

THÉORIE SPECTRALE ET PROBLÈMES NON-LINEAIRES

(SPECTRAL THEORY AND NONLINEAR PROBLEMS)

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Abstract. We present a Lie algebra theoretical schema leading to integrable systems, based on the Kostant-Kirillov coadjoint action. Many problems on Kostant-Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac-Moody Lie algebras) yield large classes of extended Lax pairs. A general statement leading to such situations is given by the Adler-Kostant-Symes theorem and the van Moerbeke-Mumford linearization method provides an algebraic map from the complex invariant manifolds of these systems to the Jacobi variety (or some subabelian variety of it) of the spectral curve. The complex flows generated by the constants of the motion are straight line motions on these varieties. We study the isospectral deformation of periodic Jacobi matrices and general difference operators from an algebraic geometrical point of view and their relation with the Kac-Moody extension of some algebras. We will present in detail the Griffith's approach and his cohomological interpretation of linearization test for solving integrable systems without reference to Kac-Moody algebras. We will discuss several examples of integrable systems of relevance in mathematical physics.

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