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UNBOUNDED SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEM OF SECOND-ORDER ON THE HALF-LINE

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Abstract. The purpose of this paper is to establish the existence of solutions for an integral boundary value problem of second-order set on the infinite interval. The arguments we have used are based upon the nonlinear alternative of Leray-Schauder type. As applications, two examples are included to show the applicability of our result.

1 Introduction

In this paper, we will consider the boundary value problem (bvp for brevity)

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t \to +\infty} x'(t) = \alpha \int_0^\eta x(s)ds, \end{cases} \tag{1.1}$$

where $\eta \in (0, +\infty)$, $\alpha \in \mathbb{R}$ such that $0 < \alpha < \frac{2}{\eta^2}$.

Boundary value problems on unbounded intervals arise in many applications in physics, biology, dynamic of population, ... Most of mathematical problems are naturally posed on finite intervals of the real line. We refer the reader, e.g., to ([1], [3]-[15]) and references therein.

For instance, Tariboon and Sitthiwirattham in [12] proved the existence of positive solutions for the three-points bvp with an integral condition:

$$\begin{cases} x'' + a(t)f(x) = 0, & t \in (0, 1), \\ x(0) = 0, & x(1) = \alpha \int_0^\eta x(s)ds, \end{cases}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{2}{\eta^2}$. They have employed the Krasnosel'skii fixed point theorem in cones.

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Galvis, Rojas and Sinitsyn in [5] have considered the same problem on the interval $[0,\gamma]$. Making use of Schauder's fixed point theorem, they have also proved existence of positive solutions.

We however notice that less papers have dealt with bvps posed on infinite domains. For instance, in [1] Agarwal, Bohner and O'Regan discussed the time scale bvp:

$$\begin{cases} x''(t) + f(t, x(\sigma(t))) = 0, & t \in [a, +\infty), \\ x(a) = 0, \quad x(t) \text{ is bounded for } t \in [a, +\infty), \end{cases}$$

where $a \in \mathbb{T}$ is fixed and $\mathbb{T}$ (a time scale) is a closed subset of $\mathbb{R}$. The forward (respectively, backward) jump operator at t for $t < \sup \mathbb{T}$ (respectively, $t > \inf \mathbb{T}$) is defined by $\sigma(t) = \inf\{\tau > t : \ \tau \in \mathbb{T}\}$ (respectively, $\sigma(t) = \sup\{\tau < t : \ \tau \in \mathbb{T}\}$) for all $t \in \mathbb{T}$. The authors have employed a diagonal procedure together with a fixed point approaches.

We also mention paper [8] where the authors have studied the existence of multiple positive solutions of the following bvp:

$$\begin{cases} (\varphi(x'(t))' + a(t)f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = \sum\limits_{i=1}^{m-2} \alpha_i x(\xi_i), & x'(\infty) = 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing positive homeomorphism with $\varphi(0) = 0$ and $\xi_i \in (0, +\infty)$ such that $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < +\infty$. The coefficients α_i satisfy $\alpha_i \in [0, +\infty)$, $0 < \sum\limits_{i=1}^{m-1} \alpha_i < 1$.

O'Regan, Yan, and Agarwal in [10] established the existence of an unbounded solutions for the second order bvp on the half line:

$$\begin{cases} x''(t) + \Phi(t)f(t, x(t)) = 0, & t \in (a, +\infty), \\ x(a) = 0, & \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $f \in C([a, +\infty) \times (0, +\infty), [0, +\infty))$ with $\lim\limits_{x\to 0^+} f(t, x) = +\infty$ for each $t \in (a, +\infty)$ and $\Phi \in C((a, +\infty), (0, +\infty))$. Also the existence of multiple unbounded positive solutions is discussed by means of the theory of the fixed point index.

In [15] Zima studied the existence of at least one positive solution to the following bvp for the second-order differential equation posed on the half-line:

$$\begin{cases} x''(t) - k^2 x(t) + f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t\to+\infty} x(t) = 0, \end{cases}$$

where $k > 0$ and f is a continuous nonnegative function. The existence results are based on application of the Krasnosel'skii fixed point theorem of cone compression and expansion.

The aim of this work is to discuss the existence of solution to the nonlocal bvp (1.1). The lack of compactness is compensated here by a Corduneanu type compactness lemma. Some preliminaries are given in section 2 while the main existence theorem is given in Section 3. The paper ends with two examples of application in Section 4.

