Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **6** (2011), 117 – 126

COMMON FIXED POINTS BY A GENERALIZED ITERATION SCHEME WITH ERRORS

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Abstract. In this paper, we introduce a generalized iteration scheme with errors for convergence to common fixed points of two nonexpansive mappings. This scheme contains a wide variety of existing iteration schemes as its special cases. The main feature of this scheme is that its special cases can handle both strong convergence like Halpern-type and weak convergence like Ishikawa-type iteration schemes. Our main theorem will in particular generalize a recent result by Kim and Xu [9].

1 Introduction

Rich literature exists on the convergence of various iteration schemes for approximating fixed points of different types of mappings. Some well known iteration schemes include Halpern type, Mann type, Ishikawa type for one mapping and Das-Debata type for two mappings. Let us have a look at these schemes one by one. From now on, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty convex subset of a normed linear space E and $T : C \to C$ be a mapping. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $[0, 1]$. The initial guess x_1 is arbitrary but fixed.

Halpern's scheme [5] is as follows.

$$x_1 = x \in C,\ x_{n+1} = a_n x + (1 - a_n)Tx_n,\ n \in \mathbb{N}. \tag{1.1}$$

An important fact about this scheme is that even if we do not have the strong convergence of the sequence $\{Tx_n\}$, we get the strong convergence of a convex combination of Tx_n and x_1.

On the other hand, Mann [11] introduced the following so-called "Mann iteration scheme".

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n Tx_n + \beta_n x_n,\ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

2010 Mathematics Subject Classification: 47H10; 54H25.
Keywords: Generalized iteration scheme; Common fixed point; Nonexpansive mapping; Weak convergence; Strong convergence.

where $\alpha_n + \beta_n = 1$. It is to be noted that this scheme usually gives only weak convergence even in a Hilbert space. See, for example, [4].

Ishikawa [6] generalized this scheme as follows:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n T x_n + \beta'_n x_n, \\ x_{n+1} = \alpha_n T z_n + \beta_n x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha'_n + \beta'_n = 1 = \alpha_n + \beta_n$. This is known as Ishikawa iteration scheme.

These were the schemes containing only one mapping. However, Das and Debata [3] introduced the following scheme for two mappings T and S:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n S x_n + \beta'_n x_n, \\ x_{n+1} = \alpha_n T z_n + \beta_n x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\alpha'_n + \beta'_n = 1 = \alpha_n + \beta_n$. We shall call this scheme the Das-Debata scheme.

Keeping in view the importance and intensive study of these schemes, the author introduced the following scheme in his doctoral thesis which later appeared also in [7]:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n S x_n + \beta'_n x_n, \\ y_n = \alpha_n T z_n + \beta_n x_n, \\ x_{n+1} = a_n x + (1 - a_n) y_n, \ n \in \mathbb{N}. \end{cases} \tag{1.5}$$

where $\alpha'_n + \beta'_n = 1 = \alpha_n + \beta_n$. The main features of this scheme are that it can be used for strong convergence like Halpern scheme (1.1) and contains all the above four schemes which generally produce weak convergence. As a matter of fact it reduces to

- Halpern scheme (1.1) for $\alpha'_n = 0$ and $\beta_n = 0$ (or $S = I$).
- Mann scheme (1.2) for $\alpha'_n = 0$ and $a_n = 0$ (or $S = I$).
- Ishikawa scheme (1.3) for $a_n = 0$ and $S = T$.
- Das-Debata scheme (1.4) for $a_n = 0$.

A motivation for the present work is the iteration scheme recently given by Kim and Xu [9]:

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n T x_n + \beta_n x_n, \\ x_{n+1} = a_n w + (1-a_n)\, y_n,\ n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where x_1 and w are given arbitrary but fixed usually different elements of C and $\alpha_n + \beta_n = 1$. Our scheme (1.5) reduces to (1.6) when $S = I$ and $x_1 = w$.

The scheme (1.6) can be generalized to the following Ishikawa type scheme.

