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APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING NONLINEAR CAUCHY PROBLEM

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Abstract. In this paper, by means of the homotopy analysis method (HAM), the solutions of some nonlinear Cauchy problem of parabolic-hyperbolic type are exactly obtained in the form of convergent Taylor series. The HAM contains the auxiliary parameter $\hbar$ that provides a convenient way of controlling the convergent region of series solutions. This analytical method is employed to solve linear examples to obtain the exact solutions. The results reveal that the proposed method is very effective and simple.

1 Introduction

Nonlinear partial differential equation are known to describe a wide variety of phenomena not only in physics, where applications extend over magneto fluid-dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, just to name a few, but also in biology and chemistry, and several other fields. Very few problems in physics, or indeed in any branch of natural science, can be solved directly consequently, one usually first study an ideal model, which is chosen to reflect as much as possible of the natural of the real scientific system, as an appropriate approximation and then handle other effects via some effective perturbative and/or non perturbative techniques.

Perturbation theory (PT) is widely used to investigate physical systems that can be exactly solved but contain small perturbations parameters [7, 11] when applying PT to such a system, expansion around the perturbation parameter is involved, and approximants are expressed as power series of these parameters. On the other hand, non perturbative techniques have been established to explore physical problems which do not a small physical parameter to be used as the perturbation parameter. The non-parameter expansion method [6], the optimized perturbation theory (OPT)

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[14], the variational perturbation theory (VPT) [5, 24] and the linear δ- expansion method (LDE) [1, 2, 15, 25, 26], are typical nonperturbative methods, and have been developed as a powerful tools in quantum field theory and in varies physical contexts during the past three decades. These methods do not involve perturbation series in powers of physical parameters, and the convergence of approximate is controlled by some artificial parameters which do not exists in the original problems. The artificial parameters are fixed at the end of calculations according to some criterion such as the principle of minimal sensitivity (PMS), which requires the approximants have the least dependence on these parameters over perturbation techniques. One of the most popular non perturbative techniques is homotopy analysis method (HAM), first proposed by Shi-Jun Liao [16]-[18] a powerful analytical method for solving linear and nonlinear differential and integral equations. The HAM was successfully applied to solve many nonlinear problems such as nonlinear Riccati differential equation with fractional order [8], nonlinear Vakhnenko equation [28], the Glauert-jet problem [29], fractional KdV-Burgers-Kuramoto Equation [10], a generalized Hirota-Satsuma coupled KdV equation [19], nonlinear heat transfer [20], to projectile motion with the quadratic law [9], to boundary layer flow of nanofluid past a stretching sheet [12], to the Poisson-Boltzmann equation of semiconductor devices [4], solitary solution of discrete MKdV equation [27], to the system of Fractional differential equations [13], to the Oldroyd 6- constant fluid with magnetic field [21], MHD-flow of an Oldroyd 8-constant fluid [22], to the nonlinear flows with slip boundary condition [23] and so on. In this paper we consider Cauchy problem for the nonlinear parabolic-hyperbolic equation of the following type

$$\left(\frac{\partial}{\partial t}-\triangle\right)\left(\frac{\partial^2}{\partial t^2}-\triangle\right)u=F(u), \tag{1.1}$$

with the initial conditions

$$\frac{\partial^k}{\partial t^k}(X,0)=\phi_k(X),\ \ X=(x_1,x_2,...x_i),k=0,1,2,$$

where the nonlinear term is represented by $F(u)$ and $\triangle$ is the Laplace operator in R^n Here we solve these problems by homotopy analysis method and shows that homotopy perturbation method is the special case of homotopy analysis method at $\hbar=-1$, obtained by A.Roozi et.al [3].