2 Notation and auxiliary facts

In this section, we collect some auxiliary facts which will be needed throughout this paper.
Consider the space X defined by

$$X = \left\{ x \in C([0,\ +\infty), \mathbb{R}), \quad \lim_{t \to +\infty} \frac{x(t)}{1+2t} \text{ exists} \right\},$$

with the norm

$$\|x\|_X = \sup_{t \in [0,\infty)} \left| \frac{x(t)}{1+2t} \right|.$$

Lemma 1. *$(X,\ \|.\|_X)$ is a Banach space.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $(X,\ \|.\|_X)$. Then

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } \|u_n - u_m\|_X < \epsilon \text{ for any } n, m > N.$$

As a consequence, the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ for $t \in [0, +\infty)$, is a Cauchy sequence in $\mathbb{R}$, too. Hence there exists $u(t) \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} |u_n(t) - u(t)| = 0,\ (t \geq 0)$. Then $\lim\limits_{n\to+\infty} |\frac{u_n(t)-u(t)}{1+2t}| = 0$ this implies $\lim\limits_{n\to+\infty} \|u_n - u\|_X = 0$, proving that $(X,\ \|.\|_X)$ is a Banach space. □

We recall the Leray-Schauder nonlinear alternative theorem, which is the main tool used to prove the main existence result.

Theorem 2. *([2]) Let C be a convex subset of a Banach space and U an open subset of C with $0 \in U$. Then every completely continuous map $N : \overline{U} \to C$ satisfies at least one of the following two properties:*
(A1) N has a fixed point in $\overline{U}$, or
(A2) There is an $x \in \partial U$ and $\lambda \in (0,\ 1)$ with $x = \lambda N x$.

3 Main results

First, we list some hypotheses on the linearity f:
$(H1)$ $0 < \alpha < \frac{2}{\eta^2}$.

$(H2)$ $f:[0,+\infty)\times\mathbb{R}\to[0,+\infty)$ is continuous.
$(H3)$ $|f(t,(1+2t)x)|\leqslant\varphi(t)\psi(|x|)$ on $[0,+\infty)\times\mathbb{R}$ with $\varphi\in L^1[0,+\infty)$ and $\psi\in C([0,+\infty),[0,+\infty))$ is nondecreasing.

Lemma 3. *If $\alpha\eta^2\neq 2$ and $e\in L^1[0,+\infty)$, then the bvp*

$$\begin{cases} x''(t)+e(t)=0, & t\in(0,+\infty),\\ x(0)=0, & \lim\limits_{t\to+\infty}x'(t)=\alpha\int_0^\eta x(s)ds, \end{cases} \tag{3.1}$$

has a unique solution given by

$$x(t)=-\int_0^t(t-s)e(s)ds-\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2e(s)ds+\frac{2t}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$

Proof. Integrating the equation in (3.1) twice from 0 to t yields

$$x(t)=x'(0)t-\int_0^t(t-s)e(s)ds. \tag{3.2}$$

So

$$\lim_{t\to+\infty}x'(t)=x'(0)-\int_0^\infty e(s)ds,$$

Integrating once again (3.2) from 0 to η ($\eta\in(0,+\infty)$) gives

$$\begin{aligned}\int_0^\eta x(s)ds &= x'(0)\frac{\eta^2}{2}-\int_0^\eta\left(\int_0^\tau(\tau-s)e(s)ds\right)d\tau\\ &= x'(0)\frac{\eta^2}{2}-\frac{1}{2}\int_0^\eta(\eta-s)^2e(s)ds.\end{aligned}$$

From $\lim\limits_{t\to+\infty}x'(t)=\alpha\int_0^\eta x(s)ds$, we obtain

$$x'(0)-\int_0^\infty e(s)ds=x'(0)\frac{\alpha\eta^2}{2}-\frac{\alpha}{2}\int_0^\eta(\eta-s)^2e(s)ds.$$

Thus

$$x'(0)=-\frac{\alpha}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2e(s)ds+\frac{2}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$

Then

$$x(t)=-\int_0^t(t-s)e(s)ds-\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2e(s)ds+\frac{2t}{2-\alpha\eta^2}\int_0^\infty e(s)ds.$$

