Surveys in Mathematics and its Applications

**ISSN** 1842-6298 (electronic), 1843-7265 (print)
Volume **4** (2009), 215 – 238

# FIXED POINT THEOREMS FOR GENERALIZED WEAKLY CONTRACTIVE MAPPINGS

Ramendra Krishna Bose and Mrinal Kanti Roychowdhury

**Abstract**. In this paper several fixed point theorems for generalized weakly contractive mappings in a metric space setting are proved. The set of generalized weakly contractive mappings considered in this paper contains the family of weakly contractive mappings as a proper subset. Fixed point theorems for single and multi-valued mappings, approximating scheme for common fixed point for some mappings, and fixed point theorems for fuzzy mappings are presented. It extends the work of several authors including Bose and Roychowdhury.

## 1 Introduction

Weakly contractive mappings in a Hilbert space setting was first introduced by Alber and Guerre-Delabriere (cf. [1]). Rhoades proved that most of the results in [1] hold in a Banach space setting, and Bae considered these type of multi-valued mappings (cf. [6]). Kamran, Zhang and Song, Beg and Abbas, Bose and Roychowdhury considered some generalized versions of these mappings and proved some fixed point theorems (cf. [17, 27, 7, 9]). Recently, Dutta and Choudhury have given another generalization of the weakly contractive mappings (for single-valued mappings) (cf. [12]). We have considered a family of generalized weakly contractive mappings which contains the class of mappings considered by Dutta and Choudhury, and also the class of weakly contractive mappings. We have proved several fixed point theorems for both single-valued as well as multi-valued mappings of this type which extends the work of several authors (cf. [12, 5, 7, 24, 9]).

Approximating fixed points of some mappings by an iterative scheme is an area of active research work. Mann (cf. [19]) introduced a one-step iterative scheme, Isikawa (cf. [15]) introduced a two-step iterative scheme, and Noor (cf. [21]) introduced a three-step iterative scheme. Bose and Mukherjee (cf. [8]) and others used Mann iterative scheme to approximate a fixed point of some mappings, and Ghose and Debnath (cf. [13]) and others considered Ishikawa iterative scheme to approximate a fixed point of some mappings. In this paper, we considered the

2000 Mathematics Subject Classification: 03E72; 47H04; 47H10; 54H25; 54C60.

Keywords: Fixed point; Weakly contractive mapping; Weakly compatible mapping; Mann iteration; Fuzzy mapping.

(modified) Mann iterative scheme and (modified) Ishikawa iterative scheme which were first introduced by Rhoades (cf. [24]), and later on by Beg and Abbas (cf. [7]). Azam and Shakeel (cf. [5]) considered these schemes in slightly modified form for weakly contractive mappings with respect to $f$. We have extended their results to generalized weakly contractive mappings with respect to $f$ and proved their results under less conditions, i.e., without imposing extra conditions on the constant sequences in the iterative scheme.

Azam and Beg (cf. [4]) considered weakly contractive fuzzy mappings and proved a common fixed point theorem for a pair of such fuzzy mappings. Next, Bose and Roychowdhury considered such fuzzy mappings and its two generalized versions, and proved some fixed point theorems (cf. [9]). The work in this paper extends/generalizes the work of Azam and Beg (cf. [4]), and Bose and Roychowdhury (cf. [9]).

In this paper in Section 2, we give all the basic definitions and lemmas that are used. In Section 3, we have presented all fixed point theorems for single-valued and multi-valued mappings and approximating scheme for common fixed point of some mappings. In Section 4, we have considered some fixed point theorems for generalized fuzzy weakly contractive mappings.

## 2 Basic Definitions and Lemmas

In this section first we give the following basic definitions for single and multi-valued mappings, and then that for the fuzzy mappings. $(X, d)$ always represents a metric space, $H$ represents the Haudorff distance induced by the metric $d$, and $K(X)$ the family of nonempty compact subsets of $X$. A point $x$ in a metric space $(X, d)$ is called a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in T(x)$. Note that, $x$ is a fixed point of a multi-valued mapping $T$ if and only if $d(x, T(x)) = 0$, whenever $T(x)$ is a closed subset of $X$.

**Definition 1.** *(generalized weakly contractive single-valued mappings). A mapping $T : X \to X$ is said to be generalized weakly contractive with respect to $f : X \to X$ if for all $x, y \in X$,*

$$\psi(d(T(x), T(y))) \le \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $f$ is the identity mapping on $X$, then the mapping is said to be generalized weakly contractive.*

**Remark 2.** *The class of generalized weakly contractive mappings considered by Dutta and Choudhury (cf. [12]) used an additional condition which is monotonicity of $\phi$, which is not required anywhere. If $\psi(t) = t$ for all $t \in [0, \infty)$, then the mapping*

*$T : X \to X$ satisfying the above inequality is said to be weakly contractive with respect to $f$ and if $f$ is the identity mapping, $T$ is said to be weakly contractive.*

**Definition 3.** *(generalized weakly contractive multi-valued mappings). A multi-valued mapping $T : X \to K(X)$ is said to be generalized weakly contractive with respect to $f : X \to X$ if for all $x, y \in X$,*

$$\psi(H(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $f$ is the identity mapping on $X$, then the multi-valued mapping $T : X \to K(X)$ is simply said to be generalized weakly contractive. If $\psi(t) = t$ for all $t \in [0, \infty)$, then the mapping is said to be weakly contractive (with respect to $f$).*

**Definition 4.** *Let $(X, d)$ be a metric space and let $f$ and $g$ be self-mappings of $X$. The mappings $f$ and $g$ are called $R$-weakly commuting, provided there exists some positive real number $R$ such that*

$$d(fgx, gfx) \leq Rd(fx, gx)$$

*for each $x \in X$. For details see Pant [22].*

Note that $R$-weakly commuting mappings commute at their coincidence points. Jungck and Rhoades (cf. [16]) then defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points.

**Definition 5.** *(cf. [26]) Let $(X, d)$ be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

*A metric space $X$ together with the convex structure $W$ is called a convex metric space.*

**Definition 6.** *Let $X$ be a convex metric space. A nonempty subset $C \subset X$ is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times [0, 1]$. Takahashi (cf. [26]) has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(x, y) \leq r\}$ are convex. Also if $\{C_\alpha : \alpha \in A\}$ is a family of convex subsets of $X$, then $\bigcap\{C_\alpha : \alpha \in A\}$ is convex. All normed spaces and their convex subsets are convex metric spaces.*

**Definition 7.** *(generalized modified Mann iterative scheme). Let $(X, d)$ be a convex complete metric space (or Banach space) and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Assume that $T(X) \subset f(X)$ and $f(X)$ is a convex subset of $X$. Define a sequence $\{y_n\}$ in $f(X)$ as*

$$y_n = f(x_{n+1}) = W(T(x_n), f(x_n), \alpha_n) \ (or\ (1-\alpha_n)f(x_n) + \alpha_n T(x_n)) \quad x_0 \in X, n \geq 0,$$

*where $0 \leq \alpha_n \leq 1$ for each $n \geq 0$. The sequence $\{y_n\}$ thus obtained is called generalized modified Mann iterative scheme.*

**Remark 8.** *In the above definition if we put $\psi(t) = t$ for all $t \in [0, \infty)$, then it reduces to the definition of modified Mann iterative scheme (cf. [7, 5]). If $\psi(t) = t$ for all $t \in [0, \infty)$ and $f$ is the identity mapping on $X$, then the reduced definition is called Mann iterative scheme (cf. [24]).*

