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ON THE BI UNIQUE RANGE SETS FOR DERIVATIVES OF MEROMORPHIC FUNCTIONS

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Abstract. In the paper we introduce the notion of Bi Unique Range Sets for derivatives of meromorphic functions and with the aid of the same we improve all previous results regarding derivatives of set sharing.

1 Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

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In connection with the famous "Gross Question" {see [8]} in the uniqueness literature, in 2003, the following question was asked by Lin and Yi [17].

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In course of time the research in this direction has somehow been shifted to find explicitly a set S with minimum cardinalities such that any two meromorphic functions f and g that share the set S together with the value ∞ must be equal {cf.[1]-[7], [11], [15]-[17], [23]-[24]}. In some of the papers sometimes the researchers have resorted to the variations over different deficiency conditions. But probably the actual answer of *Question A* for two finite sets in $\mathbb{C}$ has yet not been settled.

To the knowledge of the authors perhaps the following two results were first studied the uniqueness of the derivatives of meromorphic functions in the direction of *Question A*.

Theorem A. *[7] Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7), k be two positive integers. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1) = E_{g^{(k)}}(S_1)$ and $E_f(S_2) = E_g(S_2)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem B. *[23] Let S_i $i = 1, 2$ be given as in* Theorem A *and k be a positive integer. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2$ then $f^{(k)} \equiv g^{(k)}$.*

In 2001, the advent of the new notion of gradation of sharing of values and sets in [13, 14] further expedite the research in the direction of *Question A*. This notion is a scaling between CM and IM and measures how close a shared value is to being shared IM or to being shared CM. In the following we recall the definition.

Definition 1.1. *[13, 14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. *[13] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Recently some papers such as [19], [20] subtly use this or other sharing notions such as pseudo value sharing to obtain new results. But in these papers mainly the uniqueness of a meromorphic function corresponding to non-linear differential polynomials sharing some values are taken under considerations. In the current paper we shall confine our attention solely on the set sharing problem and to this end we proceed as follows.

Using weighted sharing of sets Banerjee and Bhattacharjee [4] improved *Theorems A* and *B* as follows.

Theorem C. *[4] Let S_i $i = 1, 2$ be given as in* Theorem A *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 2) = E_{g(k)}(S_1, 2)$, $E_f(S_2, 1) = E_g(S_2, 1)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem D. *[4] Let S_i $i = 1, 2$ be given as in* Theorem A *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 3) = E_{g(k)}(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$ then $f^{(k)} \equiv g^{(k)}$.*

In the next year Banerjee and Bhattacharjee [5] further improved *Theorems C* and *D* in the following manner.

Theorem E. *[5] Let S_i $i = 1, 2$ be given as in* Theorem A *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 2) = E_{g(k)}(S_1, 2)$, $E_f(S_2, 0) = E_g(S_2, 0)$ then $f^{(k)} \equiv g^{(k)}$.*

We observe that in the above mentioned results the cardinality of the main range set namely S_1 could not be diminished. Only the sharing conditions over the sets have been relaxed. So it will be interesting to investigate under which supposition the cardinality of the set S_1 can be further reduced in the above mentioned results so that it will also be commensurate with the possible answer of *Question A*. The purpose of the paper is to investigate this fact.

Gross and Yang [9] made a vital contribution by introducing the new idea of unique range set for meromorphic function (URSM in brief). In continuation of the concept of unique range sets it will be quite natural to investigate the existence of a pair of finite range sets in $\mathbb{C}$ shared by two meromorphic functions which leads them to-wards their uniqueness. This thought in fact paves the way for the following definition which is also pertinent with the possible answer of *Question A*.

Definition 1.3. *A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi unique range sets for the derivatives of meromorphic (entire) functions with weights m, k if for any two non-constant meromorphic (entire) functions f and g, $E_{f^{(k)}}(S_1, m) = E_{g^{(k)}}(S_1, m)$, $Ef^{(k)}(S_2, p) = Eg^{(k)}(S_2, p)$ implies $f^{(k)} \equiv g^{(k)}$. We write S_i's $i = 1$, 2 as BURSDMm, p (BURSDEm, p) in short. As usual if both $m = p = \infty$, we say S_i's $i = 1$, 2 as BURSDM (BURSDE).*

In the paper we shall show that Bi-Unique Range Sets for the Derivatives renders an useful tool in order to reduce the cardinality of the main range set in all the aforesaid theorems. Following two theorems are the main result of the paper.

