreducible representation of G^*.

Moreover, let $\rho\colon G \to GL(n, K)$ be a second projective representation of G; it also can be lifted to G^*, so that for some representation $\rho^*\colon G^* \to GL(n, K)$, we have $\rho^*(v_x) = \rho(x)$ for all $x \in G$.

Then, if ρ^* and π^* are equivalent representations of G^*, ρ and π are equivalent projective representations of G. Therefore, the number of inequivalent irreducible projective representations of G cannot exceed the number of inequivalent irreducible representations of G^* and this number is known to be finite by Chapter IV, [8]. $\square$

Definition 14. *([20]) Let α be a multiplier on G. An element $a \in G$ is said to be α-regular if $\alpha(x, a) = \alpha(a, x)$ for all $x \in C_G(a)$, where $C_G(a)$ is the centralizer of a in G, $C_G(a) = \{x \in G;\ \ xa = ax\}$. Every conjugate class of a is α-regular. Such a conjugate class is called an α-regular class.*

Let G be a finite abelian group of order n^m generated by m elements $x_1, \ldots, x_m$ of order n, i.e. $G \cong \mathbb{Z}_n \times \ldots \mathbb{Z}_n$ (m copies), where $\mathbb{Z}_n$ is a cyclic group of order n. Let π be a projective representation of G with the multiplier α over $\mathbb{C}$, the complex field. Let

$$\mu(i) = \prod_{j=1}^{n-1} \alpha(x_i^j, x_i), \ \ 1 \leq i \leq m$$

and

$$\beta(i,j) = \alpha(x_i, x_j)\alpha^{-1}(x_j, x_i), \ \ 1 \leq i < j \leq m.$$

We choose the multiplier α such that $\mu(i) = 1, \ \ i = 1, \ldots, m$ and $\beta(i,j), \ \ 1 \leq i < j \leq m$ is an n-th root of unity.

Theorem 15. *([16]) Let G be a finite abelian group of order n^m generated by $x_1, \ldots, x_m$ and let α be a multiplier on G over $\mathbb{C}$ such that $\mu(i) = 1, \ \ i = 1, \ldots, m$ and $\beta(i,j) = \omega, \ \ 1 \leq i < j \leq m$, where ω is a primitive n-th root of unity. Then, if $m = 2\mu$ is even, G has only one inequivalent irreducible projective representation of degree n^μ and if $m = 2\mu + 1$ is odd, G has n inequivalent irreducible projective representation of degree n^μ.*

Proof. Using Conlon's theorem in [7], who proved that the number of inequivalent projective representations of G with the multiplier α is equal with the number of α-regular classes of G, it is sufficient to determine the number of α-regular classes for the appropriate multiplier.

Let π be a projective representation of G with the multiplier α and let $\pi(x_i) = e_i, \ \ i = 1, \ldots, m$. We must determine the number of elements $a = e_{k_1}^{\alpha_1} \ldots e_{k_r}^{\alpha_r}$, where $1 \leq k_1 < k_2 < \ldots < k_r \leq m, \ \ 0 \leq \alpha_i \leq n-1, \ \ i = 1, \ldots, r$ such that $e_i^{-1} a e_i = a, \ \ i = 1, \ldots, m$.

In particular, we must have $e_{k_i}^{-1} a e_{k_i} = a, \ \ i = 1, \ldots, r$.

But, it can be easily verified that

$$e_{k_i}^{-1} a e_{k_i} = \omega^{\alpha_1 + \ldots + \alpha_{i-1} - \alpha_{i+1} - \ldots - \alpha_r} a, \ \ i = 1, \ldots, r$$

and we have $AX \equiv 0 \pmod{n}$, where A is the $r \times r$ matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & -1 & \ldots & -1 & -1 \end{pmatrix}$$

and $X^t = (\alpha_1, \ldots, \alpha_r)$.

So,

$$\det A = \begin{cases} 0, & \text{if } r \text{ is odd} \\ (-1)^\lambda, & \text{if } r = 2\lambda \text{ is even} \end{cases}$$

Thus, if r is even, the only solution is the trivial solution $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$.

On the other hand, if $r = 2\lambda + 1$ is odd, the above system of linear congruences reduces to $\alpha_1 \equiv -\alpha_2 \equiv \alpha_3 \equiv \ldots \equiv -\alpha_{2\lambda} \equiv \alpha_{2\lambda+1} \pmod{n}$.

Thus, if $x_{k_1}^{\alpha_1} \ldots x_{k_r}^{\alpha_r}$ is an α-regular element, it can only take the form

$$x_{k_1}^{i} x_{k_2}^{-i} \ldots x_{k_{r-1}}^{-i} x_{k_r}^{i}, \quad i = 0, 1, \ldots, n-1.$$

But, we must also have that when $k \neq k_i$, $e_k^{-1} a e_k = a$.

If $k \neq k_i$, $i = 1, \ldots, r$ and we put $k_0 = 1$, $k_{r+1} = m$, then $k > k_j$ and $k < k_{j+1}$ for $0 \leq j \leq r+1$ and $e_k^{-1} a e_k = \omega^{\alpha_1 + \ldots + \alpha_j - \alpha_{j+1} - \ldots - \alpha_r} a = \omega^{\pm i} a$ for $0 \leq i \leq n-1$. That is, if $r < m$, $i = 0$. Thus, it follows that if $m = 2\mu$ is even, 1 is the only α-regular element and if $m = 2\mu + 1$ is odd, then the α-regular elements are given by $x_1^i x_2^{-i} \ldots x_{2\mu}^{-i} x_{2\mu+1}^{i}$, $i = 0, 1, \ldots, n-1$. When $m = 2\mu$ is even, G has only one inequivalent irreducible projective representation whose degree must be n^{μ}. When $m = 2\mu + 1$ is odd, the n inequivalent irreducible projective representations have the same degree n^{μ}. (see [26]) □

Theorem 16. *([16]) Let G be as in Theorem 15, $n = 2\nu$ and the multiplier α satisfies $\mu(i) = 1$, $i = 1, \ldots, m$ and $\beta(i,j) = -1$, $1 \leq i < j \leq m$. Then, if $m = 2\mu$ is even, G has ν^m inequivalent irreducible projective representations of degree 2^{μ} and if $m = 2\mu + 1$ is odd, G has $2\nu^m$ inequivalent irreducible projective representations of degree 2^{μ}.*

Proof. The proof is similar to the proof of Theorem 15, replacing ω by -1. Let $\pi(x_i) = e_i$, $i = 1, \ldots, m$.

It is easily verified that if r is even, $x_{k_1}^{\alpha_1} \ldots x_{k_r}^{\alpha_r}$ is α-regular if and only if $\alpha_i \equiv 0 \pmod{2}$, that is $\alpha_i = 0, 2, 4, \ldots, n-2$.

