Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 11 (2016), 11 – 19

A NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

J. K. Prajapat

Abstract. In this work, we introduce and investigate an interesting subclass $\mathcal{X}_t(\gamma)$ of analytic and close-to-convex functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{X}_t(\gamma)$, we drive several properties including coefficient estimates, distortion theorems, covering theorems and radius of convexity.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{K}$ be the usual classes of function which are also univalent, starlike and convex, respectively. We also denote by $\mathcal{S}^*(\gamma)$ the class of starlike function of order γ, where $0 \leq \gamma < 1$.

Definition 1. *If f and g are two analytic functions in $\mathbb{U}$, then f is said to be* subordinate to g, *and write $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in* U, *then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ in $\mathbb{U}$.*

Gao and Zhou [2] introduce the following subclass $\mathcal{K}_s$ of analytic functions, which indeed a subclass of close-to-convex functions.

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s$, if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0, \quad z \in \mathbb{U}. \tag{1.2}$$

2010 Mathematics Subject Classification: 30C45
Keywords: Analytic Functions, Univalent Function, Starlike Functions, Close-to-Convex Function

Recently, Knwalczyk and Leś-Bomba [3] extended Definition 2, by introducing the following subclass of analytic functions.

Definition 3. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s(\gamma)$, $0 \le \gamma < 1$, if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, \quad z \in \mathbb{U}. \tag{1.3}$$

Motivated by above define function classes, we introduce the following subclass of analytic functions.

Definition 4. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{X}_t(\gamma)$ $(|t| \le 1, t \ne 0, 0 \le \gamma < 1)$, if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) > \gamma, \quad z \in \mathbb{U}. \tag{1.4}$$

In terms of subordination (1.4) can be written as

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+(1-2\gamma)z}{1-z}, \quad z \in \mathbb{U}. \tag{1.5}$$

We see that

$$\mathcal{X}_{-1}(\gamma) = \mathcal{K}_s(\gamma) \quad \text{and} \quad \mathcal{X}_{-1}(0) = \mathcal{K}_s.$$

We now present an example of functions belonging to this class.

Example 5. *The function*

$$f_1(z) = \frac{t+1-2\gamma}{(t-1)^2}\, ln\frac{1-z}{1-tz} - \frac{2(1-2\gamma)z}{(1-t)(1-z)}, \quad z \in \mathbb{U}. \tag{1.6}$$

belongs to the class $\mathcal{X}_t(\gamma)$. Indeed, f_1 is analytic in $\mathbb{U}$ and $f_1(0) = 0$. Moreover,

$$f_1'(z) = \frac{1+(1-2\gamma)z}{(1-tz)(1-z)^2}, \quad z \in \mathbb{U}.$$

If we put

$$g_1(z) = \frac{z}{1-z}, \quad z \in \mathbb{U}, \tag{1.7}$$

then $g_1 \in \mathcal{S}^*(\frac{1}{2})$ and

$$\Re\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) = \Re\left(\frac{1+(1-2\gamma)z}{1-z}\right) > \gamma, \quad z \in \mathbb{U}.$$

This means that $f_1 \in \mathcal{X}_t(\gamma)$ and is generated by g_1.

Gao and Zhou [2] and Knwalczyk and Leś-Bomba [3], have obtained properties for the function classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$, respectively. Moreover, some other interesting subclasses of $\mathcal{A}$ related to the function classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$ were considered in [4, 5]. In the present paper, we obtained coefficient estimates, distortion theorems, covering theorems and radius of convexity of the function class defined by (1.4).

2 Section

We first prove the following result.

Theorem 6. *Let* $g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *and given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{U}, \tag{2.1}$$

If we put

$$F(z) = \frac{g(z)\, g(tz)}{tz} = z + \sum_{n=2}^{\infty} c_n z^n, \quad z \in \mathbb{U}, \tag{2.2}$$

then

$$c_n = b_n + b_2 b_{n-1} t + b_3 b_{n-2} t^2 + \ldots + b_{n-1} b_2 t^{n-2} + b_n t^{n-1}, \tag{2.3}$$

and $F(z) \in \mathcal{S}^*$.