**Definition 9.** *(generalized modified Ishikawa iterative scheme). Let $(X, d)$ be a convex complete metric space (or Banach space) and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Assume that $T(X) \subset f(X)$ and $f(X)$ is a convex subset of $X$. Suppose two sequences $\{y_n\}$ and $\{z_n\}$ in $f(X)$ are defined as*

$$\begin{aligned} z_n &= f(x_{n+1}) = W(T(v_n), f(x_n), \alpha_n) \ (or\ (1-\alpha_n)f(x_n) + \alpha_n T(v_n)) \\ y_n &= f(v_n) = W(T(x_n), f(x_n), \beta_n) \ (or\ (1-\beta_n)f(x_n) + \beta_n T(x_n)), \end{aligned}$$

*where $x_0 \in X$ and $0 \leq \alpha_n, \beta_n \leq 1$ for each $n \geq 0$. Then the sequence $\{z_n\}$ thus obtained is called generalized modified Ishikawa iterative scheme.*

**Remark 10.** *In the above definition if we put $\psi(t) = t$ for all $t \in [0, \infty)$, then the reduced definition we call as modified Ishikawa iterative scheme. If $\psi(t) = t$ for all $t \in [0, \infty)$ and $f$ is the identity mapping on $X$, then the reduced definition is called Ishikawa iterative scheme (cf. [24]).*

**Lemma 11.** *(cf. [20]) Let $A$ and $B$ be nonempty compact subsets of a metric space $(X, d)$. If $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

A real linear space $X$ with a metric $d$ is called a metric linear space if $d(x+z, y+z) = d(x, y)$ and $\alpha_n \to \alpha$, $x_n \to x \implies \alpha_n x_n \to \alpha x$. Let $(X, d)$ be a metric linear space. A fuzzy set $A$ in a metric linear space $X$ is a function from $X$ into $[0, 1]$. If $x \in X$, the function value $A(x)$ is called the grade of membership of $x$ in $A$. The $\alpha$-level set of $A$, denoted by $A_\alpha$, is defined by

$$\begin{aligned} A_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1], \\ A_0 &= \overline{\{x : A(x) > 0\}}. \end{aligned}$$

Here $\bar{B}$ denotes the closure of the (non-fuzzy) set $B$.

**Definition 12.** *A fuzzy set $A$ is said to be an approximate quantity if and only if $A_\alpha$ is compact and convex in $X$ for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(x) = 1$.*

When $A$ is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, $A$ is identified with an approximation of $x_0$. For $x \in X$, let $\{x\} \in W(X)$ with membership function equal to the characteristic function $\chi_x$ of the set $\{x\}$.

Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in $X$ and $W(X)$ be a sub-collection of all approximate quantities.

**Definition 13.** *Let $A, B \in W(X)$, $\alpha \in [0, 1]$. Then we define*

$$\begin{aligned} p_\alpha(A, B) &= \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x, y), \\ p(A, B) &= \sup_\alpha p_\alpha(A, B), \\ D_\alpha(A, B) &= H(A_\alpha, B_\alpha), \\ D(A, B) &= \sup_\alpha D_\alpha(A, B). \end{aligned}$$

*where $H$ is the Hausdorff distance induced by the metric $d$.*

The function $D_\alpha(A, B)$ is called an $\alpha$-distance between $A, B \in W(X)$, and $D$ a metric on $W(X)$. We note that $p_\alpha$ is a non-decreasing function of $\alpha$ and thus $p(A, B) = p_1(A, B)$. In particular if $A = \{x\}$, then $p(\{x\}, B) = p_1(x, B) = d(x, B_1)$. Next we define an order on the family $W(X)$, which characterizes the accuracy of a given quantity.

**Definition 14.** *Let $A, B \in W(X)$. Then $A$ is said to be more accurate than $B$, denoted by $A \subset B$ (or $B$ includes $A$), if and only if $A(x) \leq B(x)$ for each $x \in X$.*

The relation $\subset$ induces a partial order on the family $W(X)$.

**Definition 15.** *Let $X$ be an arbitrary set and $Y$ be any metric linear space. $F$ is called a fuzzy mapping if and only if $F$ is a mapping from the set $X$ into $W(Y)$.*

**Definition 16.** *For $F : X \to W(X)$, we say that $u \in X$ is a fixed point of $F$ if $\{u\} \subset F(u)$, i.e. if $u \in F(u)_1$.*

**Lemma 17.** *(cf. [14]) Let $x \in X$ and $A \in W(X)$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

**Remark 18.** *Note that from the above lemma it follows that for $A \in W(X)$, $\{x\} \subset A$ if and only if $p(\{x\}, A) = 0$. If no confusion arises instead of $p(\{x\}, A)$ we will write $p(x, A)$.*

**Lemma 19.** *(cf. [14]) $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for each $x, y \in X$.*

**Lemma 20.** *(cf. [14]) If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W(X)$.*

**Lemma 21.** *(cf. [18]) Let $(X, d)$ be a complete metric linear space, $F : X \to W(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

**Remark 22.** *Let $f : X \to X$ be a self map and $T : X \to W(X)$ be a fuzzy mapping such that $\cup\{T(X)\}_\alpha \subseteq f(X)$ for $\alpha \in [0, 1]$. Then from Lemma 21, it follows that for any chosen point $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. Here $T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\}$.*

**Definition 23.** *(generalized weakly contractive fuzzy mappings). A fuzzy mappings $T : X \to W(X)$ is said to be generalized weakly contractive with respect to $f : X \to X$ if for all $x, y \in X$,*

$$\psi(D(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $f$ is the identity mapping on $X$, then the fuzzy mapping $T : X \to W(X)$ is simply said to be generalized weakly contractive. If $\psi(t) = t$ for all $t \in [0, \infty)$, then the mapping is said to be weakly contractive (with respect to $f$).*

## 3 Fixed point theorems for single and multi-valued mappings

In this section we prove the following main theorems of this paper concerning single and multi-valued mappings.

**Theorem 24.** *Let $(X, d)$ be a complete metric space and let $T, S : X \to X$ be self-mappings such that for all $x, y \in X$*

$$\psi(d(T(x), S(y))) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{3.1}$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Then there exists a unique point $u \in X$ such that $u = T(u) = S(u)$.*

*Proof.* Let $x_0 \in X$, we construct a sequence $\{x_k\}$ in $X$ by taking $x_{2k+1} = T(x_{2k})$ and $x_{2k+2} = S(x_{2k+1})$ for all $k \geq 0$. Hence by (3.1) we have,

$$\begin{aligned} \psi(d(x_{2k+1}, x_{2k+2})) &= \psi(d(T(x_{2k}), S(x_{2k+1}))) \\ &\leq \psi(d(x_{2k}, x_{2k+1})) - \phi(d(x_{2k}, x_{2k+1})) \\ &\leq \psi(d(x_{2k}, x_{2k+1})). \end{aligned} \tag{3.2}$$

Again by (3.1) we have,

$$\begin{aligned} \psi(d(x_{2k+2}, x_{2k+3})) &= \psi(d(S(x_{2k+1}), T(x_{2k+2}))) \\ &= \psi(d(T(x_{2k+2}), S(x_{2k+1}))) \\ &\leq \psi(d(x_{2k+2}, x_{2k+1})) - \phi(d(x_{2k+2}, x_{2k+1})) \\ &\leq \psi(d(x_{2k+2}, x_{2k+1})) \\ &= \psi(d(x_{2k+1}, x_{2k+2})). \end{aligned} \tag{3.3}$$

Thus for $n \geq 0$ we have, $\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1}))$. As $\psi$ is monotonically increasing, from this inequality we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, which implies that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. By (3.2) and (3.3) for any $n \geq 0$ we have,

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

Taking $n \to \infty$ and using the continuity of $\psi$ and $\phi$, we obtain

$$\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0,$$

which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now proceeding in the same way as in Theorem 3.4 in [9], it can be shown that the sequence $\{x_n\}$ is a Cauchy sequence. From the completeness of $X$, it follows that there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Moreover, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$. Now we show that $u = T(u) = S(u)$. We know $x_{2n+1} = T(x_{2n})$ and $x_{2n+2} = S(x_{2n+1})$. Note that,

$$\psi(d(x_{2n+1}, S(u))) = \psi(d(T(x_{2n}), S(u))) \leq \psi(d(x_{2n}, u)) - \phi(d(x_{2n}, u)).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(u, S(u))) \leq \psi(0) - \phi(0) = 0 \implies \psi(d(u, S(u))) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(u, S(u))) = 0 \implies d(u, S(u)) = 0 \implies u = S(u)$. Similarly we can show $u = T(u)$. Thus, we have $u = T(u) = S(u)$.