If r is odd and $a = e_{k_1}^{\alpha_1} \ldots e_{k_r}^{\alpha_r}$, $e_{k_i}^{-1} a e_{k_i} = a$, $i = 1, \ldots, r$ implies that

$$\alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_r \pmod{2}.$$

If $k \neq k_i$ for any $i = 1, \ldots, r$, then $e_k^{-1} a e_k = a$ implies that $\alpha_1 + \alpha_2 + \ldots + \alpha_r \equiv 0 \pmod{2}$ or $r\alpha_r \equiv 0 \pmod{2}$.

Since r is odd, $\alpha_r \equiv 0 \pmod{2}$. Thus, if r is odd and $r < m, x_{k_1}^{\alpha_1} \ldots x_{k_r}^{\alpha_r}$ is α-regular if and only if $\alpha_i = 0, 2, 3, \ldots, n-2$.

Hence, if m is even, the α-regular elements are $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$, where $\alpha_i = 0, 2, \ldots, n-2$, $i = 1, \ldots, m$, that is, there are $\left(\frac{n}{2}\right)^m$ α-regular classes. If m is odd, the α-regular elements are $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$, where $\alpha_i = 0, 2, \ldots, n-2$, $i = 1, \ldots, m$ and $\alpha_i = 1, 3, \ldots, n-1$, $i = 1, \ldots, m$, that is, there are $2\left(\frac{n}{2}\right)^m$ α-regular classes.

Since all irreducible projective representations of an abelian group (with a fixed multiplier) have the same degree, it can be verified that these representations have the degrees given by the theorem. □

Definition 17. *([17]) A twisted group algebra of a finite group G over an arbitrary field K is an associative K-algebra B with basis $\{u_x; x \in G\}$ such that $u_x u_y = \alpha(x,y)u_{xy}$, for all $x, y \in G$, where α is a multiplier on G We set $B = (KG)_\alpha$ to specify the multiplier α. If α belongs to $c \in H^2(G; K^*)$, then we say that $(KG)_\alpha$ belongs to c.*

Remark 18. *([17]) A (finite-dimensional) $(KG)_\alpha$-module affords a projective representation of G with the multiplier α and conversely.*

Definition 19. *([17]) Let H be a normal subgroup of G. Then $(KH)_\alpha = \sum_{y \in H} K u_y$, the subalgebra of $(KG)_\alpha$ which has the associated multiplier the restriction of α to H, is a twisted group algebra of H. Let F be a $(KH)_\alpha$-module. We form the induced module $F^G = (KG)_\alpha \otimes_{(KH)_\alpha} F = \sum_{i=1}^{s} u_{x_i} \otimes F$, where $\{x_1, \ldots, x_s\}$ is a full set of coset representatives of H in G.*

Let $x \in G$. Then, for each $i = 1, \ldots, s$, we have $xx_i = x_j h$ for $h \in H$ and for $1 \le j \le s$ depending on i. Since $x_j^{-1} x x_i \in H$, we see that $u_{x_j}^{-1} u_x u_{x_i} \in (KH)_\alpha$. Therefore, $(KG)_\alpha$ acts on F^G as follows

$$u_x(u_{x_i} \otimes l) = (u_{x_j} u_{x_j}^{-1} u_x u_{x_i}) \otimes l = u_{x_j} \otimes (u_{x_j}^{-1} u_x u_{x_i} l), \;\; l \in F \tag{2.2}$$

From (2.2), we find the induced projective representation π^G, afforded by the $(KG)_\alpha$-module F^G, from the projective representation π afforded by the $(KH)_\alpha$-module F.

Let G be a finite group and we assume that the algebraically closed field K is such that char$K$ does not divide $|G|$.

Theorem 20. *([17]) Let π be an irreducible projective representation of G belonging to $c \in H^2(G; K^*)$. Let N be an abelian normal subgroup of G such that $res_N c = 1$, where res_N is the restriction map from $H^2(G; K^*)$ into $H^2(N; K^*)$. Then the degree of π divides $[G : N]$.*

Proof. The proof is by induction on $|G|$.

If $|G| = 1$, the theorem is trivial.

We may assume that the theorem holds for groups whose orders are less than $|G|$.

The case $N = \{e\}$ is just the result of Schur ([22]).

We suppose that $N \neq \{e\}$.

If the projective representation π is faithful, then, by Theorem 1, [17], there is a proper subgroup H of G and a projective representation ρ of H such that $N \subseteq H \subseteq G$ and $\pi = \rho^G$. Clearly, ρ is irreducible. The groups H and N and the projective representation ρ satisfy now the assumptions of the theorem. Therefore, by the induction hypothesis, we have deg$\rho|[H : N]$. This shows that deg$\pi|[G : N]$.

We may now assume that π is not faithful, that is $|\ker \pi| > 1$. Let $\overline{G} = G/\ker \pi$. Then we have $|\overline{G}| < |G|$ and the induction hypothesis can be applied to $\overline{G}$. Clearly, π induces a projective representation π' of $\overline{G}$ such that degπ' = degπ. The groups $\overline{G}$ and $\overline{N}$ and the projective representation π' also satisfy the assumption of the theorem, where $^-$ denotes images in $\overline{G}$.

By induction hypothesis, we have deg$\pi'|[\overline{G} : \overline{N}]$. However, $[\overline{G} : \overline{N}]|[G : N]$. Therefore deg$\pi|[G : N]$. □

Definition 21. *A free group on a set $S = \{x_i\}$ is a group where each element can be uniquely described as a finite length product of the form $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ where each x_i is an element of S and $x_i \neq x_{i+1}$ for any i and each a_i is any non-zero integer.*

Definition 22. *Let S be a set and let $\langle S\rangle$ be the free group on S. Let R be a set of words on S, so R is a subset of $\langle S\rangle$. Let N be the normal closure of R in $\langle S\rangle$, which is defined as the smallest normal subgroup in $\langle S\rangle$ which contains R. The group $\langle S|R\rangle$ is then defined as the quotient group $\langle S \mid R\rangle = \langle S\rangle/N$ and is called a presentation of the group G. The elements of S are called the generators of $\langle S|R\rangle$ and the elements of R are called the relators. A group G is said to have the presentation $\langle S|R\rangle$ if G is isomorphic to $\langle S|R\rangle$.*

If $\langle \widetilde{F}|\widetilde{R}\rangle$ is a free presentation of G, we may define the groups $F := \widetilde{F}/[\widetilde{F}, \widetilde{R}]$ and $R := \widetilde{R}/[\widetilde{F}, \widetilde{R}]$, where $[\widetilde{F}, \widetilde{R}]$ is the commutator of $\widetilde{F}$ and $\widetilde{R}$, i.e. the group generated by $[f, r] = frf^{-1}r^{-1}$ for all $f \in \widetilde{F}, r \in \widetilde{R}$. Then $R \subseteq Z(F)$ and $\widetilde{\phi}$ denotes the surjection of $\widetilde{F}$ on G with kernel $\widetilde{R}$, then $\widetilde{\phi}$ induces a surjection $\phi\colon F \to G$ with kernel R. Thus (R, F, ϕ) is a central extension for G. Extensions of this type were introduced by Schur in his description of finite covering groups and as outlined in the next lemma, which follows easily from the freeness of $\widetilde{F}$, they have a particular universal property amongst all central extensions for G. For this reason we shall refer to them as generic central extensions for G.