Proof. Result (2.2) can be found easily. Also $|tz| \leq |z| < 1$, then from the definitions of starlike function, we have

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \frac{1}{2} \quad \text{and} \quad \Re\left(\frac{tz\, g'(tz)}{g(tz)}\right) > \frac{1}{2}.$$

Therefore

$$\begin{aligned}\Re\left(\frac{zF'(z)}{F(z)}\right) &= \Re\left(\frac{zg'(z)}{g(z)}\right) + \Re\left(\frac{tz\, g'(tz)}{g(tz)}\right) - 1 \\ &> \frac{1}{2} + \frac{1}{2} - 1 = 0.\end{aligned}$$

This proves the Theorem 2.1. □

Remark 7. *From the definition of the class* $\mathcal{X}_t(\gamma)$ *and Theorem 6, we have*

$$\Re\left(\frac{zf'(z)}{F(z)}\right) > \gamma \qquad (0 \leq \gamma < 1;\ z \in \mathbb{U}),$$

thus

$$\mathcal{X}_t(\gamma) \subset \mathcal{K}_s(\gamma) \subset \mathcal{K}_s \subset \mathcal{S}.$$

Theorem 8. *Let* $0 \le \gamma < 1$. *If the function* $f \in \mathcal{X}_t(\gamma)$, *then*

$$|a_n| \ \le \ \frac{1}{n}\left\{|c_n| + 2(1-\gamma)\left(1 + \sum_{k=2}^{n-1}|c_k|\right)\right\}, \quad k \in \mathbb{N}. \tag{2.4}$$

Proof. By setting

$$\frac{1}{1-\gamma}\left(\frac{zf'(z)}{F(z)} - \gamma\right) = h(z), \quad z \in \mathbb{U}, \tag{2.5}$$

or equivalently

$$zf'(z) = \left[1 + (1-\gamma)(h(z)-1)\right]F(z), \tag{2.6}$$

we get

$$h(z) = 1 + d_1 z + d_2 z^2 + \cdots, \quad z \in \mathbb{U}, \tag{2.7}$$

where $\Re(h(z)) > 0$. Now using (2.2) and (2.7) in (2.6), we get

$$\begin{aligned}
&2a_2 = (1-\gamma)d_1 + c_2\\
&3a_3 = (1-\gamma)(d_2 + d_1c_2) + c_3\\
&4a_4 = (1-\gamma)(d_3 + d_2c_2 + d_1c_3) + c_4\\
&\vdots\\
&na_n = (1-\gamma)(d_{n-1} + d_{n-2}c_2 + \cdots + d_1c_{n-1}) + c_n.
\end{aligned}$$

Since $\Re(h(z)) > 0$, then $|d_n| \le 2, \ n \in \mathbb{N}$. Using this property, we get

$$\begin{aligned}
2|a_2| &\le |c_2| + 2(1-\gamma),\\
3|a_3| &\le |c_3| + 2(1-\gamma)\left\{1 + |c_2|\right\}
\end{aligned}$$

and

$$4|a_4| \le |c_4| + 2(1-\gamma)\left\{1 + |c_2| + |c_3|\right\},$$

respectively. Using the principle of mathematical induction, we obtain (2.4). This completes proof of Theorem 8. □

Corollary 9. *Let* $0 \le \gamma < 1$. *If the function* $f \in \mathcal{X}_t(\gamma)$, *then*

$$|a_n| \ \le \ 1 + (n-1)(1-\gamma). \tag{2.8}$$

Proof. From Theorem 6, we know that $F(z) \in \mathcal{S}^*$, thus $|c_n| \le n$. The assertion (2.8), can now easily derived from Theorem 8. □

Remark 10. *Setting* $t = -1$ *in* (2.3) *we find that*

$$\begin{aligned}
&c_{2n} = 0, \ n \in \mathbb{N},\\
&c_3 = 2b_3 - b_2^2, \quad c_5 = 2b_5 - 2b_2b_4 + b_3^2, \quad c_7 = 2b_7 - 2b_2b_6 + 2b_3b_5 - b_4^2, \cdots
\end{aligned}$$

thus

$$c_{2n-1} = B_{2n-1},\ n = 2, 3, \cdots ,$$

where

$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1} b_n^2,\ n = 2, 3, \cdots .$$

Therefore, setting $t = -1$ in Theorem 8 and using the known inequality [2, Theorem B]

$$|B_{2n-1}| \leq 1,\ n = 2, 3, \cdots ,$$

we get the corresponding result due to Gao and Zhou [2].