If there exists another point $v \in X$ such that $v = T(v) = S(v)$, then we have $\psi(d(u, v)) = \psi(d(T(u), S(v))) \leq \psi(d(u, v)) - \phi(d(u, v)) \implies \phi(d(u, v)) \leq 0$, and hence $u = v$. Thus, the proof is complete.

$\square$

**Corollary 25.** *If we take $T = S$, then we have Theorem 2.1 of Dutta and Choudhury (cf. [12]).*

**Corollary 26.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then Theorem 24 reduces to usual weak contraction theorem (cf. [24, Theorem 1]).*

**Theorem 27.** *Let $(X, d)$ be a complete metric space and $T, S : X \to K(X)$ be two mappings such that for all $x, y \in X$*

$$\psi(H(Tx, Sy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{3.4}$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Then $T$ and $S$ have a common fixed point.*

*Proof.* Let $\{x_k\}$ be a sequence in $X$ such that $x_{2k+1} \in T(x_{2k})$ and $x_{2k+2} \in S(x_{2k+1})$ for all $k \geq 0$, and $d(x_{2k+1}, x_{2k+2}) \leq H(T(x_{2k}), S(x_{2k+1}))$ (by Lemma 11). Hence,

$$\begin{aligned} \psi(d(x_{2k+1}, x_{2k+2})) &\leq \psi(H(T(x_{2k}), S(x_{2k+1}))) \\ &\leq \psi(d(x_{2k}, x_{2k+1})) - \phi(d(x_{2k}, x_{2k+1})) \\ &\leq \psi(d(x_{2k}, x_{2k+1})). \end{aligned}$$

Similarly we can show that

$$\psi(d(x_{2k+2}, x_{2k+3})) \leq \psi(d(x_{2k+1}, x_{2k+2})) - \phi(d(x_{2k+1}, x_{2k+2})) \leq \psi(d(x_{2k+1}, x_{2k+2})).$$

Thus for $n \geq 0$ we have $\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1}))$. As $\psi$ is monotonically increasing, from this inequality it follows that for $n \geq 0$, $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Now proceeding in the same way as in Theorem 3.4 in [9], it can be shown that the sequence $\{x_n\}$ is a Cauchy sequence in $X$. Hence from the completeness of $X$ it follows that $x_n \to u$ for some $u \in X$. Moreover, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$. Now we show that $u \in T(u)$ and $u \in S(u)$. We know $x_{2n+1} \in T(x_{2n})$ and $x_{2n+2} \in S(x_{2n+1})$. Note that,

$$\psi(d(x_{2n+1}, S(u))) \leq \psi(d(T(x_{2n}), S(u))) \leq \psi(d(x_{2n}, u)) - \phi(d(x_{2n}, u)).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(u, S(u))) \leq \psi(0) - \phi(0) = 0 \implies \psi(d(u, S(u))) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(u, S(u))) = 0 \implies d(u, S(u))) = 0 \implies u \in S(u)$. Similarly, we can show $u \in T(u)$, i.e., $T$ and $S$ have a common fixed point.

□

**Corollary 28.** *If $T = S$, then $T$ has a fixed point.*

**Theorem 29.** *Let $(X,d)$ be a metric space and $T,S,f$ be self-mappings on $X$ such that for all $x,y \in X$,*

$$\psi(d(T(x),S(y))) \leq \psi(d(f(x),f(y))) - \phi(d(f(x),f(y)),$$

*where $\psi,\phi : [0,\infty) \to [0,\infty)$ are both continuous functions such that $\psi(t),\phi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $T(X) \cup S(X) \subset f(X)$ and $f(X)$ is a complete subspace of $X$, then there exists a point $p \in X$ such that $f(p) = T(p) = S(p)$ (the point $p$ is unique if $f$ is one-to-one).*

*Proof.* Let $x_0$ be a point in $X$. Choose a point $x_1 \in X$ such that $f(x_1) = T(x_0)$. This can be done since $T(X) \subset f(X)$. Similarly, choose $x_2 \in X$ such that $f(x_2) = S(x_1)$. In general, having chosen $x_k \in X$, we obtain $x_{k+1} \in X$ such that $f(x_{2k+1}) = T(x_{2k})$ and $f(x_{2k+2}) = S(x_{2k+1})$ for any integer $k \geq 0$. Hence, by the given hypothesis

$$\begin{aligned}\psi(d(f(x_{2k+1}),f(x_{2k+2}))) &= \psi(d(T(x_{2k}),S(x_{2k+1}))) \\ &\leq \psi(d(f(x_{2k}),f(x_{2k+1}))) - \phi(d(f(x_{2k}),f(x_{2k+1}))) \\ &\leq \psi(d(f(x_{2k}),f(x_{2k+1}))),\end{aligned} \tag{3.5}$$

and similarly,

$$\psi(d(f(x_{2k+2}),f(x_{2k+3}))) = \psi(d(S(x_{2k+1}),T(x_{2k+2}))) \leq \psi(d(f(x_{2k+1}),f(x_{2k+2}))). \tag{3.6}$$

As $\psi$ is monotonically increasing, from (3.5) and (3.6) it follows $\{d(f(x_n),f(x_{n+1}))\}$ is a non-increasing sequence of positive real numbers and therefore, tends to a limit $\ell \geq 0$. By (3.5) and (3.6),

$$\psi(d(f(x_{n+1}),f(x_{n+2}))) \leq \psi(d(f(x_n),f(x_{n+1}))) - \phi(d(f(x_n),f(x_{n+1}))) \text{ for all } n \geq 0.$$

Taking $n \to \infty$ and using the continuity of $\psi$ and $\phi$, we obtain

$$\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0,$$

which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore $\ell = 0$, i.e., $\lim_{n\to\infty} d(f(x_n),f(x_{n+1})) = 0$. Now proceeding in the same way as in Theorem 3.4 in [9], it can be shown that the sequence $\{f(x_n)\}$ is a Cauchy sequence in $f(X)$. As $f(X)$ is a complete subspace of $X$, $\{f(x_n)\}$ has a limit $q$ in $f(X)$. Consequently, we obtain a point $p$ in $X$ such that $f(p) = q$. Thus, $f(x_{2k+1}) \to q$ and $f(x_{2k}) \to q$ as $k \to \infty$. Now by the given hypothesis,

$$\psi(d(f(x_{2k+1}),S(p))) \leq \psi(d(f(x_{2k}),f(p))) - \phi(d(f(x_{2k}),f(p))).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(q, S(p))) \leq \psi(d(q, f(p))) - \phi(d(q, f(p))) = \psi(0) - \phi(0) = 0 \implies \psi(d(q, S(p))) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(q, S(p))) = 0 \implies d(q, S(p)) = 0 \implies q = S(p)$. Similarly, we can show $q = T(p)$. Thus, we have $q = f(p) = T(p) = S(p)$ for a point $p \in X$. Clearly the point $p$ is unique if $f$ is one-to-one. □

**Remark 30.** *The above theorem extends Theorem 2.1 of Beg and Abbas (cf. [7]), and Theorem 2.3 of Azam and Shakeel (cf. [5])*