Lemma 23. *([19]) Let (A, H, θ) be a central extension for G and let (R, F, ϕ) be as above. Then there is a homomorphism $\psi\colon F \to H$ for which the following diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\psi|_R"'] & F \arrow[r, "\phi"] \arrow[d, "\psi"'] & G \arrow[r] \arrow[d, "id"'] & 1 \\
1 \arrow[r] & A \arrow[r] & H \arrow[r, "\theta"] & G \arrow[r] & 1
\end{tikzcd}$$

Definition 24. *([19]) Let $\eta \in Hom(R, K^*)$ and let μ be a section for G in F. Then define tra η, called the transgression map, to be the class in $H^2(G; K^*)$ of the multiplier η' defined for $x, y \in G$ by η'^{-1}).*

Lemma 25. *([19]) Let π be an irreducible projective representation of G with the multiplier α belonging to the class $c \in H^2(G; K^*)$. Let $\widetilde{\pi}$ be a lift of π to F and let $\eta = \widetilde{\pi}|_R$ regarded as a homomorphism of R into K^*. Then $c = tra\ \eta$.*

Proof. Let μ be a section for G in F and use it to define the multiplier η' as in Definition 24.

For $g, h \in G$, we have

$$\eta'(g,h) = \eta(\mu(g)\mu(h)\mu(gh)^{-1}) = \widetilde{\pi}(\mu(g)\mu(h)\mu(gh)^{-1}) = \widetilde{\pi}(\mu(g))\widetilde{\pi}(\mu(h))\widetilde{\pi}(\mu(gh)^{-1})$$

and $\alpha(g,h) = \pi(g)\pi(h)\pi(gh)^{-1}$.

Define a map $\chi\colon G \to K^*$ by $\chi(g) = \pi(g)^{-1}\widetilde{\pi}(\mu(g)) = \widetilde{\pi}(\mu(g))\pi(g)^{-1}$. Then

$$\alpha(g,h)^{-1}\eta'(g,h) = \pi(gh)\pi(h)^{-1}\pi(g)^{-1}\widetilde{\pi}(\mu(g))\widetilde{\pi}(\mu(h))\widetilde{\pi}(\mu(gh)^{-1}) = \chi(g)\chi(h)\chi(gh)^{-1}.$$

Thus $\alpha^{-1}\eta' \in B^2(G; K^*)$ and α and η' belong to the same class in $H^2(G; K^*)$; $c = \mathrm{tra}\ \eta$. $\square$

Let $\eta \in Hom(R, \overline{K}^*)$, where $\overline{K}$ is an algebraically closed field of characteristic zero. Then $\eta \in \ker(\mathrm{tra})$ if and only if the restriction of η to $F' \cap R$ is trivial, where F' is the commutator subgroup of F. Suppose now that π is an irreducible projective representation of G over K, whose multiplier belongs to the class $c \in H^2(G; \overline{K}^*)$. Then if $\widetilde{\pi}$ is a lift to F of π, $\widetilde{\pi}|_{F'\cap R}$ (as a homomorphism into $\overline{K}^*$) depends only on c. We denote this homomorphism by θ_c. Furthermore, since θ_c uniquely determines c by the transgression map, we obtain a bijective correspondence between $H^2(G; \overline{K}^*)$ and $Hom(F' \cap R, \overline{K}^*)$ given by $c \longleftrightarrow \theta_c$. For each $c \in H^2(G; \overline{K}^*)$ we let I_c denote the kernel of θ_c, a subgroup of $F' \cap R$.

Definition 26. *([19] Let $\alpha \in Z^2(G; \overline{K}^*)$. If $x \in G$ is α-regular, then so is each of its conjugates in G and it is also α'-regular, whenever α and α' represent the same element of $H^2(G; \overline{K}^*)$. Thus we may define for $c \in H^2(G; \overline{K}^*)$ the notion of a c-regular conjugate class of G.*

We denote by S_F the set of conjugacy classes of F contained in F' and by S_G the set of conjugacy classes of G contained in G'. We shall say that $J \in S_F$ lies over $J \in S_G$ if J is the image of J under ϕ. For each $J \in S_G$, we define a subset Z_J of $F' \cap R$ by choosing $J \in S_F$ lying over J, then choosing $x \in J$ and a preimage X for x in J and setting $Z_J = \left\{Z \in F' \cap R; ZX \in J\right\}$. It is easy to check that Z_J is a group and that it does not depend on the choice of J or on the choices of x or X. Suppose $Z \in Z_J$. Then, since ZX is conjugate to X in F, $Z = Y^{-1}XYX^{-1}$, for some $Y \in F$. Since $Z \in R$, $Y \in \phi^{-1}(C_G(x))$. On the other hand, it is clear that $Y^{-1}XYX^{-1} \in Z_J$, for any $Y \in \phi^{-1}(C_G(x))$, hence $Z_J = \left\{Y^{-1}XYX^{-1}; Y \in \phi^{-1}(C_G(x))\right\}$.

Theorem 27. *([19]) Let $\overline{K}$ be an algebraically closed field of characteristic zero and let π_1 and π_2 be irreducible projective representations of G over $\overline{K}$. Then π_1 and π_2 belong to the same component of $\overline{K}F$ if and only if they are projectively equivalent over $\overline{K}$.*

Proof. By Lemma 3.1, [19], it is sufficient to show that the number of components of $\overline{K}F$ is equal to the number $\sum_{c\in H^2(G;\overline{K}^*)} n_c$ of mutually (projectively) inequivalent irreducible projective representations of G over $\overline{K}$, where n_c denotes the number of such representations having multiplier representing c. From Theorem 3.4, [19], we have $n_c = |\{J \in \mathcal{S}_G; J \text{ is } c-\text{regular}\}|$.

$J \in \mathcal{S}_G$ is c-regular if and only if $Z_J \subseteq I_c$. Since $\{\theta_c\}_{c\in H^2(G;\overline{K}^*)} = Hom(F' \cap R, \overline{K}^*)$, the number of elements of $H^2(G;\overline{K}^*)$ with respect to which $J \in \mathcal{S}_G$ is regular is $|Hom(F'\cap R/Z_J, \overline{K}^*)| = [F' \cap R : Z_J]$, since $\overline{K}$ is algebraically closed. Then counting the ordered pairs of the form (J, c), where $c \in H^2(G;\overline{K}^*)$ and $J \in \mathcal{S}_G$ is c-regular leads to the equality $\sum_{c\in H^2(G;\overline{K}^*)} n_c = \sum_{J\in\mathcal{S}_G} [F' \cap R : Z_J]$.