Theorem 1.1. *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0, -a\frac{n-1}{n}\}$ where $n(\geq 5)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Then S_i's $i = 1,\ 2$ are* BURSDM3, 0.

Theorem 1.2. *Let S_i, $i = 1,\ 2$ be given as in* Theorem 1.1 *where $n(\geq 5)$ be an integer. Then S_i's $i = 1,\ 2$ are* BURSDM2, 1.

Though for the standard definitions and notations of the value distribution theory we refer to [10], we now explain some notations which are used in the paper.

Definition 1.4. *[12] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ ($\overline{N}(r, a; f \mid\geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.5. *[14] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.*

Definition 1.6. *[13, 14] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and in particular if f and g share (a, p) then $\overline{N}_*(r, a; f, g) \leq \overline{N}(r, a; f \mid\geq p + 1) = \overline{N}(r, a; g \mid\geq p + 1)$.*

Definition 1.7. *Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid\ g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.*

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F = \frac{\left(f^{(k)}\right)^{n-1}\left(f^{(k)} + a\right)}{-b}, \qquad G = \frac{\left(g^{(k)}\right)^{n-1}\left(g^{(k)} + a\right)}{-b}, \tag{2.1}$$

where $n(\geq 2)$ and k are two positive integers. Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{2.2}$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{2.3}$$

Lemma 2.1. *([14], Lemma 1) Let F, G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1 and S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,0) = E_{g^{(k)}}(S_1,0)$, $E_{f^{(k)}}(S_2,p) = E_{g^{(k)}}(S_2,p)$, where $0 \leq p < \infty$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \quad &\leq \quad \overline{N}(r,0;f^{(k)} \mid\geq p+1) + \overline{N}\left(r,-a\frac{n-1}{n};f^{(k)} \mid\geq p+1\right) + \overline{N}_*(r,1;F,G) \\ &\quad +\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_0(r,0;f^{(k+1)}) + \overline{N}_0(r,0;g^{(k+1)}), \end{aligned}$$

where $\overline{N}_0(r,0;f^{(k+1)})$ is the reduced counting function of those zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)}\left(f^{(k)} - a\frac{n-1}{n}\right)(F-1)$ and $\overline{N}_0(r,0;g^{(k+1)})$ is similarly defined.

Proof. We note that

$$F' = \frac{(f^{(k)})^{n-2}(nf^{(k)} + a(n-1))f^{(k+1)}}{-b},$$

$$G' = \frac{(f^{(k)})^{n-2}(ng^{(k)} + a(n-1))g^{(k+1)}}{-b}$$

and $F'' = \frac{(f^{(k)})^{n-2}(nf^{(k)}+a(n-1))f^{(k+2)}+(f^{(k)})^{n-3}(n(n-1)f^{(k)}+a(n-1)(n-2))(f^{(k+1)})^2}{-b}$,
$G'' = \frac{(g^{(k)})^{n-2}(ng^{(k)}+a(n-1))g^{(k+2)}+(g^{(k)})^{n-3}(n(n-1)g^{(k)}+a(n-1)(n-2))(g^{(k+1)})^2}{-b}$.
So

$$\begin{aligned} H = \quad & \frac{(n-1)(nf^{(k)} + a(n-2))f^{(k+1)}}{f^{(k)}(nf^{(k)} + a(n-1))} - \frac{(n-1)(ng^{(k)} + a(n-2))g^{(k+1)}}{g^{(k)}(ng^{(k+1)} + a(n-1))} \\ & + \frac{f^{(k+2)}}{f^{(k+1)}} - \frac{g^{(k+2)}}{g^{(k+1)}} - \left(\frac{2F'}{F-1} - \frac{2G'}{G-1}\right). \end{aligned}$$