Theorem 11. *Let $0 \leq \gamma < 1$. If the function $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{|na_n - c_n| + (1-\gamma)|c_n|\} \ \leq \ 1-\gamma, \ \ z \in \mathbb{U}, \tag{2.9}$$

then $f(z) \in \mathcal{X}_t(\gamma)$

Proof. If f satisfies (1.4), then

$$\left| \frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 \right| < 1 - \gamma, \ \ z \in \mathbb{U}. \tag{2.10}$$

Evidently, since

$$\frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 = \frac{z + \sum_{n=2}^{\infty} n\ a_n z^n}{z + \sum_{n=2}^{\infty} c_n z^n} - 1$$

$$= \frac{\sum_{n=2}^{\infty} (na_n - c_n) z^{n-1}}{1 + \sum_{n=2}^{\infty} c_n z^{n-1}},$$

we see that

$$\left| \frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 \right| \ \leq \ \frac{\sum_{n=2}^{\infty} |na_n - c_n|}{1 - \sum_{n=2}^{\infty} |c_n|}.$$

Therefore, if $f(z)$ satisfies (2.9), then we have (2.10). This completes the proof of Theorem 11. □

Theorem 12. *Let $f \in \mathcal{X}_t(\gamma)$. Then the unit disk $\mathbb{U}$ is mapped by $f(z)$ on a domain that contain the disk $|w(z)| < \dfrac{1}{4-\gamma}$.*

Proof. Suppose that $f(z) \in \mathcal{X}_t(\gamma)$, and let w_0 be any complex number such that $f(z) \neq w_0$ for $z \in \mathbb{U}$. Then $w_0 \neq 0$ and

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \cdots \tag{2.11}$$

is univalent in $\mathbb{U}$. This leads to

$$\left|a_2 + \frac{1}{w_0}\right| \leq 2, \tag{2.12}$$

on the other hand, from Corollary 9, we know that

$$|a_2| \leq 2 - \gamma, \quad 0 \leq \gamma < 1. \tag{2.13}$$

Combining (2.12) and (2.13), we deduce that

$$|w_0| \geq \frac{1}{|a_2| + 2} \geq \frac{1}{4 - \gamma}. \tag{2.14}$$

This completes the proof of Theorem 12. □

Theorem 13. *Let $f \in \mathcal{X}_t(\gamma)$, then we have*

$$\frac{1 - (1 - 2\gamma)r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + (1 - 2\gamma)r}{(1 - r)^3} \quad (|z| = r, 0 \leq r < 1) \tag{2.15}$$

and

$$\int_0^r \frac{1 - (1 - 2\gamma)\tau}{(1 + \tau)^3} d\tau \leq |f(z)| \leq \int_0^r \frac{1 + (1 - 2\gamma)\tau}{(1 - \tau)^3} d\tau \quad (|z| = r, 0 \leq r < 1). \tag{2.16}$$

Proof. Suppose that $f(z) \in \mathcal{X}_t(\gamma)$. From the definition of subordination between analytic functions, we deduce that

$$\begin{aligned} \frac{1 - (1 - 2\gamma)r}{1 + r} &\leq \frac{1 - (1 - 2\gamma)|w(z)|}{1 + |w(z)|} \leq \left|\frac{tz^2 f'(z)}{g(z)g(tz)}\right| = \left|\frac{zf'(z)}{F(z)}\right| \\ &\leq \frac{1 - (1 - 2\gamma)|w(z)|}{1 + |w(z)|} \leq \frac{1 + (1 - 2\gamma)r}{1 - r} \quad (|z| = r,\ 0 \leq r < 1). \end{aligned} \tag{2.17}$$

where w is Schwarz function with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{U}$. Since

$$F(z) = \frac{g(z)g(tz)}{tz}$$

is an starlike function, it is well known [1], that

$$\frac{r}{(1 + r)^2} \leq |F(z)| \leq \frac{r}{(1 - r)^2} \quad (|z| = r,\ 0 \leq r < 1). \tag{2.18}$$

Now it follows from (2.17) and (2.18), that

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3} \quad (|z|=r,\, 0 \leq r < 1).$$