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n T x_n + \beta'_n x_n, \\ y_n = \alpha_n T z_n + \beta_n x_n, \\ x_{n+1} = a_n w + (1-a_n)\, y_n,\ \ n \in \mathbb{N}, \end{cases} \tag{1.7}$$

and then its Das-Debata type generalization will be

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n S x_n + \beta'_n x_n, \\ y_n = \alpha_n T z_n + \beta_n x_n, \\ x_{n+1} = a_n w + (1-a_n)\, y_n,\ n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\alpha'_n + \beta'_n = 1 = \alpha_n + \beta_n$. The above two schemes, to the best of our knowledge, have not yet been studied. However, we will give here a still more general scheme.

In 1998, Xu [16] introduced the following iteration scheme involving error terms:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T z_n + \beta_n x_n + \gamma_n u_n,\ \ n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $\alpha'_n + \beta'_n + \gamma'_n = 1 = \alpha_n + \beta_n + \gamma_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C. This contains both (1.2) and (1.3) as special cases.

Khan and Fukhar [8] went a step ahead to consider the two mappings case:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n S x_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n T z_n + \beta_n x_n + \gamma_n u_n,\ \ n \in \mathbb{N}, \end{cases} \tag{1.10}$$

where $\alpha'_n + \beta'_n + \gamma'_n = 1 = \alpha_n + \beta_n + \gamma_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C.

We now introduce the iteration scheme which we shall use to prove a strong convergence theorem to approximate the common fixed points of the two nonexpansive mappings S and T:

$$\begin{cases} x_1 = x \in C, \\ z_n = \alpha'_n S x_n + \beta'_n x_n + \gamma'_n v_n, \\ y_n = \alpha_n T z_n + \beta_n x_n + \gamma_n u_n, \\ x_{n+1} = a_n w + (1 - a_n)\, y_n,\ n \in \mathbb{N}, \end{cases} \tag{1.11}$$

where $\alpha'_n + \beta'_n + \gamma'_n = 1 = \alpha_n + \beta_n + \gamma_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in C.

This scheme surely contains all the ten schemes (1.1)-(1.10) with suitable choice of parameters. We can actually obtain

- (1.10) when $a_n = 0$.
- (1.9) when $a_n = 0$ and $S = T$.
- (1.8) when $\gamma'_n = 0 = \gamma_n$.
- (1.7) when $\gamma'_n = 0 = \gamma_n$ and $S = T$.
- (1.6) when $\gamma'_n = \gamma_n = 0$ and $S = I$.
- (1.5) when $\gamma'_n = \gamma_n = 0$ and $w = x$.
- (1.4) when $\gamma'_n = \gamma_n = a_n = 0$.
- (1.3) when $\gamma'_n = \gamma_n = a_n = 0$ and $S = T$.
- (1.2) when $\alpha'_n = \gamma'_n = \gamma_n = a_n = 0$.
- (1.1) when $\alpha'_n = \gamma'_n = \beta_n = \gamma_n = 0$ and $w = x$.

2 Preliminaries and Preparatory Lemmas

Let E be a real Banach space and E^* its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. The duality mapping J of E into E^* is defined as

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, x \in E$$

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for each x and y in U. For the following lemma, see for example [2].

Lemma 1. *If E is uniformly smooth then the duality mapping is single-valued and norm to norm uniformly continuous on each bounded subset of E.*

If A and B are nonempty subsets of a Banach space E such that A is a closed, convex subset of E and $B \subset A$, then a map $Q : A \to B$ is called a *retraction* from A onto B provided $Q(x) = x$ for all $x \in B$. A retraction $Q : A \to B$ is called *sunny* provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in A$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in A$. If E is a smooth Banach space, then $Q : A \to B$ is a *sunny nonexpansive retraction* if and only if $\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in A$ and $y \in B$. See [1] and [12].

In what follows, we denote the set of fixed points of a mapping T by $F(T)$. Reich [13] has proved the following.

Lemma 2. *Let E be a uniformly smooth Banach space and C a nonempty, closed, convex subset of E. Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \phi$. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ as $Qu = \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies:*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, \quad u \in C \ , \quad z \in F(T).$$

We shall also use the following lemmas.