**Theorem 31.** *In the above theorem Theorem 29, if further we have the pairs $(f, T)$ and $(f, S)$ are weakly compatible (or R-weakly commuting) (see Definition 4), then $f, T$ and $S$ have a common fixed point (and the point is unique if $f$ is one-to-one).*

*Proof.* Note that $R$-weakly commuting mappings commute at their coincidence point. We have $q = f(p) = T(p) = S(p)$ and this implies that $f(T(p)) = T(f(p))$ and $f(S(p)) = S(f(p))$. Also we have $T(q) = f(q) = S(q)$. We claim that $f(q) = q$. We have,

$$\begin{aligned} \psi(f(q), q) = \psi(T(q), S(p)) &\leq \psi(d(f(q), f(p))) - \phi(d(f(q), f(p))) \\ \implies \phi(d(f(q), f(p))) \leq 0 &\implies \phi(d(f(q), f(p))) = 0 \\ \implies d(f(q), f(p)) = 0 &\implies f(q) = f(p) = q, \end{aligned}$$

which implies $q$ is a fixed point of $f$, and then from $f(T(p)) = T(f(p))$ and $f(S(p)) = S(f(p))$ we have $q$ is also a fixed point of both $T$ and $S$. Hence, $f, T$ and $S$ have a common fixed point. Clearly the point is unique if $f$ is one-to-one. □

**Corollary 32.** *In the above theorem if $T = S$, then we have that $f$ and $T$ have a common (unique if $f$ is one-to-one) fixed point in $X$.*

**Remark 33.** *The above theorem extends Theorem 2.5 of Beg and Abbas (cf. [7]), and Theorem 2.5 of Azam and Shakeel (cf. [5]).*

Proceeding in the same way as Theorem 29 (taking the sequence $\{x_n\}$ such that $x_{n+1} = T_{n+1}(x_n)$ for $n = 0, 1, 2, \cdots$), the following theorem can also be proved.

**Theorem 34.** *Let $(X, d)$ be a metric space and $\{T_i\}_{i=1}^{\infty}$ be a sequence of self-mappings on $X$ such that for all $x, y \in X$,*

$$\psi(d(T_i(x), T_j(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y)) \textit{ for all } i, j \geq 1,$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $T_i(X) \subset f(X)$ for all $i$ and $f(X)$ is a complete subspace of $X$, then there exists a point $p \in X$ such that $f(p) = T_i(p)$ for all $i \geq 1$ (the point $p$ is unique if $f$ is one-to-one).*

**Theorem 35.** *Let $(X, d)$ be a convex metric space and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing, and $\psi$ is both monotonically increasing (strictly) and convex. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $T(X) \subset f(X)$ and $f(X)$ is a convex and complete subspace of $X$, then the generalized modified Mann iterative scheme (see Definition 7) converges to a common fixed point of $f$ and $T$.*

*Proof.* By Corollary 32, we obtain a common fixed point $q$ of $f$ and $T$. Now consider

$$\begin{aligned}
\psi(d(y_n, q) &= \psi(d(f(x_{n+1}), f(p))) = \psi(d(W(T(x_n), f(x_n), \alpha_n), f(p))) \\
&\leq \psi((1-\alpha_n)d(f(x_n), f(p)) + \alpha_n d(T(x_n), f(p))) \\
&\leq (1-\alpha_n)\psi(d(f(x_n), f(p))) + \alpha_n \psi(d(T(x_n), f(p))) \\
&\leq (1-\alpha_n)\psi(d(f(x_n), f(p))) + \alpha_n(\psi(d(f(x_n), f(p))) - \phi(d(f(x_n), f(p))) \\
&\leq \psi(d(f(x_n), f(p))) - \phi(d(f(x_n), f(p))) \qquad (3.7) \\
&= \psi(d(y_{n-1}, q)).
\end{aligned}$$

As $\psi$ is monotonically increasing from the above inequality it follows that $d(y_n, q) \leq d(y_{n-1}, q)$, i.e., $\{d(y_n, q)\}$ is a non-increasing sequence of positive real numbers, which gives $\lim_{n\to\infty} d(y_n, q) = \ell \geq 0$. Now if $\ell > 0$, then from (3.7) as both $\psi$ and $\phi$ are continuous, taking $n \to \infty$ we have, $\lim_{n\to\infty} \psi(d(y_n, q)) \leq \lim_{n\to\infty} \psi(d(y_{n-1}, q)) - \phi(d(y_{n-1}, q)) \implies \psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$ and $\phi(t) > 0$ for $t > 0$. Therefore, $\ell = 0$. Hence, the generalized modified Mann iterative scheme converges to a common fixed point of $f$ and $T$.

□

**Corollary 36.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the modified Mann iterative scheme converges to a common fixed point of $f$ and $T$, which is the work of Azam and Shakeel (cf. [5, Theorem 2.7]) without the condition $\sum \alpha_n = \infty$.*

**Remark 37.** *With reference to Theorem 31 (when $f$ is one-to-one), one can show that the generalized modified Mann iterative scheme (either for the mapping $T$ or for the mapping $S$) converges to a unique common fixed point of $f$, $T$ and $S$.*

**Theorem 38.** *Let $(X, d)$ be a normed linear space and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing, and $\psi$ is both* *monotonically increasing (strictly) and convex. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of $X$, then the generalized modified Mann iterative scheme (see Definition 7) converges to a common fixed point of $f$ and $T$.*

*Proof.* By Corollary 32, we obtain a common fixed point $q$ of $f$ and $T$. Writing $d(y_n, q) = ||y_n - q||$ and $y_n = (1 - \alpha_n)f(x_n) + \alpha_n T(x_n)$, and then proceeding in the same way as in Theorem 35 we can show that $\lim_{n\to\infty} ||y_n - q|| = 0$, that is, the generalized modified Mann iterative scheme converges to a common fixed point of $f$ and $T$. □

**Corollary 39.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the modified Mann iterative scheme converges to a common fixed point of $f$ and $T$, which is the work of Beg and Abbas (cf. [7, Theorem 2.6]) without the condition $\sum \alpha_n = \infty$.*

**Corollary 40.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$ and $f$ is the identity mapping on $X$, then it reduces to the work of Rhoades (cf. [24]) without the condition $\sum \alpha_n = \infty$.*

**Remark 41.** *Rhoades, Beg and Abbas, Azam and Shakeel in their proofs used an extra condition $\sum \alpha_n = \infty$ on the sequence $\{\alpha_n\}$, which we do not need in our proof.*

We can also prove the following two theorems.

**Theorem 42.** *Let $(X, d)$ be a convex metric space and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing, and $\psi$ is both monotonically increasing (strictly) and convex. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $T(X) \subset f(X)$ and $f(X)$ is a convex and complete subspace of $X$, then the generalized modified Ishikawa iterative scheme (see Definition 9) converges to a common fixed point of $f$ and $T$.*

**Corollary 43.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the modified Ishikawa iterative scheme converges to a common fixed point of $f$ and $T$, which is the work of Azam and Shakeel (cf. [5, Theorem 2.8]) without the condition $\sum \alpha_n \beta_n = \infty$.*

**Theorem 44.** *Let $(X, d)$ be a normed linear space and let $T, f$ be self-mappings on $X$ such that for all $x, y \in X$*

$$\psi(d(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing, and $\psi$ is both* *monotonically increasing (strictly) and convex. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of $X$, then the generalized modified Ishikawa iterative scheme (see Definition 9) converges to a common fixed point of $f$ and $T$.*

**Corollary 45.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the modified Ishikawa iterative scheme converges to a common fixed point of $f$ and $T$, which is the work of Beg and Abbas (cf. [7, Theorem 2.7]) without the condition $\sum \alpha_n \beta_n = \infty$ on the sequence $\{\alpha_n\}$.*