□

Lemma 4. *Under the assumptions of Lemma 3, the unique solution of problem (3.1) can be written as*

$$x(t) = \int_0^\infty G(t,s)e(s)ds$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{2-\alpha\eta^2}\begin{cases} 2s - \alpha t s^2 - \alpha\eta^2 s + 2\alpha t\eta s, & s \leqslant \min\{t,\,\eta\} \\ 2t - \alpha t(\eta - s)^2, & t \leqslant s \leqslant \eta \\ 2s + \alpha\eta^2 t - \alpha\eta^2 s, & \eta \leqslant s \leqslant t \\ 2t, & \max\{t,\,\eta\} \leqslant s. \end{cases}$$

Proof. We distinguish between two cases:
(a) If $t \leqslant \eta$, the solution of problem (3.1) can be expressed as

$$\begin{aligned} x(t) &= -\int_0^t (t-s)e(s)ds \\ &\quad -\frac{\alpha t}{2-\alpha\eta^2}\left[\int_0^t (\eta - s)^2 e(s)ds + \int_t^\eta (\eta-s)^2 e(s)ds\right] \\ &\quad +\frac{2t}{2-\alpha\eta^2}\left[\int_0^t e(s)ds + \int_t^\eta e(s)ds + \int_\eta^\infty e(s)ds\right] \\ &= \int_0^t \frac{2s - \alpha t s^2 - \alpha\eta^2 s + 2\alpha t\eta s}{2-\alpha\eta^2} e(s)ds \\ &\quad + \int_t^\eta \frac{2t - \alpha t(\eta - s)^2}{2-\alpha\eta^2} e(s)ds + \int_\eta^\infty \frac{2t}{2-\alpha\eta^2} e(s)ds \\ &= \int_0^\infty G(t,s)e(s)ds. \end{aligned}$$

(b) If $t \geqslant \eta$, the solution of problem (3.1) can be expressed as

$$\begin{aligned} x(t) &= -\int_0^\eta (t-s)e(s)ds - \int_\eta^t (t-s)e(s)ds \\ &\quad -\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta - s)^2 e(s)ds \\ &\quad +\frac{2t}{2-\alpha\eta^2}\left[\int_0^\eta e(s)ds + \int_\eta^t e(s)ds + \int_t^\infty e(s)ds\right] \\ &= \int_0^\eta \frac{2s - \alpha t s^2 - \alpha\eta^2 s + 2\alpha t\eta s}{2-\alpha\eta^2} e(s)ds \\ &\quad + \int_\eta^t \frac{2s + \alpha\eta^2 t - \alpha\eta^2 s}{2-\alpha\eta^2} e(s)ds + \int_t^\infty \frac{2t}{2-\alpha\eta^2} e(s)ds \\ &= \int_0^\infty G(t,s)e(s)ds. \end{aligned}$$

□

Lemma 5. *If* $(H1)$ *holds, then the Green's function of problem (3.1) satisfies*

$$0 \leqslant G(t,s) \leqslant \frac{2t}{2-\alpha\eta^2}, \text{ for } t,s \in [0,+\infty).$$

Proof. **Claim 1.** We show that $G(t,s) \geqslant 0$.
For $0 \leqslant s \leqslant \min\{t,\eta\}$, we have

$$\begin{aligned} G(t,s) &= \frac{2s-\alpha ts^2-\alpha\eta^2 s+2\alpha t\eta s}{2-\alpha\eta^2} \\ &= \frac{s(2-\alpha\eta^2)+\alpha ts(2\eta-s)}{2-\alpha\eta^2} \\ &\geqslant 0. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant \eta$,

$$\begin{aligned} G(t,s) &= \frac{2t-\alpha t(\eta-s)^2}{2-\alpha\eta^2} \\ &= \frac{t(2-\alpha\eta^2)+\alpha ts(2\eta-s)}{2-\alpha\eta^2} \\ &\geqslant 0. \end{aligned}$$

For $0 \leqslant \eta \leqslant s \leqslant t$,

$$\begin{aligned} G(t,s) &= \frac{2s+\alpha\eta^2 t-\alpha\eta^2 s}{2-\alpha\eta^2} \\ &= \frac{2s+\alpha\eta^2(t-s)}{2-\alpha\eta^2} \\ &\geqslant 0. \end{aligned}$$