Since $E_{f^{(k)}}(S_2,0) = E_{g^{(k)}}(S_2,0)$ it follows that if z_0 is a 0-point of $f^{(k)}$ ($g^{(k)}$) then either $g^{(k)}(z_0) = 0$ ($f^{(k)}(z_0) = 0$) or $g^{(k)}(z_0) = -a\frac{n-1}{n}$ ($f^{(k)}(z_0) = -a\frac{n-1}{n}$). Clearly F and G share $(1,0)$. Since H has only simple poles, the lemma can easily be proved by simple calculation. □

Lemma 2.3. *[5] Let f and g be two meromorphic functions sharing $(1,m)$, where $1 \leq m < \infty$. Then*

$$\begin{aligned} &\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ \leq\ & \frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\end{aligned}$$

Lemma 2.4. *[18] Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n \neq 0$ and $b_m \neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let S_1 and S_2 be defined as in* Theorem 1.1 *with $n \geq 3$ and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,m)=E_{g^{(k)}}(S_1,m)$, $E_{f^{(k)}}(S_2,p)=E_{g^{(k)}}(S_2,p)$, $0 \leq p < \infty$ and $\Phi \not\equiv 0$ then*

$$\begin{aligned} &(2p+1)\left\{\overline{N}\left(r,0;f^{(k)}\mid\geq p+1\right)+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\mid\geq p+1\right)\right\}\\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}).\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi=\frac{(f^{(k)})^{n-2}\left(nf^{(k)}+a(n-1)\right)f^{(k+1)}}{-b(F-1)}-\frac{(g^{(k)})^{n-2}\left(ng^{(k)}+a(n-1)\right)g^{(k+1)}}{-b(G-1)}.$$

Let z_0 be a zero or a $-a\frac{n-1}{n}$- point of $f^{(k)}$ with multiplicity r. Since $E_{f^{(k)}}(S_1,p)=E_{g^{(k)}}(S_1,p)$ then that would be a zero of Φ of multiplicity $min\,\{(n-2)r+r-1,\ r+r-1\}$ i.e., of multiplicity $min\,\{(n-1)r-1,\ 2r-1\}$ if $r \leq p$ and a zero of multiplicity at least $min\{(n-2)(p+1)+p,\ p+1+p\}$ i.e., a zero of multiplicity at least $min\{(n-1)p+(n-2),\ 2p+1\}$ if $r > p$. So using *Lemma 2.4* by a simple calculation we can write

$$\begin{aligned} &\min\{(n-1)p+(n-2),(2p+1)\}\left\{\overline{N}(r,0;f^{(k)}\mid\geq p+1)+\overline{N}(r,-a\tfrac{n-1}{n};f^{(k)}\mid\geq p+1)\right\}\\ \leq\ & N(r,0;\Phi)\\ \leq\ & T(r,\Phi)\\ \leq\ & N(r,\infty;\Phi)+S(r,F)+S(r,G)\\ \leq\ & \overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 2.6. *Let S_1, S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,m)=E_{g^{(k)}}(S_1,m)$, $E_{f^{(k)}}(S_2,p)=E_{g^{(k)}}(S_2,p)$, where $0\le p<\infty$, $2\le m<\infty$ and $H\not\equiv 0$, then*

$$\begin{aligned}
&(n+1)\left\{T(r,f^{(k)})+T(r,g^{(k)}\right\}\\
\le\ & 2\left\{\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)\right\}+\overline{N}\left(r,0;f^{(k)}\mid\ge p+1\right)\\
&+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\mid\ge p+1\right)+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\
&+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
&(n+1)\{T(r,f^{(k)})+T(r,g^{(k)})\}\\
\le\ & \overline{N}(r,1;F)+\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)+\overline{N}(r,\infty;f)\\
&+\overline{N}(r,1;G)+\overline{N}(r,0;g^{(k)})+\overline{N}\left(r,-a\frac{n}{n-1};g^{(k)}\right)+\overline{N}(r,\infty;g)\\
&-N_0(r,0;f^{(k+1)})-N_0(r,0;g^{(k+1)})+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}\tag{2.4}$$

Using *Lemmas 2.1*, *2.2*, *2.3* and *2.4* we note that

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\overline{N}(r,0;f^{(k)}\mid\ge p+1)\\
&+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\mid\ge p+1\right)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f^{(k+1)})+\overline{N}_0(r,0;f^{(k+1)})\\
&+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}\tag{2.5}$$