Let $z = re^{i\theta}$ $(0 < r < 1)$. If $\mathcal{L}$ denotes that *closed* line segment in the complex ζ-plane from $\zeta = 0$ and $\zeta = z$, then we have

$$f(z) = \int_{\mathcal{L}} f'(\zeta)d\zeta = \int_0^r f'(\tau e^{i\theta})e^{i\theta}d\tau \quad (|z|=r,\, 0 \leq r < 1).$$

Thus by using upper estimate in (2.15), we have

$$|f(z)| = \left|\int_0^z f'(\zeta)d\zeta\right| \leq \int_0^r |f'(\tau e^{i\theta})|d\tau \leq \int_0^r \frac{1+(1-2\gamma)\tau}{(1-\tau)^3}d\tau \quad (|z|=r,\, 0 \leq r < 1),$$

which yields the right hand side of the inequality in (2.16). In order to prove the lower bound in (2.16), it is sufficient to show that it holds true for z_0 nearest to zero, where $|z_0| = r\,(0 < r < 1)$. Moreover, we have

$$|f(z)| \geq |f(z_0)| \quad (|z|=r,\, 0 \leq r < 1).$$

Since $f(z)$ is close-to-convex function in the open unit disk $\mathbb{U}$, it is univalent in $\mathbb{U}$. We deduce that the original image of the closed line segment $\mathcal{L}_0$ in the complex $\zeta-$plane from $\zeta = 0$ and $\zeta = f(z_0)$ is a piece of arc Γ in the disk $\mathbb{U}_r$, given by

$$\mathbb{U}_r = \{z : \; z \in \mathbb{C} \; \text{ and } \; |z| \leq r \; (0 \leq r < 1)\}.$$

Since, in accordance with (2.15), we have

$$|f(z)| = \int_{f(\Gamma)} |dw| = \int_{\Gamma} |f'(z)||dz| \geq \int_0^r \frac{1-(1-2\gamma)\tau}{(1+\tau)^3}d\tau \quad (|z|=r,\, 0 \leq r < 1).$$

This completes the proof of Theorem 13. □

Theorem 14. *Let $f \in \mathcal{X}_t(\gamma)$, then $f(z)$ is convex in $|z| < r_0 = 2-\sqrt{3}$.*

Proof. When $f(z) \in \mathcal{X}_t(\gamma)$, there exists $g(z) \in \mathcal{S}^*(1/2)$ such that (1.4) holds, then $F(z)$ defined by (2.2) is a starlike function, so from (1.4) we have

$$zf'(z) = F(z)p(z), \tag{2.19}$$

where $p(0) = 1$ and $\Re(p(z)) > 0$. From (2.19), we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zF'(z)}{F(z)} + \frac{zp'(z)}{p(z)},$$

so on using well know estimates [1], we have

$$\begin{aligned}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= \Re\left\{\frac{zF'(z)}{F(z)}\right\}+\Re\left\{\frac{zp'(z)}{p(z)}\right\}\\ &\geq \frac{1-r}{1+r}-\left|\frac{zp'(z)}{p(z)}\right|\\ &\geq \frac{1-r}{1+r}-\frac{2r}{1-r^2}=\frac{r^2-4r+1}{1-r^2}. \end{aligned} \tag{2.20}$$

It is easily seen that, if $r^2-4r+1>0$, then $\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$. Let

$$H(r)=r^2-4r+1, \tag{2.21}$$

since $H(0)=1, H(1)=-2$, and $H'(r)=2r-4<0,\quad 0\leq r<1$, this shows that $H(r)$ is monotonically decreasing function and thus equation $H(r)=r^2-4r+1$ has a root r_0 in interval (0,1). On solving equation (2.21), we get $r_0=2-\sqrt{3}$.

Thus when $r<r_0$, $\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$, that is, $f(z)$ is convex in $|z|<r_0$. This completes the proof of Theorem 14. □

Acknowledgement. The author is grateful to the referees for the valuable comments.

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J. K. Prajapat
Department of Mathematics,
Central University of Rajasthan,
Bandarsindri, Kishangarh-305817,
Dist.-Ajmer, Rajasthan, INDIA.
e-mail: jkprajapat@curaj.ac.in
http://wikieducator.org/User:Jkprajapat

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