2 Homotopy analysis method

In order to show the basic idea of HAM, consider the following differential equation

$$N\left[u(x,t)\right]=0, \tag{2.1}$$

where N is a nonlinear operator, x and t denote the independent variables and u is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation.

$$(1-q)\,L\left[\phi(x,t;q)-u_0(x,t)\right]=q\,\hbar\,H(x,t)\,N\left[\phi(x,t;q)\right], \tag{2.2}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of and $H(x,t)$ denotes a nonzero auxiliary function. It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (2.2) becomes

$$\phi(x,t;0)=u_0(x,t),\quad \phi(x,t;1)=u(x,t),$$

respectively. Thus as q increases from 0 to 1, the solution varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q , one has

$$\phi(x,t;q)=u_0(x,t)+\sum_{\mathrm{m}=1}^{\infty}u_m(x,t)q^m, \tag{2.3}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}|q=0.$$

The convergence of the series (2.3) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q=1$, one has

$$u(x,t)=u_0(x,t)+\sum_{\mathrm{m}=1}^{\infty}u_m(x,t),$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [16]-[18]. Define the vectors

$$\vec{u_n}=\left(u_0(x,t),u_1(x,t),\ldots,u_n(x,t)\right).$$

Differentiate the zeroth-order deformation equation (2.1) m-times with respect to q and then dividing them by $m!$ and finally setting $q=0$, we get the following m th-order deformation equation:

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar\,\Re_m(\vec{u_{m-1}}), \tag{2.4}$$

where

$$\Re_m(\vec{u_{m-1}})=\frac{1}{m!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}|q=0,$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1.\end{cases}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (2.4) with linear boundary conditions that comes form the original problem, which can be solved by the symbolic computation software such as Mathematica or Maple. For the convergence of the above method we refer the reader to Liao. If equation (2.1) admits unique solution, then this method will produce the unique solution. If equation (2.1) does not posses a unique solution, the HAM will give a solution among many other possible solutions.

3 Applications

In this section the applicability of HAM shall be demonstrated by the following examples:

3.1 Example 1

Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=-\left(\frac{1}{3}\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{1}{6}\frac{\partial^2 u}{\partial t^2}\right)^3-16.u, \tag{3.1}$$

with the initial conditions

$$u(x,0)=-x^4,\ \frac{\partial u}{\partial t}(x,0)=0,$$

and

$$\frac{\partial^2 u}{\partial t^2}(x,0)=0.$$

To solve equation (3.1) by means of the homotopy analysis method let us consider the following linear operator:

$$L\left[\phi(x,t;q)\right]=\frac{\partial^3\phi(x,t;q)}{\partial t^3},$$

with the property that

$$L\left[c_1+c_2t+c_3\frac{t^2}{2}\right],$$

which implies that

$$L^{-1}(.)=\int_0^t\int_0^t\int_0^t(.)\ dt\ dt\ dt,$$

we now define the nonlinear operator as

$$\begin{aligned} N\left[\phi(x,t;q)\right] &= \frac{\partial^3\phi(x,t;q)}{\partial t^3}-\frac{\partial^3\phi(x,t;q)}{\partial t\partial x^2}-\frac{\partial^4\phi(x,t;q)}{\partial t^2\partial x^2}+\frac{\partial^4\phi(x,t;q)}{\partial x^4} \\ &\quad +\frac{1}{216}\left(\frac{\partial^2\phi(x,t;q)}{\partial t^2}\right)^3-\frac{1}{9}\left(\frac{\partial^2\phi(x,t;q)}{\partial x^2}\right)^2+16\phi(x,t;q). \end{aligned}$$

Using the above definition, we construct the zeroth- order deformation equation by

$$(1-q)\,L\left[\phi(x,t;q)-u_0(x,t)\right]=q\,\hbar\,H(x,t)\;N\left[\phi(x,t;q)\right], \tag{3.2}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of and $H(x,t)$ denotes a nonzero auxiliary function. It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (3.2) becomes

$$\phi(x,t;0)=u_0(x,t),\quad \phi(x,t;1)=u(x,t),$$

then we obtain the m th-order deformation equation:

$$\begin{aligned} L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right] &= \hbar\Re_m(\vec{u_{m-1}}) \\ &\Longrightarrow \\ u_m(x,t) &= \chi_m u_{m-1}(x,t)+\hbar L^{-1}\left(H(x,t)\Re_m(\vec{u_{m-1}})\right), \end{aligned}$$