**Corollary 46.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$ and $f$ is the identity mapping on $X$, then it reduces to the work of Rhoades (cf. [24]) without the condition $\sum \alpha_n \beta_n = \infty$.*

**Remark 47.** *Rhoades, Beg and Abbas, Azam and Shakeel in their proofs used an extra condition $\sum \alpha_n \beta_n = \infty$, which we do not need in our proof.*

**Theorem 48.** *Let $(X, d)$ be a metric space, $f : X \to X$ be a self-mapping and $T, S : X \to K(X)$ be multi-valued mappings such that for all $x, y \in X$,*

$$\psi(H(T(x), S(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y)),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $\cup_{x \in X} T(x) \subset f(X)$, $\cup_{x \in X} S(x) \subset f(X)$ and $f(X)$ is a complete subspace of $X$, then there exists a point $p \in X$ such that $f(p) \in T(p)$ and $f(p) \in S(p)$.*

*Proof.* Let $x_0$ be a point in $X$. Choose a point $x_1 \in X$ such that $f(x_1) \in T(x_0)$. This can be done since $\cup_{x \in X} T(x) \subset f(X)$. Similarly, choose $x_2 \in X$ such that $f(x_2) \in S(x_1)$ and such that $d(f(x_1), f(x_2)) \leq H(T(x_0), S(x_1))$. In general, having chosen $x_k \in X$, we obtain $x_{k+1} \in X$ such that $f(x_{2k+1}) \in T(x_{2k})$, $f(x_{2k+2}) \in S(x_{2k+1})$, and such that $d(f(x_{2k+1}), f(x_{2k+2})) \leq H(T(x_{2k}), S(x_{2k+1}))$ for any integer $k \geq 0$ (by Lemma 11). Hence, by the given hypothesis

$$\begin{aligned}\psi(d(f(x_{2k+1}), f(x_{2k+2}))) &\leq \psi(H(T(x_{2k}), S(x_{2k+1}))) \\ &\leq \psi(d(f(x_{2k}), f(x_{2k+1}))) - \phi(d(f(x_{2k}), f(x_{2k+1}))) \qquad (3.8) \\ &\leq \psi(d(f(x_{2k}), f(x_{2k+1}))),\end{aligned}$$

and similarly,

$$\psi(d(f(x_{2k+2}), f(x_{2k+3}))) \leq \psi(H(S(x_{2k+1}), T(x_{2k+2}))) \leq \psi(d(f(x_{2k+1}), f(x_{2k+2}))), \qquad (3.9)$$

which show that $\{d(f(x_n), f(x_{n+1}))\}$ is a non-increasing sequence of positive real numbers and therefore, tends to a limit $\ell \geq 0$. By (3.8) and (3.9),

$$\psi(d(f(x_{n+1}), f(x_{n+2}))) \leq \psi(d(f(x_n), f(x_{n+1}))) - \phi(d(f(x_n), f(x_{n+1}))) \text{ for all } n \geq 0.$$

Taking $n \to \infty$ and using the continuity of $\psi$ and $\phi$, we obtain

$$\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0,$$

which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore $\ell = 0$, i.e., $\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$. Now proceeding in the same way as in Theorem 3.4 in [9], it can be shown that the sequence $\{f(x_n)\}$ is a Cauchy sequence in $f(X)$. As $f(X)$ is a complete subspace of $X$, $\{f(x_n)\}$ has a limit $q$ in $f(X)$. Consequently, we obtain $p$ in $X$ such that $f(p) = q$. Thus, $f(x_{2k+1}) \to q$ and $f(x_{2k}) \to q$ as $k \to \infty$. Now by the given hypothesis,

$$\psi(d(f(x_{2k+1}), S(p))) \leq \psi(d(f(x_{2k}), f(p))) - \phi(d(f(x_{2k}), f(p))).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(q, S(p))) \leq \psi(d(q, f(p))) - \phi(d(q, f(p))) = \psi(0) - \phi(0) = 0 \implies \psi(d(q, S(p))) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(q, S(p))) = 0 \implies d(q, S(p)) = 0 \implies q \in S(p)$. Similarly, we can show $q \in T(p)$ and hence is the theorem. □

**Remark 49.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the above theorem reduces to Theorem 3.7 of Bose and Roychowdhury (cf. [9]).*

Proceeding in the same way as Theorem 48 (taking the sequence $\{x_n\}$ such that $x_{n+1} \in T_{n+1}(x_n)$ for $n \in \mathbb{Z}^+$), the following theorem can also be proved.

**Theorem 50.** *Let $(X, d)$ be a metric space, $f : X \to X$ be a self-mapping and $\{T_i : X \to K(X)\}_{i\in\mathbb{Z}^+}$ be a sequence of multi-valued mappings such that for all $x, y \in X$,*

$$\psi(H(T_i(x), T_j(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y)) \text{ for all } i, j \geq 1,$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). If $\cup_{x\in X} T_i(x) \subset f(X)$ for all $i$ and $f(X)$ is a complete subspace of $X$, then there exists a point $p \in X$ such that $f(p) \in T_i(p)$ for all $i$.*

**Theorem 51.** *Let $(X, d)$ be a complete metric space and $T, S : X \to K(X)$ be two mappings such that for all $x, y \in X$*

$$\psi(H(Tx, Sy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{3.10}$$

*where*

$$M(x,y)=\max\{d(x,y),d(T(x),x),d(S(y),y),\frac{1}{2}[d(y,T(x))+d(x,S(y))]\},$$

*and* $\psi,\phi:[0,\infty)\to[0,\infty)$ *are both continuous functions such that* $\psi(t),\phi(t)>0$ *for* $t\in(0,\infty)$ *and* $\psi(0)=0=\phi(0)$. *In addition,* $\phi$ *is non-decreasing and* $\psi$ *is monotonically increasing (strictly). Then there there exists a point* $u\in X$ *such that* $u\in T(u)$ *and* $u\in S(u)$.

*Proof.* Clearly $M(x,y)=0$ if and only if $x=y$ is a common fixed point of $T$ and $S$. Let $\{x_k\}$ be a sequence in $X$ such that $x_{2k+1}\in T(x_{2k})$ and $x_{2k}\in S(x_{2k+1})$ for all $k\geq 0$, and $d(x_{2k+1},x_{2k})\leq H(T(x_{2k}),S(x_{2k+1}))$. Hence as $\psi$ is monotonically increasing we have,

$$\begin{aligned}\psi(d(x_{2k+1},x_{2k+2}))&\leq\psi(H(T(x_{2k}),S(x_{2k+1})))\\&\leq\psi(M(x_{2k},x_{2k+1}))-\phi(M(x_{2k},x_{2k+1}))\\&\leq\psi(M(x_{2k},x_{2k+1}))\\&\leq\psi(d(x_{2k},x_{2k+1}))\ \ [\text{as } M(x_{2k},x_{2k+1})\leq d(x_{2k},x_{2k+1})]\\\implies d(x_{2k+1},x_{2k+2})&\leq M(x_{2k},x_{2k+1})\leq d(x_{2k},x_{2k+1}).\end{aligned}$$