Lemma 3. *[10] Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be nonnegative sequences satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} s_n = 0, \lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$, then $\lim_{n\to\infty} r_n = 0$.

Lemma 4. *[15] Let $\{r_n\}$and $\{k_n\}$ be nonnegative sequences satisfying*

$$r_{n+1} \leq r_n + k_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} k_n < \infty$, then $\lim_{n\to\infty} r_n = 0$.

Lemma 5. *[9] If E is a Banach space, then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y,\ j(x + y)\rangle$$

holds where $j(x + y) \in J(x + y)$.

3 Convergence by a generalized iteration scheme

We shall use our new iteration scheme (1.11) to prove the following strong convergence theorem to approximate the common fixed points of two nonexpansive mappings S and T. This will end up with a generalization of Theorem 1 of Kim and Xu [9] among others. See the remarks after the proof of the following theorem.

Theorem 6. *Let E be a uniformly smooth Banach space and C a closed, convex subset of E. Let S and T be nonexpansive mappings from C into itself such that $F = F(T) \cap F(S) \neq \phi$. Further, let $\{x_n\}$ be defined by (1.11) where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. The parameters $\alpha_n, \beta_n, \alpha'_n, \beta'_n, \gamma_n$ and γ'_n are in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$. Let a_n be in $(0,1)$ and the sequences $a_n, \alpha_n, \beta_n, \alpha'_n, \beta'_n$ all converge to zero. Also assume that*

$$\begin{cases} \sum_{n=1}^{\infty} a_n = \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \\ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} \left|\alpha'_{n+1} - \alpha'_n\right| < \infty, \\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} \left|\beta'_{n+1} - \beta'_n\right| < \infty, \\ \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty. \end{cases}$$

Then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. Let $q \in F$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C so without loss of generality we may assume that there exists $M > 0$ such that

$$\max(\sup_{k\geq 1} \|u_k - q\|, \sup_{k\geq 1} \|v_k - q\|, \|w - q\|, \|x_1 - q\|) \leq M.$$

We shall first prove that $\{x_n\}$ is a bounded sequence. To do this, we shall prove by mathematical induction that $\|x_n - q\| \leq M$ holds for all $n \in \mathbb{N}$. The assertion is clearly true for $n = 1$. Suppose that the assertion is true for $n = k$ for some positive integer k. That is, suppose $\|x_k - q\| \leq M$ for some positive integer k. We now prove that $\|x_{k+1} - q\| \leq M$. We have

$$\begin{aligned} \|z_k - q\| &= \left\|\alpha'_k S x_k + \beta'_k x_k + \gamma'_k v_k - q\right\| \\ &\leq \alpha'_k \|x_k - q\| + \beta'_k \|x_k - q\| + \gamma'_k \|v_k - q\| \\ &\leq M \end{aligned}$$

and

$$\begin{aligned} \|y_k - q\| &\leq \alpha_k \|z_k - q\| + \beta_k \|x_k - q\| + \gamma_k \|u_k - q\| \\ &\leq M \end{aligned}$$

so that

$$\begin{aligned} \|x_{k+1} - q\| &\leq a_k \|w - q\| + (1 - a_k) \|y_k - q\| \\ &\leq M. \end{aligned}$$

This shows that not only $\{x_n\}$ but also $\{y_n\}$ and $\{z_n\}$ are bounded.

Next consider

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|a_n w + (1 - a_n) y_n - a_{n-1} w - (1 - a_{n-1}) y_{n-1}\| \\ &= \left\| \begin{array}{c} (a_n - a_{n-1}) w + (1 - a_n)(y_n - y_{n-1}) \\ +(1 - a_n) y_{n-1} - (1 - a_{n-1}) y_{n-1} \end{array} \right\| \\ &\leq |a_n - a_{n-1}| (\|w\| + \|y_{n-1}\|) + (1 - a_n) \|y_n - y_{n-1}\|. \end{aligned}$$