For $s \geqslant \max\{t,\eta\} \geqslant 0$,

$$G(t,s) = \frac{2t}{2-\alpha\eta^2} \geqslant 0.$$

Claim 2. From $0 < \alpha < \frac{2}{\eta^2}$ we will show that $G(t,s) \leqslant \frac{2t}{2-\alpha\eta^2}$.
For $0 \leqslant s \leqslant \min\{t,\eta\}$, we have

$$\begin{aligned} G(t,s) &= \frac{2s - \alpha t s^2 - \alpha\eta^2 s + 2\alpha t\eta s}{2-\alpha\eta^2} \\ &= \frac{s(2-\alpha\eta^2) + \alpha t s(2\eta - s)}{2-\alpha\eta^2} \\ &\leqslant \frac{t(2-\alpha\eta^2) + \alpha t s(2\eta - s)}{2-\alpha\eta^2} \\ &\leqslant \frac{2t - \alpha t(\eta - s)^2}{2-\alpha\eta^2} \\ &\leqslant \frac{2t}{2-\alpha\eta^2}. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant \eta$,

$$\begin{aligned} G(t,s) &= \frac{2t - \alpha t(\eta - s)^2}{2-\alpha\eta^2} \\ &\leqslant \frac{2t}{2-\alpha\eta^2}. \end{aligned}$$

For $0 \leqslant \eta \leqslant s \leqslant t$,

$$\begin{aligned} G(t,s) &= \frac{2s + \alpha\eta^2 t - \alpha\eta^2 s}{2-\alpha\eta^2} \\ &= \frac{(2-\alpha\eta^2)s + \alpha\eta^2 t}{2-\alpha\eta^2} \\ &\leqslant \frac{(2-\alpha\eta^2)t + \alpha\eta^2 t}{2-\alpha\eta^2} \\ &\leqslant \frac{2t}{2-\alpha\eta^2}. \end{aligned}$$

For $s \geqslant \max\{t,\eta\} \geqslant 0$,

$$G(t,s) = \frac{2t}{2-\alpha\eta^2}.$$

Thus

$$0 \leqslant G(t,s) \leqslant \frac{2t}{2-\alpha\eta^2}, \text{ for } t,s \in [0,+\infty).$$

□

Now, define the operator T by

$$(Tx)(t) = \int_0^\infty G(t,s) f(s, x(s)) ds, \quad \text{for } x \in X.$$

Bvp (1.1) has a solution x if and only if x solves the operator equation $x = Tx$. We will study the existence of a fixed point of T. For this, we verify that the operator T satisfies all conditions of Theorem 2.
Since the Arzela-Ascoli theorem fails to work in the space X, we need a modified compactness criterion to prove T is compact. The proof of the following compactness criterion can be found in [9].

Lemma 6. *([9]) Let $B = \{x \in X,\ \|x\|_X < l\}$ such that $l > 0$,*
$B_1 = \{\frac{x(t)}{1+2t},\ x \in B\}$.
If B_1 is equicontinuous on any compact intervals of $[0, +\infty)$ and equiconvergent at infinity, then B is relatively compact on X.

Remark 7. *B_1 is called equiconvergent at infinity if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\left|\frac{x(t)}{1+2t} - \lim_{t\to+\infty}\frac{x(t)}{1+2t}\right| < \epsilon \quad \text{for any } x \in B \text{ and } t > \delta.$$

Lemma 8. *Under Assumptions $(H1)-(H3)$, the operator $T : X \to X$ is completely continuous.*

Proof. **Claim 1.** We show that operator $T : X \to X$ is continuous.
Let $x_n \to x$ as $n \to +\infty$, then there exists $r_0 > 0$ such that

$$\max\left\{\|x\|_X,\ \sup_{n\in\mathbb{N}}\|x_n\|_X\right\} < r_0.$$

From Lemma 5 and $(H3)$, we have

$$\begin{aligned}
&\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s)) - f(s,x(s))|ds \\
&\quad\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty (|f(s,x_n(s))| + |f(s,x(s))|)ds \\
&\quad\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \left(\left|f\left(s,\frac{x_n(s)(1+2s)}{1+2s}\right)\right| + \left|f\left(s,\frac{x(s)(1+2s)}{1+2s}\right)\right|\right) ds \\
&\quad\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \left(\varphi(s)\psi\left(\left|\frac{x_n(s)}{1+2s}\right|\right) + \varphi(s)\psi\left(\left|\frac{x(s)}{1+2s}\right|\right)\right) ds \\
&\quad\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty (\varphi(s)\psi(\|x_n\|_X) + \varphi(s)\psi(\|x\|_X))\, ds \\
&\quad\leqslant \frac{2\psi(r_0)}{2-\alpha\eta^2}\int_0^\infty \varphi(s)ds.
\end{aligned}$$