If $\mathcal{J} \in \mathcal{S}_F$, let $\widehat{\mathcal{J}}$ denotes the element $\sum_{x\in\widetilde{J}} x$ of $\overline{K}F$. Then $\{\widehat{\mathcal{J}}\}_{\mathcal{J}\in\mathcal{S}_F}$ has the same cardinality as the set $\mathcal{I}$ of primitive central idempotents of $\overline{K}F$, since each is a basis for the same vector space over $\overline{K}$, namely $Z(\overline{K}F) \cap \overline{K}F'$. Thus the number of components of $\overline{K}F$ is $|\mathcal{S}_F|$. Now let $\mathcal{J} \in \mathcal{S}_F$ lie over $J \in \mathcal{S}_G$. Then it is easily observed that the elements of $\mathcal{S}_F$ lying over J are precisely those of the form $r\mathcal{J}$, where $r \in F' \cap R$. Furthermore, if $r \in F' \cap R$, then $r\mathcal{J} = \mathcal{J}$ if and only if $r \in Z_J$. Thus the number of elements of $\mathcal{S}_F$ lying over $J \in \mathcal{S}_G$ is $[F' \cap R : Z_J]$ and $|\mathcal{S}_F| = \sum_{J\in\mathcal{S}_G} [F' \cap R : Z_J] = \sum_{c\in H^2(G;\overline{K}^*)} n_c$. $\square$

3 Faithful irreducible projective representations of finite (nilpotent) groups

Definition 28. *([26], [18]) Let G be a finite group and let H_1 and H_2 be subgroups of G such that $[H_1, H_2] = 1$.*

Let α be a multiplier belonging to $c \in H^2(G; K^*)$. The map $\varphi\colon H_1 \times H_2 \to K^*$ defined by

$$\varphi(x, y) = \alpha(x, y)\alpha(y, x)^{-1}$$

for all $x \in H_1, y \in H_2$ is independent of the choice of α in c. It satisfies

$$\varphi(x, y_1y_2) = \varphi(x, y_1)\varphi(x, y_2)$$

$$\varphi(x_1x_2, y) = \varphi(x_1, y)\varphi(x_2, y)$$

for all $x_1, x_2, x \in H_1$ and $y, y_1, y_2 \in H_2$.

We call φ the bilinear pairing determined by c.

Definition 29. *([18]) Let $z \in G$. Define $\varphi_z \colon C_G(z) \to K^*$ by*

$$\varphi_z(y) = \varphi(z, y)$$

for all $y \in C_G(z)$. Then φ_z is a group homomorphism. When φ_z is trivial, we say that z is a c-normal element.

Proposition 30. *([26]) If the finite group G has a faithful irreducible projective representation belonging to $c \in H^2(G; K^*)$, then the center of G, $Z(G)$, contains no nontrivial c-normal elements.*

Proof. Suppose that there is a c-normal element $y \neq e$, $y \in Z(G)$. Then we have, for $\alpha \in c$,

$$\alpha(x, y)\alpha(y, x)^{-1} = 1, \text{ for all } x \in G.$$

Let π be a faithful irreducible projective representation belonging to c. Since $y \in Z(G)$ and $x \in G$, we have

$$\alpha(x, y)\pi(xy) = \alpha(y, x)\pi(yx).$$

Then $\pi(x)\pi(y) = \pi(y)\pi(x)$ for all $x \in G$.

By Schur's lemma, $\pi(y) \in K^*I$, so π is not faithful. Contradiction! □

Proposition 31. *([18]) Let G be a finite group and let K be a field. Let π be a projective representation of G belonging to $c \in H^2(G; K^*)$. Then each element $g \in \ker \pi$ is c-normal.*

Proof. Let $y \in C_G(g)$. Since $g \in \ker \pi$, $\pi(g)$ is a scalar matrix and we have

$$\pi(y)\pi(g) = \pi(g)\pi(y)$$

Let φ be the bilinear pairing determined by c.

Since π is a projective representation with the multiplier α, we have

$$\pi(g)\pi(y) = \alpha(g, y)\pi(gy) = \varphi(g, y)\alpha(y, g)\pi(yg) = \tag{3.1}$$

$$= \varphi(g, y)\pi(y)\pi(g) = \varphi(g, y)\pi(g)\pi(y).$$

From (3.1), it results that $\varphi(g, y) = 1$. Therefore, g is c-normal. □

Definition 32. *A series $\{H_1, \ldots, H_n\}$ of a group G is called a central series if H_i, $i = \overline{1, n}$ are normal subgroups of G and for all $i = \overline{1, n}$, $H_{i-1}/H_i \leq Z(G/H_i)$.*

Definition 33. *A group G is called nilpotent if G has at least a central series.*

Proposition 34. *([18]) Let G be a nilpotent group and $c \in H^2(G; K^*)$. If the center of G contains no nontrivial c-normal elements, then each projective representation of G belonging to c is faithful.*

Proof. Let π be a projective representation of G belonging to c.

By Proposition 31, each element in $\ker \pi$ is c-normal.

By [12], since G is nilpotent and $\ker \pi$ is a normal subgroup of G, we have $\ker \pi \cap Z(G) \neq \{e\}$ if $\ker \pi \neq \{e\}$.

By the assumption that $Z(G)$ contains no nontrivial c-normal elements, it results that $\ker \pi = \{e\}$, so π is faithful. □

Corollary 35. *([18]) Let G be a nilpotent group and let K be the complex field. Let $c \in H^2(G; K^*)$. An irreducible projective representation of G belonging to c is faithful if and only if the center of G contains no nontrivial c-normal elements.*

Proof. The proof results from Proposition 31 and Proposition 34 □

4 Faithful irreducible projective representations of metacyclic groups

Definition 36. *([18]) A group G is called metacyclic if it has a cyclic normal subgroup whose factor group is also cyclic. Hence, G can be generated by two elements a and b with relations $a^m = e$, $b^s = a^t$ and $ba = a^r b$, where the positive integers m, r, s, t satisfy $r^s \equiv 1 \pmod{n}$ and $m | t(r-1)$.*

Definition 37. *A group G is called supersolvable if there exists a normal series*

$$\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{s-1} \triangleleft H_s = G$$

such that each quotient group H_{i+1}/H_i is cyclic and each H_i is normal in G.

In the following we assume that K is an algebraically closed field.

Theorem 38. *([17]) Let G be a metacyclic group with a normal subgroup N such that G/N is cyclic. Let π be an irreducible projective representation of G over K. Then there is a subgroup H of G and a 1-dimensional projective representation ρ of H such that $N \subseteq H$ and $\pi = \rho^G$. If we further assume that π is faithful, then $H = N$ and $\deg\ \pi = [G : N]$.*

Proof. Since all metacyclic groups are supersolvable, the assertion for π not necessarily faithful follows from Theorem 3, [17].