Put $K = \sup[\|Sx_n\|, \|x_n\|, \|Tz_n\|, \|w\| + \|y_n\|, \|u_n\|, \|v_n\|]$. Then

$$\begin{aligned}\|y_n - y_{n-1}\| &= \left\| \begin{array}{c} \alpha_n T z_n + \beta_n x_n + \gamma_n u_n - \alpha_{n-1} T z_{n-1} \\ -\beta_{n-1} x_{n-1} - \gamma_{n-1} u_{n-1} \end{array} \right\| \\ &= \left\| \begin{array}{c} \alpha_n (Tz_n - Tz_{n-1}) + (\alpha_n - \alpha_{n-1}) Tz_{n-1} \\ +\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} \\ +\gamma_n u_n + \gamma_{n-1} u_{n-1} \end{array} \right\| \\ &\le \begin{array}{c} \alpha_n \|z_n - z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tz_{n-1}\| \\ + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\gamma_n| \|u_n\| + |\gamma_{n-1}| \|u_{n-1}\| \end{array} \\ &\le \alpha_n \|z_n - z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + \delta_n K\end{aligned}$$

where $\delta_n = |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}|$.

Similarly,

$$\begin{aligned}\|z_n - z_{n-1}\| &\le \left|\alpha'_n + \beta'_n\right| \|x_n - x_{n-1}\| + \delta'_n K \\ &= \left|1 - \gamma'_n\right| \|x_n - x_{n-1}\| + \delta'_n K \\ &\le \|x_n - x_{n-1}\| + \delta'_n K\end{aligned}$$

where $\delta'_n = \left|\alpha'_n - \alpha'_{n-1}\right| + \left|\beta'_n - \beta'_{n-1}\right| + \left|\gamma'_n\right| + \left|\gamma'_{n-1}\right|$. Thus

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \alpha_n[\|x_n - x_{n-1}\| + \delta'_n K] + \beta_n \|x_n - x_{n-1}\| + \delta_n K \\ &\le (\alpha_n + \beta_n) \|x_n - x_{n-1}\| + \left(\delta_n + \delta'_n\right) K \\ &\le \|x_n - x_{n-1}\| + \left(\delta_n + \delta'_n\right) K\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1} - x_n\| &\le |a_n - a_{n-1}| (\|w\| + \|y_n\|) + (1 - a_n) \left[\begin{array}{c} \|x_n - x_{n-1}\| \\ + (\delta_n + \delta'_n) K \end{array} \right] \\ &\le \|x_n - x_{n-1}\| + \left[|a_n - a_{n-1}| + \left(\delta_n + \delta'_n\right)\right] K\end{aligned}$$

Applying Lemma 4, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Next, having in mind Lemma 2, let $p = Q(w) = \lim_{t\to 0} p_t$ where p_t is the unique fixed point of the contraction $x \mapsto tu + (1-t)Tx$. See also Shioji and Takahashi [14]. Now we shall prove that $\limsup_{n\to\infty} \langle w - p, J(x_n - p)\rangle \le 0$. Since $p_t = tw + (1-t)Tp_t$, therefore $p_t - x_n = t(w - x_n) + (1-t)(Tp_t - x_n)$ and so by Lemma 5,

$$\begin{aligned}\|p_t - x_n\|^2 &\le (1-t)^2 \|Tp_t - x_n\|^2 + 2t \langle w - x_n, J(p_t - x_n)\rangle \\ &\le (1 - 2t + t^2) \|p_t - x_n\|^2 \\ &\quad + (1-t)^2 \|Tx_n - x_n\| [\|Tx_n - x_n\| + 2\|p_t - x_n\|] \\ &\quad + 2t \|p_t - x_n\|^2 + 2t \langle w - p_t, J(p_t - x_n)\rangle .\end{aligned}$$

Thus

$$\begin{aligned} 0 &\leq t^2 \|p_t - x_n\|^2 + (1-t)^2 \|Tx_n - x_n\| [\|Tx_n - x_n\| + 2\|p_t - x_n\|] \\ &\quad -2t \langle p_t - w, J(p_t - x_n)\rangle . \end{aligned}$$

But

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| + \|y_n - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + a_n \|x - y_n\| + \alpha_n \|Tz_n\| \\ &\quad +\beta_n \|x_n - Tx_n\| + \gamma_n u_n \\ &\to 0, \end{aligned}$$

therefore

$$\limsup_{n\to\infty} \langle p_t - w, J(p_t - x_n)\rangle \leq \frac{t}{2} L \tag{3.1}$$

where $L \geq \|p_t - x_n\|^2 > 0$ for all $t \in (0,1)$ and $n \in \mathbb{N}$.