Using (2.5) in (2.4) and noting that

$$\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)=\overline{N}(r,0;g^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};g^{(k)}\right)$$

the lemma follows. □

Lemma 2.7. *Let $f^{(k)}$, $g^{(k)}$ be two non-constant meromorphic functions such that $E_{f^{(k)}}(\{0,-a\frac{n-1}{n}\},0)=E_{g^{(k)}}(\{0,-a\frac{n-1}{n}\},0)$. Then, $\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)\equiv\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)$ implies $f^{(k)}\equiv g^{(k)}$, where $n\ (\geq 2)$ is an integer, k is a positive integer and a is a nonzero finite constant.*

Proof. Let z_0 be a zero of $f^{(k)}$ $(g^{(k)})$. Then z_0 must be either a 0-point or a $-a\frac{n-1}{n}$ point of $g^{(k)}$ $(f^{(k)})$. But from the given condition if z_0 is not a zero of $g^{(k)}$, then it must be a zero of $g^{(k)}+a$, which is impossible. So we conclude that here $f^{(k)}$ and $g^{(k)}$ share $(0,\infty)$ and f, g share (∞,∞). We also note that $\Theta\left(\infty;f^{(k)}\right)+\Theta\left(\infty;g^{(k)}\right)\geq 2-\frac{2}{k+1}=\frac{2k}{k+1}>0$. Now the lemma can be proved in the line of proof of *Lemma 3* [16]. □

Lemma 2.8. *Let f, g be two non-constant meromorphic functions such that $E_f(\{0,-a\frac{n-1}{n}\},0)=E_g(\{0,-a\frac{n-1}{n}\},0)$ and suppose $n\ (\geq 3)$ be an integer. Then*

$$(f^{(k)})^{n-1}(f^{(k)}+a)\;(g^{(k)})^{n-1}(g^{(k)}+a)\not\equiv b^2,$$

where a, b are finite nonzero constants.

Proof. If possible, let us suppose

$$(f^{(k)})^{n-1}(f^{(k)}+a)\;(g^{(k)})^{n-1}(g^{(k)}+a)\equiv b^2. \tag{2.6}$$

Let z_0 be a zero of $f^{(k)}$ $(g^{(k)})$. Then z_0 must be either a 0-point or a $-a\frac{n-1}{n}$ point of $g^{(k)}$ $(f^{(k)})$, which is impossible from (2.6). It follows that $f^{(k)}$ $(g^{(k)})$ has no zero.

Next let z_0 be a zero of $f^{(k)}+a$ with multiplicity p. Then z_0 is a pole of $g^{(k)}$ with multiplicity q such that $p=(n-1)q+q=nq\geq n$.

Since the poles of f can be the zeros of $g^{(k)}+a$ only, we get

$$\overline{N}(r,\infty;f)\leq\overline{N}(r,-a;g^{(k)})\leq\frac{1}{n}T(r,g^{(k)}).$$

By the second fundamental theorem we get

$$\begin{aligned}T(r,f^{(k)}) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f^{(k)})+\overline{N}(r,-a;f^{(k)})+S(r,f^{(k)})\\ &\leq \frac{1}{n}\,N(r,-a;f^{(k)})+\frac{1}{n}T(r,g^{(k)})+S(r,f^{(k)})\\ &\leq \frac{1}{n}T(r,f^{(k)})+\frac{1}{n}T(r,g^{(k)})+S(r,f^{(k)}).\end{aligned}$$

i.e.,

$$(1-\frac{1}{n})\;T(r,f^{(k)})\leq\frac{1}{n}\;T(r,g^{(k)})+S(r,f^{(k)}). \tag{2.7}$$

Similarly

$$(1-\frac{1}{n})\,T(r,g^{(k)})\leq\frac{1}{n}\,T(r,f^{(k)})+S(r,g^{(k)}) \tag{2.8}$$