We assume now that π is faithful.

If $G = \{e\}$, there is nothing to prove.

We further assume that $G \neq \{e\}$. We suppose that $N \subset H$. Let $b \in G$ such that the coset bN is a generator of G/N. Let $s = [G : N]$ and $r = [G : H]$. Then $s > r$ and $b^r \in H - N$.

We may assume that ρ^G is obtained from H with respect to the coset representatives $e, b, \ldots, b^{q-1}$ of H in G. Then $\rho^G(b^r) = \rho(b^r) I_r$, by equation (2.2).

But $b^r \neq e$, therefore π is not faithful and the proof is completed. □

Corollary 39. *([17]) Let G be a metacyclic group with a faithful irreducible projective representation over an algebraically closed field. If N is a normal subgroup of G such that both N and G/N are cyclic, then $|N|$ is unique.*

Remark 40. *([17]) The converse of Corollary 39 is not true. Let G be the quaternion group of order 8. Then the irreducible representation of G of degree 2 is not faithful as a projective representation.*

Definition 41. *Let G be a group, let N be a normal subgroup of G and let H a subgroup of G. The following statements are equivalent:*

1. *$G = NH$ and $N \cap H = \{e\}$ (with e being the identity element of G);*
2. *$G = HN$ and $N \cap H = \{e\}$;*
3. *Every element of G can be written as a unique product of an element of N and an element of H;*
4. *Every element of G can be written as a unique product of an element of H and an element of N;*
5. *The natural embedding $H \longrightarrow G$, composed with the natural projection $G \longrightarrow G/N$, yields an isomorphism between H and the quotient group G/N;*
6. *There is a homomorphism $G \longrightarrow H$ which is the identity on H and whose kernel is N.*

If one (and therefore all) of these statements hold, we say that G is a semidirect product of N and H.

Proposition 42. *([18]) Let G be a metacyclic group with generators a and b. If G has a faithful irreducible projective representation over K, then G is the semidirect product of the subgroups $\langle a\rangle$ and $\langle b\rangle$.*

Proof. If the $\langle a\rangle\langle b\rangle$ is not semi-direct, then $\langle a\rangle \cap \langle b\rangle \neq \{e\}$, so there are positive integers s and t such that $b^s = a^t \neq e$.

Let $c \in H^2(G; K^*)$. Clearly $a^t \in Z(G)$. We show that a^t is a c-normal element.

Let φ be the bilinear pairing determined by c. Then

$$\varphi(a^t, b) = \varphi(b^s, b) = \varphi(b^{s-1}b, b) = \varphi(b^{s-1}, b)\varphi(b, b) =$$

$$\varphi(b^{s-2}, b)\varphi(b, b)\varphi(b, b) = \ldots = \varphi(b, b)^s = [\alpha(b, b)\alpha(b, b)^{-1}]^s = 1$$

In the same way we can show that $\varphi(a^t, a) = \varphi(a, a)^t = 1$.

So, $Z(G)$ contains a nontrivial c-normal element for each $c \in H^2(G; K^*)$. This contradicts the assumption that G has a faithful irreducible projective representation, by Proposition 34. Therefore, $G = \langle a\rangle\langle b\rangle$ is a semi-direct product. □

Theorem 43. *([18]) Let $G = \langle a, b; a^m = b^s = e, ba = a^r b\rangle$ be a metacyclic group. The following conditions are equivalent:*

(1) G has a faithful irreducible projective representation over K;

(2) the center $Z(G)$ contains no nontrivial c-normal elements, for some $c \in H^2(G; K^*)$;

(3) the integer s is at least positive integer for which

$$1 + r + \ldots + r^{s-1} \equiv 0 \pmod{m} \tag{4.1}$$

Proof. (1) $\Longrightarrow$ (2)

Results from Proposition 34.

(2) $\Longrightarrow$ (3)

We consider first the case $(r-1, m) = 1$. Then $H^2(G; K^*) = 1$, by Theorem 3.6, [18], so that (2) implies $Z(G) = \{e\}$. Hence, by the form of G, s is the least positive integer such that $r^s \equiv 1 \pmod{m}$. Since $(r-1, m) = 1$, it follows that s is the least positive integer such that $1 + r + \ldots + r^{s-1} \equiv 0 \pmod{m}$.

Now we may assume that $(r-1, m) > 1$.

Let $B = \sum\limits_{x \in G} K u_x$ be the twisted group algebra belonging c. Following the notations of Proposition 3.2, [18], we see that

$$u_b u_a^\gamma = \xi^\gamma u_a^\gamma u_b,$$

where $\gamma = \frac{m}{(r-1,m)}$.

By the hypothesis, the central subgroup $\langle a^\gamma\rangle$ contains no nontrivial c-normal elements. Since the cyclic group $\langle a^\gamma\rangle$ has order $(r-1, m)$, it follows from relation (1.4), [18] and the above equation that ξ^γ is a primitive $(r-1, m)$-th root of unity. But ξ is a β-th root of unity and $\gamma|\beta$ (as is stated in the proof of Proposition 3.6, [18]), so ξ^γ is a $\frac{\beta}{\gamma}$-th root of unity. Hence, $(r-1, m)$ is a factor of $\frac{\beta}{\gamma}$, that is $m = \gamma(r-1, m)|\beta$. This shows that $\beta = (1 + r + \ldots + r^{s-1}, m)$ equals m. Therefore, (4.1) holds.

Now it remains to show that s is the least positive integer for which (4.1) holds. We assume the contrary and so there is an integer t such that $1 \leq t < s$ and $1 + r + \ldots + r^{t-1} \equiv 0 \pmod{m}$. This implies that $b^t \in Z(G)$. By Proposition 3.2, [17], $u_{b^t} u_a = u_a u_{b^t}$, so that b^t is c-normal by (1.4), [18]. This contradicts (2), so (3) holds.

(3) $\Longrightarrow$ (1)

Condition (4.1) implies that $\beta = (1 + r + \ldots + r^{s-1}, m) = m$. Let ξ be a primitive m-th root of unity. We define the following $s \times s$ matrices over K:

$$U_a = \begin{pmatrix} \xi & 0 & \dots & 0 & 0 \\ 0 & \xi^{1+r} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \xi^{1+r+\dots+r^{s-1}} \end{pmatrix}, U_b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is easy to check that $U_{a^m} = U_{b^s} = I_s$ and $U_b U_a = \xi U_{a^r} U_b$.

Hence, we get a projective representation π of G by putting $\pi(a^i b^j) = U_{a^i} U_{b^j}$, $1 \le i \le m, 1 \le j \le s$.

From the hypothesis, it follows that the diagonal elements in U_a are distinct. Hence, the centralizer algebra of $\{U_a, U_b\}$ contains only scalar multiples of I_s. Therefore, π is irreducible by Schur's lemma.