Since $\{z_t - x_n\}$ is a bounded set, J is norm to norm uniformly continuous on each bounded subset of E by Lemma 1 , and $p_t \to p$ as $t \to 0$, therefore by letting $t \to 0$ in (3.1), we get

$$\limsup_{n\to\infty} \langle w - p, J(x_n - p)\rangle \leq 0.$$

Our last task is to prove that $x_n \to p$. Again we shall use Lemma 5.

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-a_n)(y_n - p) + a_n (w-p)\|^2 \\ &\leq (1-a_n)^2 \|y_n - p\|^2 + 2a_n \langle w - p, J(x_{n+1} - p)\rangle \\ &\leq (1-a_n) \|y_n - p\|^2 + 2a_n \langle w - p, J(x_{n+1} - p)\rangle \end{aligned}$$

But a simple calculation leads to the fact that

$$\begin{aligned} \|y_n - p\| &\leq \|x_n - p\| + \gamma_n \|v_n - p\| + \gamma'_n \|u_n - p\| \\ &\leq \|x_n - p\| + (\gamma_n + \gamma'_n) \|x_1 - p\| \end{aligned}$$

so that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + (\gamma_n + \gamma'_n)^2 \|x_1 - p\|^2 \\ &\quad +2(\gamma_n + \gamma'_n) \|x_n - p\| \|x_1 - p\| \\ &\leq \|x_n - p\|^2 + (\gamma_n + \gamma'_n)^2 \|x_1 - p\|^2 \\ &\quad +2(\gamma_n + \gamma'_n) \max(\|w - p\|, \|x_1 - p\|) \|x_1 - p\| \\ &\leq \|x_n - p\|^2 + \lambda_n D^2 \end{aligned}$$

where we choose $\max(\|w-p\| \|x_1-p\|, \|x_1-p\|^2) \leq D^2$ and $\lambda_n = (\gamma_n+\gamma_n')^2 + 2(\gamma_n+\gamma_n')$. Hence

$$\|x_{n+1}-p\|^2 \leq (1-a_n)\|x_n-p\|^2 + 2a_n \langle w-p, J(x_{n+1}-p)\rangle + \lambda_n D^2.$$

Since $\sum_{n=1}^{\infty} \lambda_n D^2 < \infty$ and $\limsup_{n\to\infty} \langle w-p, J(x_n-p)\rangle \leq 0$, therefore by Lemma 3, we obtain $x_n \to p$ as required. □

This theorem generalizes Theorem 1 of Kim and Xu [9] as follows:

Corollary 7. *Let C be a closed and convex subset of a uniformly smooth Banach space E. Let T be nonexpansive mapping from C into itself such that $F(T) \neq \phi$. Further, let $\{x_n\}$ defined by (1.6) satisfy*

$$\begin{cases} 0 < a_n < 1, \ a_n \to 0, \ \sum_{n=1}^{\infty} a_n = \infty, \ \sum_{n=1}^{\infty} |a_{n+1}-a_n| < \infty, \\ 0 < b_n < 1, \ b_n \to 0, \sum_{n=1}^{\infty} b_n = \infty, \sum_{n=1}^{\infty} |b_{n+1}-b_n| < \infty, \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 8.

1. *It is also worth noting that unlike Kim and Xu [9], we do not need the condition $\sum_{n=1}^{\infty} b_n = \infty$.*
2. *Results proved under similar conditions using schemes (1.1), (1.5) − (1.8) are also covered by our above theorem.*
3. *The schemes (1.2) − (1.4), (1.9) and (1.10) requiring $a_n = 0$ are already covered by Theorem 2 proved by Khan and Fukhar [8].*

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