Adding (2.7) and (2.8) we get

$$(1-\frac{2}{n})\,\{T(r,f^{(k)})+T(r,g^{(k)})\}\leq S(r,f^{(k)})+S(r,g^{(k)}),$$

a contradiction for $n\geq 3$. This proves the lemma. □

Lemma 2.9. *Let F, G be given by (2.1) and they share $(1,m)$. Also let $\omega_1,\omega_2\ldots\omega_n$ are the members of the set $S_1=\{z: z^n+az^{n-1}+b=0\}$, where a, b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no repeated root and $n\ (\geq 3)$ is an integer. Then*

$$\overline{N}_*(r,1;F,G)\leq\frac{1}{m}\left[\overline{N}(r,0;f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};f^{(k)})\right]+S(r,f^{(k)}).$$

Proof. First we note that since S_1 has distinct elements, $-a\frac{n-1}{n}$ can not be a member of S_2. So

$$\begin{aligned}
&\overline{N}_*(r,1;F,G)\\
\leq\ &\overline{N}(r,1;F\mid\geq m+1)\\
\leq\ &\frac{1}{m}\,\big(N(r,1;F)-\overline{N}(r,1;F)\big)\\
\leq\ &\frac{1}{m}\left[\sum_{j=1}^{n}\Big(N(r,\omega_j;f^{(k)})-\overline{N}(r,\omega_j;f^{(k)})\Big)\right]\\
\leq\ &\frac{1}{m}\left[N\left(r,0;f^{(k+1)}\mid f^{(k)}\neq 0,\ -a\frac{n-1}{n}\right)\right]\\
\leq\ &\frac{1}{m}\left[\overline{N}\left(r,\infty;\frac{f^{(k)}(f^{(k)}+a\frac{n-1}{n})}{f^{(k+1)}}\right)\right]\\
\leq\ &\frac{1}{m}\left[N\left(r,\infty;\frac{f^{(k+1)}}{f^{(k)}(f^{(k)}+a\frac{n-1}{n})}\right)\right]+S(r,f^{(k)})\\
\leq\ &\frac{1}{m}\left[\overline{N}(r,0;f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};f^{(k)})\right]+S(r,f^{(k)}).
\end{aligned}$$

□

Lemma 2.10. *[21] If $H\equiv 0$, then F, G share $(1,\infty)$.*

3 Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using *Lemma 2.6* for $m=3$ and $p=0$, *Lemma 2.5* for $p=0$, *Lemma 2.4* and *Lemma 2.9* for $m=3$ we obtain

$$\begin{aligned}
& (n+1)\left\{T(r,f^{(k)})+T(r,g^{(k)})\right\} \qquad (3.1)\\
\leq\ & 3\left\{\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)\right\}\\
& +2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& -\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\ & \frac{5}{k+1}\left\{T(r,f^{(k)}+T(r,g^{(k)})\right\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& +\frac{1}{4}\left\{\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)\right\}\\
& +\frac{1}{4}\left\{\overline{N}(r,0;g^{(k)})+\overline{N}\left(r,-a\frac{n-1}{n};g^{(k)}\right)\right\}+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\ & \left(\frac{n}{2}+\frac{1}{2}+\frac{5}{k+1}\right)\left\{T(r,f^{(k)})+T(r,g^{(k)})\right\}+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

(3.1) gives a contradiction for $n \geq 5$.
Subcase 1.2 Let $H \equiv 0$. Then

$$F \equiv \frac{AG+B}{CG+D}, \qquad (3.2)$$

where A, B, C, D are constants such that $AD-BC \neq 0$. Also,

$$T(r,F)=T(r,G)+O(1).$$

i.e.,

$$nT(r,f^{(k)})=nT(r,g^{(k)})+O(1). \qquad (3.3)$$

In view of *lemma 2.10* it follows that F and G share $(1,\infty)$. We now consider the following cases.
Subcase 1.2.1. Let $AC \neq 0$. From (3.7) we get

$$\overline{N}(r,\infty;G)=\overline{N}\left(r,\frac{A}{C};F\right). \qquad (3.4)$$

Since F and G share $(1,\infty)$, it follows that $\frac{A}{C}\neq 1$. Suppose $F-\frac{A}{C}$ have no repeated zeros. So in view of *Lemma 2.5*, (3.3) and (3.4), by the second fundamental theorem we get

$$\begin{aligned}
& (n+1)T(r,f^{(k)}) \\
\leq\; & \overline{N}(r,0;f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,\infty;f)+\overline{N}\left(r,\frac{A}{C};F\right)+S(r,f) \\
\leq\; & \frac{2}{k+1}\left\{T(r,f^{(k)})+T(r,g^{(k)})\right\}+S(r,f^{(k)}) \\
\leq\; & \frac{4}{k+1}T(r,f^{(k)})+S(r,f^{(k)}),
\end{aligned}$$

which gives a contradiction for $n\geq 5$.