where

$$\begin{aligned} \Re_m(\vec{u_{m-1}}) &= \frac{\partial^3 u_{m-1}}{\partial t^3}-\frac{\partial^3 u_{m-1}}{\partial t\partial x^2}-\frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2}+\frac{\partial^4 u_{m-1}}{\partial x^4}+\frac{1}{9}\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^2 u_{m-1-i}}{\partial x^2} \\ &\quad -\frac{1}{216}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}\frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_j}{\partial t^2}\frac{\partial^2 u_{m-j-i-1}}{\partial t^2}+16u_{m-1}, \end{aligned}$$

solve the above equation under the initial conditions

$$u_m(x,0)=0 \textit{ and } \frac{\partial u_m}{\partial t}(x,0)=0.$$

For simplicity let us take $u_0(x,t)=-x^4$ and

$$\begin{aligned} &u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar\int_0^t\int_0^t\int_0^t\left(\tfrac{\partial^3 u_{m-1}}{\partial t^3}-\tfrac{\partial^3 u_{m-1}}{\partial\xi_1\partial x^2}-\tfrac{\partial^4 u_{m-1}}{\partial\xi_1^2\partial x^2}+\tfrac{\partial^4 u_{m-1}}{\partial x^4}\right)d\xi_1\;d\xi_2\;dt \\ &+\hbar\int_0^t\int_0^t\int_0^t\tfrac{1}{9}\textstyle\sum_{i=0}^{m-1}\tfrac{\partial^2 u_i}{\partial x^2}\tfrac{\partial^2 u_{m-1-i}}{\partial x^2}d\xi_1\;d\xi_2\;dt \\ &-\hbar\int_0^t\int_0^t\int_0^t\left(\tfrac{1}{216}\textstyle\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}\tfrac{\partial^2 u_i}{\partial t^2}\tfrac{\partial^2 u_j}{\partial t^2}\tfrac{\partial^2 u_{m-j-i-1}}{\partial t^2}-16u_{m-1}\right)d\xi_1\;d\xi_2\;dt, \end{aligned}$$

the other components are given by

$$\begin{aligned}
u_1(x,t) &= 0\\
u_2(x,t) &= 0\\
u_3(x,t) &= -\hbar\ 4\ t^3\\
u_4(x,t) &= u_5(x,t) = ... = 0.
\end{aligned}$$

Therefore the approximate solution is given by at $\hbar = -1$

$$u(x,t) = -x^4 + 4\ t^3$$

which is an exact solution and is same as obtained by and is same as obtained by A.Roozi et.al [3].

3.2 Example 2

Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2\ u^2,$$

with the initial conditions

$$u(x,0) = e^x,\ \frac{\partial u}{\partial t}(x,0) = e^x \text{ and } \frac{\partial^2 u}{\partial t^2}(x,0) = e^x,$$

we now define the nonlinear operator as

$$\begin{aligned}
N\left[\phi(x,t;q)\right] &= \frac{\partial^3\phi(x,t;q)}{\partial t^3} - \frac{\partial^3\phi(x,t;q)}{\partial t\partial x^2} - \frac{\partial^4\phi(x,t;q)}{\partial t^2\partial x^2} + \frac{\partial^4\phi(x,t;q)}{\partial x^4}\\
&+ \left(\frac{\partial^2\phi(x,t;q)}{\partial t^2}\right)^2 - \left(\frac{\partial^2\phi(x,t;q)}{\partial x^2}\right)^2 + 2\phi^2(x,t;q),
\end{aligned}$$

then we obtain the m th-order deformation equation:

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar\ \Re_m(\vec{u_{m-1}}),$$

where

$$\begin{aligned}
\Re_m(\vec{u_{m-1}}) &= \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t\partial x^2} - \frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} + \sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^2 u_{m-1-i}}{\partial x^2}\\
&- \sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_{m-1-i}}{\partial t^2} + 2\sum_{i=0}^{m-1} u_i u_{m-1-i},
\end{aligned}$$

then the linear differential equation can be written as

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t \int_0^t \int_0^t \left(\frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial \xi_1 \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial \xi_1^2 \partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} \right) d\xi_1 \ d\xi_2 \ dt$$
$$+\hbar \int_0^t \int_0^t \int_0^t \left(\sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial x^2} \frac{\partial^2 u_{m-1-i}}{\partial x^2} - \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial t^2} \frac{\partial^2 u_{m-1-i}}{\partial t^2} + 2 \sum_{i=0}^{m-1} u_i \ u_{m-1-i} \right) d\xi_1 \ d\xi_2 \ dt,$$