We show that π is faithful. Let $a^i b^j \in \ker \pi$. Then $U_{a^i} U_{b^j}$ is a scalar matrix. We can assume that $j \equiv 0 \pmod{s}$, since otherwise, $U_{a^i} U_{b^j}$ is not diagonal. But then, $\xi^i = \xi^{i(1+r)}$, so that $\xi^{ir} = 1$. This implies that $ir \equiv 0 \pmod{m}$. Since $(r, m) = 1$ by (4.1), the above equation yields $i \equiv 0 \pmod{m}$, that is $a^i b^j = 1$. Therefore, π is faithful.

□

5 Faithful irreducible projective representations of p-groups

Definition 44. *A periodic group is a group in which each element has finite order.*

All finite groups are periodic. All finite cyclic groups are periodic.

Definition 45. *Given a prime number p, a p-group is a periodic group in which each element has a power of p as its order. That is, for each element g of the group, there is a nonnegative integer n such that $g^{p^n} = e$, where e is the identity element of the group.*

A finite group is a p-group if and only if its order is a power of p.

Definition 46. *([2]) Let G be a finite group, let $Z(G)$ be the center of G and let $\alpha \in Z^2(G; K^*)$. The set $\{g \in Z(G);\ \forall\ a \in G,\ \alpha(a,g) = \alpha(g,a)\}$ forms a subgroup of G, called the α-center of G.*

Theorem 47. *(Clifford) Let $\pi\colon G \to GL(n, K)$ be an irreducible representation with K a field. Then the restriction of π to a normal subgroup N breaks up into a direct sum of inequivalent irreducible representations of N of equal dimensions. These irreducible representations of N lie in one orbit for the action of G by conjugation on the equivalence classes of irreducible representations of N. In particular the number of distinct summands is no greater than the index of N in G.*

Proposition 48. *([2]) Let G be a finite p-group, let K be a field of characteristic $p \geq 0$, let $\alpha \in Z^2(G; K^*)$ and let N be the α-center of G. The group G has a faithful irreducible projective representation with the multiplier α if and only if N has a faithful irreducible projective representation with the multiplier α.*

Proof. If π is an irreducible projective representation of G with the multiplier α, then by Clifford's theorem $\pi|_N = \nu \oplus \ldots \oplus \nu$, where ν is an irreducible projective representation of N with the multiplier α. It follows that if π is faithful, then so is ν.

Conversely, let ν be a faithful irreducible projective representation of N with the multiplier α and let π be an irreducible component of the induced representation ν^G. If π is not faithful, then $\ker \pi \cap Z(G) \neq \{e\}$. Let $b \in \ker \pi \cap Z(G)$ and $b \neq e$. Then $\pi(b) = kI$ $(k \in K^*)$ and for each $g \in G$ we have $\pi(b)\pi(g) = \pi(g)\pi(b)$ and $\alpha(b, g)\pi(bg) = \alpha(g, b)\pi(gb)$. Since $bg = gb$, we conclude that $\alpha(b, g) = \alpha(g, b)$. Therefore, b is a non-identity element of the α-center N. Since $\nu^G|_N = \nu \oplus \ldots \oplus \nu$, it follows that $\pi|_N = \nu \oplus \ldots \oplus \nu$. Finally, we get $\nu(b) = kI$. This is a contradiction. Hence, π is a faithful representation. □

Definition 49. *The socle of a group G, denoted by Soc (G), is the subgroup generated by the minimal non-trivial normal subgroups of G. The socle is a direct product of minimal normal subgroups.*

Definition 50. *The Jacobson radical of a ring R, denoted by $J(R)$ has the following equivalent characterizations:*

1. *$J(R)$ equals the intersection of all maximal right ideals within the ring;*
2. *$J(R)$ equals the intersection of all maximal left ideals within the ring;*
3. *$J(R)$ equals the intersection of all annihilators of simple right R-modules, where a (left or right) module S over a ring R is called simple or irreducible if it is not the zero module 0 and if its only submodules are 0 and S and the annihilator of S is the set of all elements r in R such that for each s in S, $rs = 0$;*
4. *$J(R)$ equals the intersection of all annihilators of simple left R-modules;*
5. *$J(R)$ is the intersection of all primitive ideals within R, where a left primitive ideal in ring theory is the annihilator of a simple left module;*
6. *$J(R)$ is the unique right ideal of R maximal with the property that every element is right quasiregular, this means that r is said to be quasiregular, if $1 - r$ is a unit in R.*

Let $G = \langle a_1 \rangle \times \ldots \langle a_s \rangle$. The elements $u_{a_1}, \ldots, u_{a_s}$ of the natural K-basis of the algebra $(KG)_\alpha$ are generators of this algebra. Therefore, if $u_{a_i}^{o(a_i)} = \lambda_i$ ($\lambda_i \in K^*$, $i = 1, \ldots, s$) (where $o(a_i)$ is the order of $a_i \in G$), then we denote the algebra $(KG)_\alpha$ also by $[G, K, \lambda_1, \ldots, \lambda_s]$.

We often denote the restriction of $\alpha \in Z^2(G; K^*)$ to a subgroup H of G by α as well. We identify u_e with the unity of the field K. Therefore, we write γ instead of γu_e ($\gamma \in K$).

Proposition 51. *([2]) Let G be a finite p-group and let $\alpha \in Z^2(G; K^*)$. If G has a faithful irreducible projective representation with the multiplier α, then G is abelian. Let H be the socle of an abelian p-group G. Then the following conditions are equivalent:*

1. *G has a faithful irreducible projective representation with the multiplier α;*
2. *H has a faithful irreducible projective representation with the multiplier α;*
3. *if $(KH)_\alpha = [H, K, \delta_1, \ldots, \delta_m]$, then none of the products $\delta_1^{t_1} \ldots \delta_m^{t_m}$ ($0 \leq t_i < p$, $t_1 + \ldots + t_m \neq 0$) is the p-th power of an element of K.*

Proof. From [14], it is known that an irreducible projective representation π of G with the multiplier α is realized in the field $(KG)_\alpha / J((KH)_\alpha)$, where $J((KH)_\alpha)$ is the Jacobson radical of the algebra $(KH)_\alpha$. Hence, $\pi(a)\pi(b) = \pi(b)\pi(a)$ for all $a, b \in G$. If π is faithful, then from the equality $\pi(a^{-1}b^{-1}ab) = \gamma\pi(a)^{-1}\pi(b)^{-1}\pi(a)\pi(b) = \gamma I$ ($\delta \in K^*$) it follows that $a^{-1}b^{-1}ab = e$, i.e. $ab = ba$ for all $a, b \in G$. Therefore, G is abelian.

Let π be an irreducible projective representation of an abelian group G with the multiplier α. If π is not faithful, then $\pi(a) = \gamma I$, $\gamma \in K$ for some non-identity element $a \in H$. Since $\pi(a)^p = \alpha(a, a)\alpha(a, a^2) \ldots \alpha(a, a^{p-1})I$, we have $\alpha(a, a)\alpha(a, a^2) \ldots \alpha(a, a^{p-1}) = \gamma^p$.