Next suppose $F-\frac{A}{C}$ has one repeated zero at $-a\frac{n-1}{n}$. In view of *Lemma 2.5*, (3.3) and (3.4), by the second fundamental theorem we get

$$\begin{aligned}
& (n-1)T(r,f^{(k)}) \\
\leq\; & \overline{N}(r,0;f^{(k)})+\overline{N}(r,\infty;f)+\overline{N}\left(r,\frac{A}{C};F\right)+S(r,f) \\
\leq\; & \left(1+\frac{2}{k+1}\right)T(r,f^{(k)})+S(r,f^{(k)}),
\end{aligned}$$

which is a contradiction for $n\geq 5$.

Subcase 1.2.2. Let $A\neq 0$ and $C=0$. Then $F=\alpha_0 G+\beta_0$, where $\alpha_0=\frac{A}{D}$ and $\beta_0=\frac{B}{D}$.

We note that 1 can not be an exceptional value Picard (e.v.P.) of F (G). For, if it happens, then $f^{(k)}$ $(g^{(k)})$ omits $n\geq 5$ values which is a contradiction.

So F and G have some 1-points. Then $\alpha_0+\beta_0=1$ and so

$$F\equiv \alpha_0 G+1-\alpha_0. \tag{3.5}$$

Suppose $\alpha_0\neq 1$. If $F-(1-\alpha_0)$ have no repeated zero, then using *Lemma 2.5*, (3.3) and the second fundamental theorem we get

$$\begin{aligned}
& (n+1)T(r,f^{(k)}) \\
\leq\; & \overline{N}(r,0;f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,\infty;f)+\overline{N}(r,1-\alpha_0;F)+S(r,f^{(k)}) \\
\leq\; & \frac{1}{k+1}\left\{2T(r,f^{(k)})+T(r,g^{(k)})\right\}+2T(r,g^{(k)})+S(r,f^{(k)}) \\
\leq\; & \left(2+\frac{3}{k+1}\right)T(r,f^{(k)})+S(r,f^{(k)}),
\end{aligned}$$

which implies a contradiction in view of *Lemma 2.4* and $n \geq 5$. If $F-(1-\alpha_0)$ have a repeated zero, in view of *Lemma 2.5*, (3.3) and (3.4), by the second fundamental theorem we get

$$\begin{aligned} & (n-1)T(r,f^{(k)}) \\ \leq\ & \overline{N}(r,0;f^{(k)})+\overline{N}(r,\infty;f)+\overline{N}(r,1-\alpha_0;F)+S(r,f^{(k)}) \\ \leq\ & \left(3+\frac{1}{k+1}\right)T(r,f^{(k)})+S(r,f^{(k)}), \end{aligned}$$

which implies a contradiction in view of *Lemma 2.4* and $n \geq 5$. So $\alpha_0 = 1$ and hence $F \equiv G$, which contradicts $\Phi \not\equiv 0$.

Subcase 1.2.3. Let $A=0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma_0 G+\delta_0}$, where $\gamma_0 = \frac{C}{B}$ and $\delta_0 = \frac{D}{B}$.

Clearly 1 can not be an e.v.P. of F and so of G.

Since F and G have some 1-points we have $\gamma_0+\delta_0 = 1$ and so

$$F \equiv \frac{1}{\gamma_0 G+1-\gamma_0}. \tag{3.6}$$

Suppose $\gamma_0 \neq 1$. Now noting that $\overline{N}(r,0;G) = \overline{N}(r,\frac{1}{1-\gamma_0};F)$, proceeding in the same way as done in *Subcase 1.2.2.* we can deal the two cases where $F-\frac{1}{1-\gamma_0}$ has distinct zeros or one repeated zero and in both the cases we get contradictions. Here we omit the detail. So we must have $\gamma_0 = 1$ then $FG \equiv 1$, which is impossible by *Lemma 2.8*. This completes the proof of the theorem.