we start with the initial approximation

$$u_0(x,t) = \left(1 + t + \frac{t^2}{2}\right) e^x,$$

solve the above equation under the initial conditions

$$u_m(x,0) = 0 \ and \ \frac{\partial u_m}{\partial t}(x,0) = 0,$$

we get

$$\begin{aligned} u_1(x,t) &= -\frac{\hbar \ t^3 \ e^x}{6} \\ u_2(x,t) &= -\frac{\hbar^2 \ t^4 \ e^x}{24} \\ u_3(x,t) &= -\frac{\hbar^3 \ t^5 \ e^x}{120}, \end{aligned}$$

and so on therefore the approximate solution is given by at $\hbar = -1$

$$u(x,t) = e^{x+t},$$

which is an exact solution and is same as obtained by and is same as obtained by A.Roozi et.al [3].

3.3 Example 3

Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = u\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x},$$

with the initial conditions

$$u(x,0) = cos(x), \ \frac{\partial u}{\partial t}(x,0) = -sin(x) \ and \ \frac{\partial^2 u}{\partial t^2}(x,0) = -cos(x),$$

we now define the nonlinear operator as

$$\begin{aligned} N\left[\phi(x,t;q)\right] &= \frac{\partial^3 \phi(x,t;q)}{\partial t^3} - \frac{\partial^3 \phi(x,t;q)}{\partial t \partial x^2} - \frac{\partial^4 \phi(x,t;q)}{\partial t^2 \partial x^2} + \frac{\partial^4 \phi(x,t;q)}{\partial x^4} \\ &\quad -\phi(x,t;q)\frac{\partial \phi(x,t;q)}{\partial t} - \frac{\partial^2 \phi(x,t;q)}{\partial t^2}\frac{\partial \phi(x,t;q)}{\partial x}. \end{aligned}$$

Then we obtain the m th-order deformation equation:

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar\, \Re_m(\vec{u_{m-1}}), \tag{3.3}$$

where

$$\begin{aligned}\Re_m(\vec{u_{m-1}}) &= \frac{\partial^3 u_{m-1}}{\partial t^3}-\frac{\partial^3 u_{m-1}}{\partial t\partial x^2}-\frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2}+\frac{\partial^4 u_{m-1}}{\partial x^4}+\sum_{i=0}^{m-1} u_i\frac{\partial u_{m-1-i}}{\partial t}\\ &\quad -\sum_{i=0}^{m-1}\frac{\partial^2 u_{m-1}}{\partial t^2}\frac{\partial u_{m-1-i}}{\partial x},\end{aligned}$$

we start with the initial approximation

$$u_0(x,t)=cos(x)-tsin(x)-\left(\frac{t^2}{2}\right)sin(x),$$

and

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar\int_0^t\int_0^t\int_0^t\left(\frac{\partial^3 u_{m-1}}{\partial t^3}-\frac{\partial^3 u_{m-1}}{\partial\xi_1\partial x^2}-\frac{\partial^4 u_{m-1}}{\partial\xi_1^2\partial x^2}+\frac{\partial^4 u_{m-1}}{\partial x^4}\right)d\xi_1\, d\xi_2\, dt$$
$$+\hbar\int_0^t\int_0^t\int_0^t\left(-u_{m-1}\frac{\partial u_{m-1}}{\partial t}-\frac{\partial^2 u_{m-1}}{\partial t^2}\frac{\partial u_{m-1}}{\partial x}\right)d\xi_1\, d\xi_2\, dt,$$

solve the above equation under the initial conditions

$$u_m(x,0)=0 \ and\ \frac{\partial u_m}{\partial t}(x,0)=0,$$

we get

$$\begin{aligned}u_1(x,t) &= -\frac{\hbar\, t^3\, sin(x)}{6}-\frac{\hbar\, t^4\, cos(x)}{24}\\ u_2(x,t) &= -\frac{\hbar^2\, t^5\, sin(x)}{120}-\frac{\hbar^2\, t^6\, cos(x)}{720}\\ u_3(x,t) &= -\frac{\hbar^3\, t^7\, sin(x)}{5040}-\frac{\hbar^2\, t^8\, cos(x)}{40320}\end{aligned}$$

and so on therefore the approximate solution is given by at $\hbar=-1$

$$u(x,t)=cos(x+t), \tag{3.4}$$

which is an exact solution and is same as obtained by and is same as obtained by A.Roozi et.al [3].