Since $\varphi \in L^1[0, +\infty)$, the term $\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s)) - f(s,x(s))|ds$ is bounded. Using the continuity of f, we obtain that

$$\|Tx_n - Tx\|_X = \sup_{t\in[0,\infty)}\int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x_n(s)) - f(s,x(s))|ds \to 0, \quad \text{as} \quad n \to +\infty,$$

proving our claim.
Claim 2. We show that operator $T : X \to X$ is relatively compact.
Let $\Omega = \{x \in X, \quad \|x\|_X \leqslant k\}$, $(k > 0)$, be any bounded subset of X.
From Lemma 5 and $(H3)$, we have

$$\begin{aligned}
\|Tx\|_X &= \sup_{t\in[0,\infty)} \int_0^\infty \frac{G(t,s)}{1+2t}|f(s,x(s))|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty |f(s,x(s))|ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \left|f\left(s,\frac{x(s)(1+2s)}{1+2s}\right)\right| ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi\left(\left|\frac{x(s)}{1+2s}\right|\right) ds \\
&\leqslant \frac{1}{2-\alpha\eta^2}\int_0^\infty \varphi(s)\psi(\|x\|_X)ds \\
&\leqslant \frac{\psi(k)}{2-\alpha\eta^2}\int_0^\infty \varphi(s)ds, \text{ for } x \in \Omega.
\end{aligned}$$

Since $\varphi \in L^1[0,+\infty)$, $\|Tx\|_X$ is bounded for $x \in \Omega$. Hence, $T\Omega$ is uniformly bounded.
To show that $T\Omega$ is equicontinuous on any compact interval of $[0,+\infty)$, let $\beta > 0$, $t_1, t_2 \in [0,\beta]$ $(t_2 > t_1)$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left|\frac{(Tx)(t_2)}{1+2t_2} - \frac{(Tx)(t_1)}{1+2t_1}\right| \\
&= \left|\int_0^\infty \frac{G(t_2,s)}{1+2t_2} f(s,x(s))ds - \int_0^\infty \frac{G(t_1,s)}{1+2t_1} f(s,x(s))ds\right| \\
&\leqslant \int_0^\infty \left|\frac{G(t_2,s)}{1+2t_2} - \frac{G(t_1,s)}{1+2t_2} + \frac{G(t_1,s)}{1+2t_2} - \frac{G(t_1,s)}{1+2t_1}\right| |f(s,x(s))|ds \\
&\leqslant \int_0^\infty \left[\frac{|G(t_2,s)-G(t_1,s)|}{1+2t_2} + \frac{2|t_1-t_2|G(t_1,s)}{(1+2t_2)(1+2t_1)}\right]\left|f\left(s,\frac{(1+2s)x(s)}{1+2s}\right)\right| ds \\
&\leqslant \psi(k)\int_0^\infty \left[\frac{|G(t_2,s)-G(t_1,s)|}{1+2t_2} + \frac{2|t_1-t_2|G(t_1,s)}{(1+2t_2)(1+2t_1)}\right]\varphi(s)ds.
\end{aligned}$$

Since for $s \in [0,+\infty)$ the function $t \mapsto G(t,s)$ is continuous on the compact interval $[0,\beta]$, then it is uniformly continuous on $[0,\beta]$. Hence

$$|G(t_2,s) - G(t_1,s)| \to 0, \text{ uniformly as } |t_1 - t_2| \to 0.$$

So

$$\left|\frac{(Tx)(t_2)}{1+2t_2} - \frac{(Tx)(t_1)}{1+2t_1}\right| \to 0, \text{ uniformly as } |t_1 - t_2| \to 0,$$

for all $x \in \Omega$. Thus $T\Omega$ is locally equicontinuous on $[0,+\infty)$.
We show that $T\Omega$ is equiconvergent at infinity. For any $x \in \Omega$,