Case 2. Suppose that $\Phi \equiv 0$. On integration we get $(F-1) \equiv A(G-1)$ for some non zero constant A. So in view of *Lemma 2.4* we have

$$T(r,f^{(k)}) = T(r,g^{(k)})+O(1). \tag{3.7}$$

Since by the given condition of the theorem $E_f(S_2,0) = E_g(S_2,0)$ we consider the following cases.

Subcase 2.1. Let us first assume $f^{(k)}$ and $g^{(k)}$ share $(0,0)$ and $(-a\frac{n-1}{n},0)$. If one of 0 or $-a\frac{n-1}{n}$ is an e.v.P. of both $f^{(k)}$ and $g^{(k)}$, then we get $A=1$ and we have $F \equiv G$, which in view of *Lemma 2.7* implies $f^{(k)} \equiv g^{(k)}$. If both 0 and $-a\frac{n-1}{n}$ are e.v.P. of $f^{(k)}$ as well as of $g^{(k)}$ then noting that here $F \equiv AG+(1-A)$, suppose $A \neq 1$. Using *Lemma 2.4*, (3.7) and the second fundamental theorem we get

$$\begin{aligned} & nT(r,f^{(k)}) \\ \leq\ & \overline{N}(r,0;F)+\overline{N}(r,1-A;F)+\overline{N}(r,\infty;F)+S(r,F) \\ \leq\ & \overline{N}(r,0;f^{(k)})+\overline{N}(r,-a;f^{(k)})+\overline{N}(r,0;G)+\overline{N}(r,\infty;f)+S(r,f^{(k)}) \\ \leq\ & (1+\frac{1}{k+1})T(r,f^{(k)})+T(r,g^{(k)})+S(r,f^{(k)}) \\ \leq\ & (2+\frac{1}{k+1})T(r,f^{(k)})+S(r,f^{(k)}), \end{aligned}$$

which implies a contradiction since $n \geq 5$.
Subcase 2.2. Here we take $A \neq 1$, since otherwise by *Lemma 2.7* we get $f^{(k)} \equiv g^{(k)}$. Next suppose that there is at least one point z_0 such that $f^{(k)}(z_0) = 0$ and $g^{(k)}(z_0) = -a\frac{n-1}{n}$. At the point z_0, we have $F(z_0) = 0$ and $G(z_0) = \beta$ (say). So $A = \frac{1}{1-\beta}$. Clearly $\beta \neq 0$. Putting this values we obtain from above

$$F \equiv \frac{1}{1-\beta}G + \frac{\beta}{\beta - 1}.$$

Since $\beta \neq 0$, noting that $\overline{N}(r, \frac{\beta}{\beta-1}; F) = \overline{N}(r, 0; G)$, we can again get a contradiction as above when $n \geq 5$.

If 0 is an e.v.P. of $f^{(k)}$ and so $-a\frac{n-1}{n}$ is an e.v.P. of $g^{(k)}$, then noting that here

$$AG \equiv F + A - 1, \tag{3.8}$$

we consider the following subcases.
Subcase 2.2.1. Suppose $F + A - 1$ has n distinct zeros, ζ_i, $i = 1, 2, \ldots, n$. Then we get from (3.8)

$$A(g^{(k)})^{n-1}(g^{(k)} + a) \equiv (f^{(k)} - \zeta_1)(f^{(k)} - \zeta_2) \ldots (f^{(k)} - \zeta_n).$$

Since none of the ζ_i's, $i = 1, 2, \ldots, n$ coincides with $-a\frac{n-1}{n}$, we get a contradiction from (3.8) for those points z_1, where $f^{(k)}(z_1) = -a\frac{n-1}{n}$ and $g^{(k)}(z_1) = 0$.
Subcase 2.2.2. Suppose $F + A - 1$ has $n - 2$ distinct zeros, ξ_i, $i = 1, 2, \ldots, n-2$ and a double zero at $-a\frac{n-1}{n}$. Then (3.8) takes the form

$$A(g^{(k)})^{n-1}(g^{(k)} + a) \equiv \left(f^{(k)} + \frac{a(n-1)}{n}\right)^2 (f^{(k)} - \xi_1)(f^{(k)} - \xi_2) \ldots (f^{(k)} - \xi_{n-2}).$$