3.4 Example 4

Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)u=\frac{\partial u}{\partial t}-2u, \tag{3.5}$$

with the initial conditions

$$u(x_1,x_2,0)=sinh(x_1+x_2),\ \frac{\partial u}{\partial t}(x_1,x_2,0)=2sinh(x_1+x_2),$$

and

$$\frac{\partial^2 u}{\partial t^2}(x_1,x_2,0)=4sinh(x_1+x_2),$$

we now define the nonlinear operator as

$$N\left[\phi(x,t;q)\right]=\frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t^3}-\frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t\partial x_1^2}-\frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t\partial x_2^2}-\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial t^2}$$
$$+\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^4}+\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial x_2^2}-\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial t^2\partial x_2^2}+\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_2^4}$$
$$-\frac{\partial\phi(x_1,x_2,t;q)}{\partial t}+2\phi(x_1,x_2,t;q).$$

Then we obtain the m th-order deformation equation:

$$L\left[u_m(x_1,x_2,t)-\chi_m u_{m-1}(x_1,x_2,t)\right]=\hbar\ \Re_m(\vec{u_{m-1}}), \tag{3.6}$$

where

$$\begin{aligned}\Re_m(\vec{u_{m-1}}) \quad &= \quad \frac{\partial^3 u_{m-1}}{\partial t^3}-\frac{\partial^3 u_{m-1}}{\partial t\partial x_1^2}-\frac{\partial^3 u_{m-1}}{\partial t\partial x_2^2}-\frac{\partial^4 u_{m-1}}{\partial x_1^2\partial t^2}+\frac{\partial^4 u_{m-1}}{\partial x_1^4}+\frac{\partial^4 u_{m-1}}{\partial x_1^2\partial x_2^2}\\ &\quad -\frac{\partial^4 u_{m-1}}{\partial t^2\partial x_2^2}+\frac{\partial^4 u_{m-1}}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 u_{m-1}}{\partial x_2^4}-\frac{\partial u_{m-1}}{\partial t}+2u_{m-1}\end{aligned}$$

Taking the initial approximation

$$u_0(x_1,x_2,t)=\left(1+2t+2t^2\right)sinh(x_1+x_2)$$

and other components are

$$\begin{aligned}u_1(x_1,x_2,t) &= -\frac{4\hbar\ t^3\ sinh(x_1+x_2)}{3}\\ u_2(x_1,x_2,t) &= \frac{2\hbar^2\ t^4\ sinh(x_1+x_2)}{3}\\ u_3(x_1,x_2,t) &= -\frac{4\hbar^3\ t^5\ sinh(x_1+x_2)}{15}\end{aligned}$$

and so on therefore the approximate solution is given by at $\hbar=-1$

$$u(x_1,x_2,t)=sinh(x_1+x_2)\ e^{2t} \tag{3.7}$$

which is an exact solution and is same as obtained by and is same as obtained by A.Roozi et.al [3].

4 Conclusions

The homotopy analysis method is used for calculating numerical solution for nonlinear Cauchy's problems. Different from all other analytical methods, it provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values of auxiliary parameter $\hbar$, auxiliary function $H(t)$ and auxiliary linear operator L. Also we showed that homotopy perturbation method is the special case of homotopy analysis method. There are some important points to make here. First, we have great freedom to choose the auxiliary parameter $\hbar$, auxiliary function $H(t)$ and auxiliary linear operator L and the initial guesses. Second the HAM was shown to be simple, yet powerful analytic-numeric scheme for solving various nonlinear problems. Numerical computation has been done by Maple 13 software package.
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