Conversely, if the last equality holds, then an irreducible projective representation ν of the subgroup $\langle a \rangle$ with the multiplier α is one-dimensional: $\nu(a^i) = \gamma^i$, $i = 0, 1, \ldots, p-1$. Hence, by Clifford's theorem, $\pi(a) = \nu(a) \oplus \ldots \oplus \nu(a) = \gamma I$.

Let $H = \langle b_1 \rangle \times \ldots \times \langle b_m \rangle$ and $a = b_1^{t_1} \ldots b_m^{t_m}$. Then $\alpha(a, a)\alpha(a, a^2) \ldots \alpha(a, a^{p-1}) \in K^p$ if and only if $\delta_1^{t_1} \ldots \delta_m^{t_m} \in K^p$. □

Definition 52. *([4]) Let $\alpha \in Z^2(G; K^*)$. We denote by $\ker \alpha$ the union of all cyclic subgroups $\langle g \rangle$ of G such that the restriction of α to $\langle g \rangle \times \langle g \rangle$ is an element of $B^2(G; K^*)$.*

Remark 53. *([4]) From Lemma 1 in [3], $G' \subset \ker \alpha$, $\ker \alpha$ is a normal subgroup of G and the restriction of α to $\ker \alpha \times \ker \alpha$ is an element of $B^2(G; K^*)$. The set $\ker \alpha$ is called the kernel of α.*

Lemma 54. *([4]) Let G be an abelian p-group and let $\alpha \in Z^2(G;K^*)$. The group G has a faithful irreducible projective representation over K with the multiplier α if and only if* $\ker \alpha = \{e\}$.

Proof. The proof results by applying Proposition 51. □

Proposition 55. *([5]) Let K be an arbitrary field of a finite characteristic p, let G be a p-group, $\alpha \in Z^2(G;K^*)$, $H = \ker \alpha$ and let π be an irreducible projective representation of G over K with the multiplier α. Then $\ker \pi = H$.*

Proof. Assume that π is a matrix representation. If $\pi(g) = kI$ for certain $g \in G$ and $k \in K^*$, then $\pi(g)^{|g|} = \alpha(g,g)\alpha(g,g^2)\ldots\alpha(g,g^{|g|-1})\pi(e) = k^{|g|}I$. It follows, by applying Lemma 1.2, [5], that $g \in H$. Hence, $\ker \pi \subset H$.

By Lemma 1.3, [5], H is a normal subgroup of G and up to cohomology $\alpha(x,y) = 1$ for all $x, y \in H$. By Clifford's theorem, $\pi|_H = \nu \oplus \ldots \oplus \nu$, where ν is the one-dimensional linear representation of H. Hence, $H \subset \ker \pi$ and consequently, $\ker \pi = H$. □

Corollary 56. *([5]) Let G be a p-group and $\alpha \in Z^2(G;K^*)$. Then G admits a faithful irreducible projective representation of G over K with the multiplier α if and only if G is abelian and* $\ker \alpha = \{e\}$.

6 The g-conjugate of an irreducible projective representation of a normal subgroup of a finite group

Definition 57. *([21]) Let α be a multiplier on G. We define*

$$f_\alpha(x,y) = \alpha(x,y)\alpha(xyx^{-1},x)^{-1}, \ \textit{for all} \ x,y \in G.$$

Definition 58. *([21]) Let α be a multiplier on G and π an irreducible projective representation of a normal subgroup H of G with the multiplier α. Define*

$$\pi^{(g)}(h) = f_\alpha(g,h)\pi(ghg^{-1})$$

for all $g \in G, h \in H$.

Lemma 59. *([21]) In the conditions of Definition 58, $\pi^{(g)}$ is an irreducible projective representation of H with the associated multiplier α.*

Proof. Applying Definition 58, Definition 57, Definition 1 and the fact that $\pi^{(g)}$ is a projective representation, we obtain :

$$\pi^{(g)}(h)\pi^{(g)}(h_1) = f_\alpha(g,h)\pi(ghg^{-1})f_\alpha(g,h_1)\pi(gh_1g^{-1}) =$$

$$f_\alpha(g,h)f_\alpha(g,h_1)\alpha(ghg^{-1},gh_1g^{-1})\pi(ghg^{-1}gh_1g^{-1}) =$$

$$f_\alpha(g,h)f_\alpha(g,h_1)\alpha(ghg^{-1},gh_1g^{-1})\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(ghg^{-1},g)^{-1}\alpha(g,h_1)\alpha(gh_1g^{-1},g)^{-1}\alpha(ghg^{-1},gh_1g^{-1})\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(ghg^{-1},g)^{-1}\alpha(g,h_1)\alpha(gh_1g^{-1},g)^{-1}\alpha(ghg^{-1},gh_1g^{-1}g)\cdot$$

$$\cdot\alpha(gh_1g^{-1},g)\alpha(ghg^{-1}gh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(ghg^{-1},g)^{-1}\alpha(g,h_1)\alpha(gh_1g^{-1},g)^{-1}\alpha(ghg^{-1},gh_1)\cdot$$

$$\cdot\alpha(gh_1g^{-1},g)\alpha(ghh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(ghg^{-1},g)^{-1}\alpha(g,h_1)\alpha(ghg^{-1},gh_1)\alpha(ghh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(ghg^{-1},g)^{-1}\alpha(ghg^{-1},g)\alpha(gh,h_1)\alpha(ghh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(g,h)\alpha(gh,h_1)\alpha(ghh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(g,hh_1)\alpha(h,h_1)\alpha(ghh_1g^{-1},g)^{-1}\pi(ghh_1g^{-1}) =$$

$$\alpha(h,h_1)f_\alpha(g,hh_1)\pi(ghh_1g^{-1}) = \alpha(h,h_1)\pi^{(g)}(hh_1)$$

Thus $\pi^{(g)}$ is a projective representation of H with the multiplier α, which is irreducible since π is irreducible. □

Definition 60. *([21]) Let π and ρ be two irreducible projective representations of a normal subgroup H of G. π and ρ are called conjugate if ρ and $\pi^{(g)}$ are equivalent, for some $g \in G$.*

Lemma 61. *([21]) Let π be an irreducible projective representation of H with the multiplier α and let $I_\pi = \left\{g \in G | \pi^{(g)} \simeq \pi\right\}$ (where $\simeq$ denotes the equivalence of representations). I_π is a group called the inertia group of π and $F_\pi = I_\pi/H$ is called the inertia factor of π.*

Theorem 62. *([21]) Let π be an irreducible projective representation of a normal subgroup H of G with the multiplier α and let I_π be the inertia group of π. Then there is an irreducible projective representation $\widetilde{\pi}$ of I_π with the multiplier β such that*

(i) $\widetilde{\pi}(g)\pi(h)\widetilde{\pi}(g)^{-1} = \pi^{(g)}(h)$;

(ii) $\widetilde{\pi}(h) = \pi(h)$;

(iii) $\pi(h)\widetilde{\pi}(g) = \alpha(h,g)\widetilde{\pi}(hg)$, *for all* $g \in I_\pi, h \in H$.