Similarly we have, $d(x_{2k+2},x_{2k+3})\leq M(x_{2k+1},x_{2k+2})\leq d(x_{2k+1},x_{2k+2})$. Thus for $n\geq 0$ we have $d(x_{n+1},x_{n+2})\leq d(x_n,x_{n+1})$, which shows that $\{d(x_n,x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Now proceeding in the same way as in the previous theorem, and in Theorem 3.4 in [9], it can be shown that $\{x_n\}$ is Cauchy sequence in $X$. Hence from the completeness of $X$ it follows that $x_n\to u$ for some $u\in X$. Now we show that $u\in T(u)$ and $u\in S(u)$. We have,

$$\begin{aligned}d(S(u),u)\leq M(x_{2k},u)&=\max\{d(x_{2k},u),d(T(x_{2k}),x_{2k}),d(S(u),u),\\&\qquad\frac{1}{2}[d(u,T(x_{2k}))+d(x_{2k},S(u))]\}\\&\leq\max\{d(x_{2k},u),d(x_{2k+1},x_{2k})+d(T(x_{2k}),x_{2k+1}),d(S(u),u),\\&\qquad\frac{1}{2}[d(u,x_{2k+1})+d(x_{2k+1},T(x_{2k}))+d(x_{2k},u)+d(u,S(u))]\}\\&=\max\{d(x_{2k},u),d(x_{2k+1},x_{2k}),d(S(u),u),\\&\qquad\frac{1}{2}[d(u,x_{2k+1})+d(x_{2k},u)+d(u,S(u))]\}.\end{aligned}$$

and hence, taking $k\to\infty$ we have

$$d(u,S(u))\leq M(x_{2k},u)\leq\max\{0,0,d(S(u),u),\frac{1}{2}[0+0+d(u,S(u))]\}=d(S(u),u),$$

and so $\lim_{k\to\infty}M(x_{2k},u)=d(S(u),u)$. We have,

$$d(u,S(u))\leq d(u,x_{2k+1})+d(x_{2k+1},S(u))\leq d(u,x_{2k+1})+H(T(x_{2k}),S(u)).$$

As $\lim_{k\to\infty} d(u, x_{2k+1}) = 0$, $\psi$ is monotonically increasing and both $\psi$ and $\phi$ are continuous, we have by the given hypothesis (3.10),

$$\begin{aligned}\psi(d(u,S(u))) &\le 0 + \lim_{k\to\infty}\psi(H(T(x_{2k}),S(u)))\\ &\le \lim_{k\to\infty}\psi(M(x_{2k},u)) - \lim_{k\to\infty}\phi(M(x_{2k},u))\\ &= \psi(d(u,S(u))) - \phi(d(u,S(u))),\end{aligned}$$

which implies $\phi(d(u,S(u))) \le 0$, i.e. $\phi(d(u,S(u))) = 0 \implies d(u,S(u)) = 0 \implies u \in S(u)$. Similarly, we can show $u \in T(u)$. □

**Remark 52.** *If we take $\psi(t) = t$ for all $t \in [0,\infty)$, then the above theorem reduces to Theorem 3.4 of Bose and Roychowdhury (cf. [9]).*

Using the techniques used in Theorem 48 and Theorem 51 of this paper, and Theorem 3.6 in [9] we can also prove the following theorem.

**Theorem 53.** *Let $K$ be a nonempty closed subset of a complete and convex metric space $(X,d)$ and $T : K \to K(X)$ be a mapping such that for all $x, y \in K$*

$$\psi(H(Tx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y)),$$

*where*

$$M(x,y) = \max\{d(x,y), d(T(x),x), d(T(y),y), \frac{1}{2}[d(y,T(x)) + d(x,T(y))]\},$$

*and $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Suppose that $T(x) \subset K$ for each $x \in \partial K$ (the boundary of $K$). Then there there exists a point $u \in K$ such that $u \in T(u)$.*

**Remark 54.** *If we take $\psi(t) = t$ for all $t \in [0,\infty)$, then the above theorem reduces to Theorem 3.6 of Bose and Roychowdhury (cf. [9]).*

## 4 Fixed point theorems concerning fuzzy mappings

In this section we give the main theorems of this paper concerning fuzzy mappings.

**Theorem 55.** *Let $(X,d)$ be a complete metric linear space and $T, S : X \to W(X)$ be a pair of fuzzy mappings such that for all $x, y \in X$*

$$\psi(D(T(x),S(y))) \le \psi(d(x,y)) - \phi(d(x,y)), \tag{4.1}$$

*where $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Then there there exists a point $u \in X$ such that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$.*

*Proof.* Let $x_0$ be an arbitrary but fixed element of $X$. We shall construct a sequence $\{x_n\}$ of points of $X$ as follows. By Lemma 21, there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$. By Lemma 21 and Lemma 11, we can choose $x_2 \in X$ such that $\{x_2\} \subset S(x_1)$ and

$$d(x_1, x_2) \leq H(T(x_0)_1, S(x_1)_1).$$

This in view of the inequality (4.1) we have,

$$\psi(d(x_1, x_2)) \leq \psi(D_1(T(x_0), S(x_1))) \leq \psi(D(T(x_0), S(x_1))) \leq \psi(d(x_0, x_1)) - \phi(d(x_0, x_1)).$$

Continuing this process, having chosen $x_k \in X$, we obtain $x_{k+1} \in X$ such that for all $k \geq 0$, $\{x_{2k+1}\} \subset T(x_{2k})$, $\{x_{2k+2}\} \subset S(x_{2k+1})$, and

$$\psi(d(x_{2k+1}, x_{2k+2})) \leq \psi(D(T(x_{2k}), S(x_{2k+1}))) \leq \psi(d(x_{2k}, x_{2k+1}))) - \phi(d(x_{2k}, x_{2k+1}))),$$

$$\psi(d(x_{2k+2}, x_{2k+3})) \leq \psi(D(S(x_{2k+1}), T(x_{2k+2}))) \leq \psi(d(x_{2k+1}, x_{2k+2}))) - \phi(d(x_{2k+1}, x_{2k+2}))).$$

As $\psi$ is monotonically increasing, from the above two inequalities it follows that for $n \geq 0$, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. We have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n), x_{n+1})) \text{ for all } n \geq 0.$$

Taking $n \to \infty$ and using the continuity of $\psi$ and $\phi$, we obtain

$$\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0,$$

which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore $\ell = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now proceeding in the same way as in Theorem 3.4 in [9], it can be shown that the sequence $\{x_n\}$ is a Cauchy sequence in $X$. From the completeness of $X$, it follows that there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Moreover, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$. Now we prove that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$. We know $\{x_{2k+1}\} \subset T(x_{2k})$ and $\{x_{2k}\} \subset S(x_{2k+1})$ for $k \geq 0$. Note that,

$$d(x_{2k+1}, S(u)_1) \leq H(T(x_{2k})_1, S(u)_1) = D_1(T(x_{2k}), S(u)) \leq D(T(x_{2k}), S(u)),$$

and hence as $\psi$ is monotonically increasing we have,

$$\psi(d(x_{2k+1}, S(u)_1)) \leq \psi(D(T(x_{2k}), S(u))) \leq \psi(d(x_{2k}, u)) - \phi(d(x_{2k}, u)).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(u, S(u)_1)) \leq \psi(d(u, u)) - \phi(d(u, u)) = \psi(0) - \phi(0) = 0 \implies \psi(d(u, S(u)_1)) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(u, S(u)_1)) = 0 \implies d(u, S(u)_1) = 0 \implies u \in S(u)_1$, i.e., $\{u\} \subset S(u)$. Similarly, we can show $\{u\} \subset T(u)$ and hence is the theorem. □

**Theorem 56.** *Let $(X, d)$ be a complete metric linear space and $T, S : X \to W(X)$ be a pair of mappings such that for all $x, y \in X$*

$$\psi(D(T(x), S(y))) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{4.2}$$

*where*

$$M(x, y) = \max\{d(x, y), p(T(x), x), p(S(y), y), \frac{1}{2}[p(y, T(x)) + p(x, S(y))]\}.$$

*and $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Then there there exists a point $u \in X$ such that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$.*

*Proof.* Note that $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of $T$ and $S$ (cf. Theorem 3.4 in [9]).