So using *Lemma 2.4* in (3.8), from the second fundamental theorem we get

$$\begin{aligned} & (n-2)T(r, f^{(k)}) \\ \leq\ & \sum_{i=1}^{n-2} \overline{N}(r, \xi_i; f^{(k)}) + \overline{N}(r, 0; f^{(k)}) + \overline{N}(r, -\frac{a(n-1)}{n}; f^{(k)}) + S(r, f^{(k)}) \\ \leq\ & \overline{N}(r, 0; g^{(k)}) + \overline{N}(r, -a; g^{(k)}) + S(r, f^{(k)}) \\ \leq\ & 2T(r, f^{(k)}) + S(r, f^{(k)}), \end{aligned}$$

a contradiction for $n \geq 5$.

If 0 and $-a\frac{n-1}{n}$ both are e.v.P. of $f^{(k)}$ and of $g^{(k)}$, then we consider the following subcases.
Subcase 2.2.3. Suppose as in *Subcase 2.2.1.*, $F + A - 1$ has n distinct zeros,

ζ_i, $i = 1, 2, \ldots, n$. Then using *Lemma 2.4* in (3.8), from the second fundamental theorem we get

$$\begin{aligned} & nT(r, f^{(k)}) \\ \leq & \sum_{i=1}^{n} \overline{N}(r, \zeta_i; f^{(k)}) + \overline{N}(r, 0; f^{(k)}) + \overline{N}(r, -\frac{a(n-1)}{n}; f^{(k)}) + S(r, f^{(k)}) \\ \leq & \overline{N}(r, -a; g^{(k)}) + S(r, f^{(k)}) \\ \leq & T(r, f^{(k)}) + S(r, f^{(k)}), \end{aligned}$$

a contradiction for $n \geq 5$.
Subcase 2.2.4. Next suppose as in *Subcase 2.2.2.*, $F + A - 1$ has $n-2$ distinct zeros, ξ_i, $i = 1, 2, \ldots, n-2$ and a double zero at $-a\frac{n-1}{n}$. This subcase can be dealt with the same method as resorted in *Subcase 2.2.2.* so we omit the detail. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1, 2)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using *Lemma 2.6* for $m = 2$ and $p = 1$, *Lemma 2.5* for $p = 0$ and $p = 1$, *Lemma 2.4* and *Lemma 2.9* for $m = 2$ we obtain

$$\begin{aligned} & (n+1)\left\{T(r, f^{(k)}) + T(r, g^{(k)})\right\} \qquad (3.9) \\ \leq & 2\left\{\overline{N}(r, 0; f^{(k)}) + \overline{N}\left(r, -a\frac{n-1}{n}; f^{(k)}\right)\right\} + \overline{N}(r, 0; f^{(k)} \mid\geq 2) \\ & + \overline{N}\left(r, -a\frac{n-1}{n}; f^{(k)} \mid\geq 2\right) + 2\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} \\ & + \frac{1}{2}\left[N(r, 1; F) + N(r, 1; G)\right] - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r, f^{(k)}) + S(r, g^{(k)}) \\ \leq & \frac{13}{3(k+1)}\left\{T(r, f^{(k)} + T(r, g^{(k)})\right\} + \frac{1}{2}\left[N(r, 1; F) + N(r, 1; G)\right] \\ & + \frac{11}{24}\left\{\overline{N}(r, 0; f^{(k)}) + \overline{N}\left(r, -a\frac{n-1}{n}; f^{(k)}\right)\right\} \\ & + \frac{11}{24}\left\{\overline{N}(r, 0; g^{(k)}) + \overline{N}\left(r, -a\frac{n-1}{n}; g^{(k)}\right)\right\} + S(r, f^{(k)}) + S(r, g^{(k)}) \\ \leq & \left(\frac{n}{2} + \frac{11}{12} + \frac{13}{3(k+1)}\right)\left\{T(r, f^{(k)}) + T(r, g^{(k)})\right\} + S(r, f^{(k)}) + S(r, g^{(k)}). \end{aligned}$$

(3.9) gives a contradiction for $n \geq 5$.
We now omit the rest of the proof since the same is similar to that of *Theorem 1.1*. □

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