Proof. By the definition of I_π, there is some matrix $\pi'(g)$ such that

$$\pi'(g)\pi(h)\pi'(g)^{-1} = \pi^{(g)}(h) \text{ for all } g \in I_\pi, h \in H.$$

Let $g_1, g_2 \in G$. We have

$$\pi'(g_1g_2)\pi(h)\pi'(g_1g_2)^{-1} = \pi^{(g_1g_2)}(h) =$$

$$\pi'(g_1)\pi'(g_2)\pi(h)(\pi'(g_1)\pi'(g_2))^{-1} = \pi'(g_1)\pi'(g_2)\pi(h)\pi'(g_2)^{-1}\pi'(g_1)^{-1}$$

for all $h \in H$.

So

$$\pi'(g_1g_2)^{-1}\pi'(g_1)\pi'(g_2)\pi(h) = \pi(h)\pi'(g_1g_2)^{-1}\pi'(g_1)\pi'(g_2) \tag{6.1}$$

By Schur's lemma, since π is irreducible and taking into account relation (6.1), there is an element $\sigma(g_1, g_2) \in K^*$ such that $\pi'(g_1g_2)^{-1}\pi'(g_1)\pi'(g_2) = \sigma(g_1, g_2)I_n$. Hence $\pi'(g_1)\pi'(g_2) = \sigma(g_1, g_2)\pi'(g_1g_2)$, where σ is some multiplier on I_π.

Now choose a set $\{x_1, x_2, \ldots, x_n\}$ of right coset representatives of H in I_π and define

$$\widetilde{\pi}(hx_i) = \alpha(h, x_i)^{-1}\pi(h)\pi'(x_i) \tag{6.2}$$

$$\widetilde{\pi}(h) = \pi(h)$$

for all $h \in H, i = 1, \ldots, n$.

It is easy to check that $\widetilde{\pi}$ is a projective representation of I_π with some multiplier β satisfying (i) and (ii). The fact that $\widetilde{\pi}$ is irreducible follows as in the linear case in [6].

Further, by (6.2), $\widetilde{\pi}(h(h_1x_i)) = \widetilde{\pi}(hh_1x_i) = \alpha(hh_1, x_i)^{-1}\pi(hh_1)\pi'(x_i) =$

$$\alpha(hh_1, x_i)^{-1}\alpha(h, h_1)^{-1}\pi(h)\pi(h_1)\pi'(x_i) = \pi(h)\alpha(h, h_1)^{-1}\alpha(hh_1, x_i)^{-1}\pi(h_1)\pi'(x_i) =$$

$$\pi(h)\alpha(h, h_1x_i)^{-1}\alpha(h_1, x_i)^{-1}\pi(h_1)\pi'(x_i) = \alpha(h, h_1x_i)^{-1}\pi(h)\widetilde{\pi}(h_1x_i)$$

Then we obtain $\pi(h)\widetilde{\pi}(g) = \alpha(h, g)\widetilde{\pi}(hg)$ for all $g \in I_\pi, h \in H$. □

Theorem 63. *([21]) Let $\pi, \alpha, I_\pi, \widetilde{\pi}, \beta$ be as above. The set*

$$\left\{(\widetilde{\pi}\otimes\rho)^G; \rho \textit{ is an irreducible projective representation of } F_\pi \textit{ with the multiplier } \alpha\beta^{-1}\right\}$$

is a set of irreducible projective representations of G with the multiplier α. Furthermore, $(\widetilde{\pi} \otimes \rho)^G \simeq (\widetilde{\pi}_1 \otimes \rho_1)^G$ if and only if

(i) $\pi_1 \simeq \pi^{(g)}$ for $g \in G$ (as a representation of H);

(ii) assuming $\widetilde{\pi}_1(x) = f_\alpha(g, x)\widetilde{\pi}(gxg^{-1})$, $x \in I_{\pi_1}$ (we can do this since $I_{\pi^{(g)}} = g^{-1}I_\pi g$), $\rho_1(x) \simeq \rho(gxg^{-1})$, as representations of F_{π_1}.

Proof. The theorem can be proved following the argument in [6]. □

Theorem 64. *([21]) Let τ be an irreducible projective representation of G with the multiplier α and let π be any conjugate of $\tau|_H$. Then $\tau \simeq (\widetilde{\pi} \otimes \rho)^G$, where $\widetilde{\pi}$ is the irreducible projective representation of I_π with the multiplier β defined in Theorem 63 and ρ is some irreducible projective representation of F_π with the multiplier $\alpha\beta^{-1}$.*

Proof. We follow the linear case in Section 6, [6]. □

Theorem 65. *([13]) Let G be a finite group, let K be an algebraically closed field and let H be a normal subgroup of G. Let τ be an irreducible projective representation of G over K with the multiplier α and let π be any conjugate of $\tau|_H$. Then*

(i) H is normal in I_π;

(ii) there is an irreducible projective representation $\widetilde{\pi}$ of I_π and ρ of I_π/H such that $\tau = (\widetilde{\pi} \otimes \widetilde{\rho})^G$, where $\widetilde{\rho}$ is an irreducible projective representation of I_π, $\widetilde{\rho} = \rho^G$ and deg $\widetilde{\pi} =$ deg π.

Proof. The proof follows from Theorem 63 and Theorem 64. □

Theorem 66. *([13]) Let τ be an irreducible projective representation of G with the multiplier α and let N be an abelian normal subgroup of G. Let $d_\alpha(N)$ denotes the common degree of irreducible projective representations of N with the multiplier α. Then deg $\tau | d_\alpha(N) \times [G : N]$.*

Proof. Let π be an irreducible conjugate of $\tau|_N$.

Using Theorem 65, there is an irreducible projective representation $\widetilde{\pi}$ of I_π and $\widetilde{\rho}$ of I_π/N such that $\tau = (\widetilde{\pi} \otimes \widetilde{\rho})^G$, where deg $\widetilde{\pi} =$ deg $\pi = d_\alpha(N)$ and deg $\widetilde{\rho} =$ deg ρ.

Now deg $\tau =$ deg $\widetilde{\pi} \times$ deg $\widetilde{\rho} \times [G : I_\pi] = d_\alpha(N) \times$ deg $\widetilde{\rho} \times [G : I_\pi]$.

Since the degree of an irreducible projective representation of a finite group G divides the order of G ([22]), deg $\rho | [I_\pi : N]$.

Hence deg $\tau | d_\alpha(N) \times [I_\pi : N] \times [G : I_\pi] = d_\alpha(N) \times [G : N]$. □

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