Let $x_0$ be an arbitrary but fixed element of $X$. We shall construct a sequence $\{x_n\}$ of points of $X$ as follows. By Lemma 21, there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$. By Lemma 21 and Lemma 11, we can choose $x_2 \in X$ such that $\{x_2\} \subset S(x_1)$ and

$$d(x_1, x_2) \leq H(T(x_0)_1, S(x_1)_1) = D_1(T(x_0), S(x_1)) \leq D(T(x_0), S(x_1)).$$

Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $\{x_{2k+1}\} \subset T(x_{2k})$, $\{x_{2k+2}\} \subset S(x_{2k+1})$ and

$$d(x_{2k+1}, x_{2k+2}) \leq D(T(x_{2k}), S(x_{2k+1})) \text{ and } d(x_{2k+2}, x_{2k+3}) \leq D(S(x_{2k+1}), T(x_{2k+2})).$$

Hence by the given hypothesis and as $\psi$ is monotonically increasing we have,

$$\psi(d(x_{2k+1}, x_{2k+2})) \leq \psi(M(x_{2k}, x_{2k+1})) - \phi(M(x_{2k}, x_{2k+1})) \leq \psi(M(x_{2k}, x_{2k+1})), \tag{4.3}$$

$$\psi(d(x_{2k+2}, x_{2k+3})) \leq \psi(M(x_{2k+1}, x_{2k+2})) - \phi(M(x_{2k+1}, x_{2k+2})) \leq \psi(M(x_{2k+1}, x_{2k+2})). \tag{4.4}$$

As done in Theorem 3.4 in [9], we have

$$M(x_{2k}, x_{2k+1}) \leq d(x_{2k}, x_{2k+1}) \text{ and } M(x_{2k+1}, x_{2k+2}) \leq d(x_{2k+1}, x_{2k+2}). \tag{4.5}$$

As $\psi$ is monotonically increasing by (4.3), (4.4) and (4.5) for $n \geq 0$ we have,

$$d(x_{n+1}, x_{n+2}) \leq M(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}), \tag{4.6}$$

which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. Now taking $n \to \infty$ from (4.6) we have, $\lim_{n\to\infty} M(x_n, x_{n+1}) = \ell$. Again by (4.3) and (4.4) for $n \geq 0$ we have,

$$\psi(d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1})).$$

Now taking $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Proceeding in the same way as in Theorem 3.4 in [9], we can show that $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of $X$, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Moreover, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$.

Now we prove that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$. We have $\{x_{2k+1}\} \subset T(x_{2k})$ and $\{x_{2k}\} \subset S(x_{2k+1})$, and

$$\begin{aligned}
p(u, S(u)) &\leq M(x_{2k}, u) \\
&= \max\{d(x_{2k}, u), p(T(x_{2k}), x_{2k}), p(S(u), u), \\
&\qquad \frac{1}{2}[p(u, T(x_{2k})) + p(x_{2k}, S(u))]\} \\
&\leq \max\{d(x_{2k}, u), d(x_{2k}, x_{2k+1}) + p(T(x_{2k}), x_{2k+1}), p(S(u), u), \\
&\qquad \frac{1}{2}[d(u, x_{2k+1}) + p(x_{2k+1}, T(x_{2k})) + d(x_{2k}, u) + p(u, S(u))]\} \\
&\leq \max\{d(x_{2k}, u), d(x_{2k}, x_{2k+1}), p(S(u), u), \\
&\qquad \frac{1}{2}[d(u, x_{2k+1}) + d(x_{2k}, u) + p(u, S(u))]\}.
\end{aligned}$$

Now taking $k \to \infty$ we have,

$$p(u, S(u)) \leq M(x_{2k}, u) \leq \max\{0, 0, p(S(u), u), \frac{1}{2}[0 + 0 + p(u, S(u))]\} = p(u, S(u)),$$

and so $\lim_{k\to\infty} M(x_{2k}, u) = p(u, S(u))$. Note that

$$p(u, S(u)) \leq d(u, x_{2k+1}) + p(x_{2k+1}, S(u)) \leq d(u, x_{2k+1}) + D(T(x_{2k}), S(u)).$$

Hence taking $k \to \infty$ we have,

$$p(u, S(u)) \leq 0 + \lim_{k\to\infty} D(T(x_{2k}), S(u)) = \lim_{k\to\infty} D(T(x_{2k}), S(u)).$$

As $\psi$ is non-decreasing, using the continuity of $\psi$ and $\phi$ and the given hypothesis (4.2) we have,

$$\begin{aligned}
\psi(p(u, S(u)) &\leq \lim_{k\to\infty} \psi(D(T(x_{2k}), S(u))) \\
&\leq \lim_{k\to\infty} \psi(M(x_{2k}, u)) - \lim_{k\to\infty} \phi(M(x_{2k}, u)) \\
&= \psi(p(u, S(u)) - \phi(p(u, S(u))) \\
\implies \phi(p(u, S(u))) &\leq 0.
\end{aligned}$$

Now using the fact $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$ we have, $\phi(p(u, S(u))) = 0 \implies p(u, S(u)) = 0 \implies \{u\} \subset S(u)$. Similarly we can show, $\{u\} \subset T(u)$.

□

**Remark 57.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the above theorem reduces to Theorem 4.1 of Bose and Roychowdhury (cf. [9]).*

**Theorem 58.** *Let $(X, d)$ be a complete metric linear space. Let $f : X \to X$ be a self-mapping, and $T : X \to W(X)$ be a fuzzy mapping such that for all $x, y \in X$*

$$\psi(D(T(x), T(y))) \leq \psi(d(f(x), f(y))) - \phi(d(f(x), f(y))),$$

*where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, $\phi$ is non-decreasing and $\psi$ is monotonically increasing (strictly). Suppose $\cup\{T(X)\}_\alpha \subseteq f(X)$ for $\alpha \in [0, 1]$, and $f(X)$ is complete. Then there exists $u \in X$ such that $u$ is a coincidence point of $f$ and $T$, that is $\{f(u)\} \subset T(u)$. Here $T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\}$.*

*Proof.* Let $x_0 \in X$ and $y_0 = f(x_0)$. Since $\cup\{T(X)\}_\alpha \subset f(X)$ for each $\alpha \in [0, 1]$, by Remark 22 for $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. Again by Remark 22 and Lemma 11, for $x_1 \in X$ there exist points $x_2, y_2 \in X$ such that $y_2 = f(x_2)$ and $\{y_2\} \subset T(x_1)$, and

$$d(y_1, y_2) \leq H(T(x_0)_1, T(x_1)_1) \leq D(T(x_0), T(x_1)).$$

By repeating this process for any $k \geq 1$, we can select points $x_k, y_k \in X$ such that $y_k = f(x_k)$ and $\{y_k\} \subset T(x_{k-1})$, and

$$d(y_k, y_{k+1}) \leq H(T(x_{k-1})_1, T(x_k)_1) \leq D(T(x_{k-1}), T(x_k)).$$

As $\psi$ is monotonically increasing, from the above inequalities for $k \geq 0$ we have,

$$\begin{aligned} \psi(d(y_{k+1}, y_{k+2})) &\leq \psi(D(T(x_k), T(x_{k+1}))) \\ &\leq \psi(d(f(x_k), f(x_{k+1}))) - \phi(d(f(x_k), f(x_{k+1}))) \\ &\leq \psi(d(y_k, y_{k+1})). \end{aligned} \tag{4.7}$$

As $\psi$ is monotonically increasing from the above inequality we have, $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$ for $n \geq 0$, which shows that $\{d(y_n, y_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. For any $n \geq 0$ by (4.7) we have, $\psi(d(y_{n+1}, y_{n+2})) \leq \psi(d(y_n, y_{n+1})) - \phi(d(y_n, y_{n+1}))$. Taking $n \to \infty$ and using the continuity of $\psi$ and $\phi$, we obtain

$$\psi(\ell) \leq \psi(\ell) - \phi(\ell) \implies \phi(\ell) \leq 0,$$

which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore $\ell = 0$, i.e., $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Now proceeding in the same way as in Theorem 3.4 in

[9], it can be shown that the sequence $\{y_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, $\{y_n\}$ converges to some point in $f(X)$. Let $y = \lim_{n\to\infty} y_n$ and $u \in X$ be such that $y = f(u)$. Note that,

$$d(y_{k+1}, T(u)_1) \leq H(T(x_k)_1, T(u)_1) = D_1(T(x_k), T(u)) \leq D(T(x_k), T(u)),$$

and hence as $\psi$ is monotonically increasing we have,

$$\psi(d(y_{k+1}, T(u)_1)) \leq \psi(D((T(x_k), T(u))) \leq \psi(d(f(x_k), f(u))) - \phi(d(f(x_k), f(u))).$$

Letting $n \to \infty$ and using the continuity of both $\psi$ and $\phi$ we have, $\psi(d(y, T(u)_1)) \leq \psi(d(y, y)) - \phi(d(y, y)) = \psi(0) - \phi(0) = 0 \implies \psi(d(y, T(u)_1)) \leq 0$. Now using the fact $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ we have, $\psi(d(y, T(u)_1)) = 0 \implies d(y, S(u)_1) = 0 \implies y = f(u) \in T(u)_1$, i.e., $\{f(u)\} \subset T(u)$, and hence is the theorem. □

**Remark 59.** *If we take $\psi(t) = t$ for all $t \in [0, \infty)$, then the above theorem reduces to Theorem 4.2 of Bose and Roychowdhury (cf. [9]).*

**Remark 60.** *Dutta and Choudhury in their paper ([12, Theorem 2.1]) assumed that $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone non-decreasing with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. In the proof they wrote that $\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) \implies d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that means by 'monotone non-decreasing' they meant that reverse implication holds. But this is not the way that monotone functions are defined. In the example they gave, the function $\phi$ is not one-to-one, which creates confusion (cf. [12, Example 2.2]). Their example clearly fits in our paper, as in our case $\psi$ is monotonically increasing (strictly), i.e., $\psi$ is one-to-one and $\phi$ is non-decreasing.*

**Example:** Let $X = [0, 1] \cup \{2, 3, 4, \cdots\}$ and

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1], x \neq y, \\ x + y, & \text{if at least one of } x \text{ or } y \notin [0, 1] \text{ and } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then $(X, d)$ is a complete metric space (cf. [11]). Let $\psi : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t > 1, \end{cases}$$

and let $\phi : [0, \infty) \to [0, \infty)$ be defined as

$$\phi(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } 0 \leq t \leq 1, \\ \frac{1}{2}, & \text{if } t > 1. \end{cases}$$

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - \frac{1}{2}x^2, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } x \in \{2, 3, 4, \cdots\}. \end{cases}$$

Then it is seen that $T$ has a unique fixed point which is 0 (cf. [12]).

## References


[1] Ya. I. Alber and S. Guerre-Delabriere, *New Results in Operator Theory and its Applications : The Israel M. Glazman Memorial Volume*, Birkhauser Verlag, 1997. MR1478463(98j:47126). Zbl 0870.00030.

[2] S. C. Arora and V. Sharma, *Fixed point theorems for fuzzy mappings*, Fuzzy Sets and Systems, **110** (2000), 127-130. MR1748116. Zbl 0988.54047.

[3] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43** (1972), 553-562. MR0341459(49 6210). Zbl 0239.54032.

[4] A. Azam and I. Beg, *Common fixed points of fuzzy maps*, Mathematical and computer modelling, **49** (2009), 1331-1336. MR2508351. Zbl 1165.54311.

[5] A. Azam and M. Shakeel, *Weakly contractive maps and common fixed points*, Matematicki Vesnik, Vol. **60**, No. 2 (2008), 101-106. MR2411009(2009e:54085). Zbl pre05619561.

[6] J. S. Bae, *Fixed point theorems for weakly contractive multivalued maps*, J. Math. Anal. Appl., **284** (2003), 690-697. MR1998661(2004f:54038). Zbl 1033.47038.

[7] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory and Applications, Volume 2006, Article ID 74503, 1-7. MR2235486(2007a:47058). Zbl 1133.54024.

[8] R. K. Bose and R. N. Mukherjee, *Approximating fixed points of some mappings,* Proc. Amer. Math. Soc., Vol. **82**, No. 4 (1981), 603-606. MR0614886(82m:47038). Zbl 0475.47043.

[9] R. K. Bose and M. K. Roychowdhury, *Fixed point theorems for multi-valued mappings and fuzzy mappings*, to appear, International Journal of Pure and Applied Mathematics.

[10] R. K. Bose and D. Sahani, *Fuzzy mappings and fixed point theorems*, Fuzzy Sets and Systems, **21** (1987), 53-58. MR0868355(87m:54018). Zbl 0609.54032.

[11] D. W. Boyd and J. S. Wong, *On nonlinear contractions*, Proc. of Amer. Math. Soc., vol. **2**, no. 2 (1969), 458-464. MR0239559(39 916). Zbl 0175.44903.

[12] P. N. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory and Applications, Volume **2008**, Article ID 406368, 1-8. MR2470177(2009m:54060). Zbl pre05534742.

[13] M. K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math. Anal. Appl., **207** (1997), 96-103. MR1433027(98c:47065). Zbl 0881.47036.

[14] S. Heilpern, *Fuzzy mappings and fixed point theorem*, J. Math. Anal. Appl., **83** (1981), 566-569. MR0641351(83a:54070). Zbl 0486.54006.

[15] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147-150. MR0336469(49 1243). Zbl 0286.47036.

[16] G. Jungck and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian Journal of Pure and Applied Mathematics, vol. **29**, no. 3 (1998), 227-238. MR1617919. Zbl 0904.54034.

[17] T. Kamran, *Multivalued $f$-weakly Picard mappings*, Nonlinear Analysis, 67 (2007), 2289-2296. MR2331879(2008d:47095). Zbl 1128.54024.

[18] B. S. Lee and S. J. Cho, *A fixed point theorem for contractive type fuzzy mappings*, Fuzzy Sets and Systems, **61** (1994), 309-312. MR1273252. Zbl 0831.54036.

[19] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506-510. MR0054846(14,988f). Zbl 0050.11603.

[20] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475-488. MR0254828. Zbl 0187.45002.

[21] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217-229. MR1790406(2001h:49019). Zbl 0964.49007.

[22] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl., **188** (1994), 436-440. MR1305460(95j:54037). Zbl 0830.54031.

[23] R. A. Rashwan and M. A. Ahmed, *Common fixed point theorems for fuzzy mappings*, Arch. Math. (Brno), **38** (2002), 219-226. MR1921593. Zbl 1068.54008.

[24] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis, vol **47**, no 4 (2001), 2683-2693. MR1972392(2004c:47125). Zbl 1042.47521.

[25] B. E. Rhoades, *A common fixed point theorem for a sequence of fuzzy mappings*, Int. J. Math. Sci., **8** (1995), 447-450. MR1331943 (96c:54072). Zbl 0840.47049.

[26] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings I*, Kodai Math. Sem. Rep., **22** (1970), 142-149. MR0267565(42 2467).

[27] Q. Zhang and Y. Song, *Fixed point theory for generalized $\phi$-weak contractions*, Applied Mathematics Letters, **22** (2009), 75-78. MR2484285. Zbl 1163.47304.

Ramendra Krishna Bose
Department of Mathematics,
The University of Texas-Pan American,
1201 West University Drive,
Edinburg, TX 78539-2999, USA.
e-mail: rkbose@utpa.edu, rkbose@hotmail.com

Mrinal Kanti Roychowdhury
Department of Mathematics,
The University of Texas-Pan American,
1201 West University Drive,
Edinburg, TX 78539-2999, USA.
e-mail: roychowdhurymk@utpa.edu