$$\int_0^\infty f(s,x(s))ds \leqslant \psi(k)\int_0^\infty \varphi(s)ds.$$

Since $\varphi \in L^1[0,+\infty)$, we get $f \in L^1[0,+\infty)$. Moreover

$$\begin{aligned}
\lim_{t\to\infty}\left(\frac{(Tx)(t)}{1+2t}\right) &= \lim_{t\to\infty}\Bigg(-\int_0^t \frac{t-s}{1+2t} f(s,x(s))ds - \frac{\alpha t}{(1+2t)(2-\alpha\eta^2)}\int_0^\eta (\eta-s)^2 f(s,x(s))ds \\
&\quad + \frac{2t}{(1+2t)(2-\alpha\eta^2)}\int_0^\infty f(s,x(s))ds\Bigg) \\
&= -\frac{1}{2}\int_0^\infty f(s,x(s))ds - \frac{\alpha}{2(2-\alpha\eta^2)}\int_0^\eta (\eta-s)^2 f(s,x(s))ds \\
&\quad + \frac{1}{2-\alpha\eta^2}\int_0^\infty f(s,x(s))ds \\
&= \frac{\alpha\eta^2}{2(2-\alpha\eta^2)}\int_0^\infty f(s,x(s))ds - \frac{\alpha}{2(2-\alpha\eta^2)}\int_0^\eta (\eta-s)^2 f(s,x(s))ds.
\end{aligned}$$

So

$$\begin{aligned}
\left|\frac{(Tx)(t)}{1+2t} - \lim_{t\to\infty}\frac{(Tx)(t)}{1+2t}\right| &= \Bigg| -\int_0^t \frac{t-s}{1+2t} f(s,x(s))ds \\
&\quad + \left(-\frac{\alpha t}{(1+2t)(2-\alpha\eta^2)} + \frac{\alpha}{2(2-\alpha\eta^2)}\right)\int_0^\eta (\eta-s)^2 f(s,x(s))ds \\
&\quad + \left(\frac{2t}{(1+2t)(2-\alpha\eta^2)} - \frac{\alpha\eta^2}{2(2-\alpha\eta^2)}\right)\int_0^\infty f(s,x(s))ds\Bigg|.
\end{aligned}$$

Hence

$$\left|\frac{(Tx)(t)}{1+2t} - \lim_{t\to\infty}\frac{(Tx)(t)}{1+2t}\right| \to 0 \quad \text{as} \quad t \to +\infty.$$

Then $T\Omega$ is equiconvergent at infinity. By Lemma 6, we conclude that $T : X \to X$ is completely continuous. □

Theorem 9. *Assume that* $(H1)-(H3)$ *and the following condition holds:*
$(H4)$ *There exists* $\rho > 0$ *such that*

$$\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^\infty \varphi(s)ds} > 1.$$

Then problem (1.1) has an unbounded solution $x = x(t)$ *such that*

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \textit{for } t \in [0,+\infty).$$

Proof. Consider the family of parameterized bvps

$$\begin{cases} x''(t) + \lambda f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = 0, & \lim\limits_{t \to +\infty} x'(t) = \alpha \int_0^{\eta} x(s)ds, \end{cases} \tag{3.3}$$

for $\lambda \in (0,\ 1)$.
Let

$$U = \{x \in X, \quad \|x\|_X < \rho\}.$$

Solving problem (1.1) is equivalent to searching for fixed point for operator T in $\overline{U}$. We will prove that $x \neq \lambda Tx$, for $x \in \partial U$ and $\lambda \in (0,\ 1)$. On the contrary, assume there exists $x \in \partial U$ with $x = \lambda Tx$; then for $\lambda \in (0,\ 1)$ we have

$$\begin{aligned} \|x\|_X &= \sup_{t \in [0,\infty)} \left| \frac{(\lambda Tx)(t)}{1+2t} \right| \\ &\leqslant \sup_{t \in [0,\infty)} \left| \frac{(Tx)(t)}{1+2t} \right| \\ &\leqslant \sup_{t \in [0,\infty)} \int_0^{\infty} \frac{G(t,s)}{1+2t} \left| f\left(s, \frac{(1+2s)x(s)}{1+2s}\right) \right| ds \\ &\leqslant \int_0^{\infty} \frac{1}{2-\alpha\eta^2} \varphi(s) \psi\left(\left| \frac{x(s)}{1+2s} \right| \right) ds \\ &\leqslant \frac{1}{2-\alpha\eta^2} \int_0^{\infty} \varphi(s)\psi(\|x\|_X) ds \\ &\leqslant \frac{\psi(\rho)}{2-\alpha\eta^2} \int_0^{\infty} \varphi(s) ds. \end{aligned}$$

So

$$\rho \leqslant \frac{\psi(\rho)}{2-\alpha\eta^2} \int_0^{\infty} \varphi(s) ds.$$

Hence

$$\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^{\infty} \varphi(s)ds} \leqslant 1,$$

which contradicts condition $(H4)$.
By Theorem 2 and Lemma 8, we deduce that bvp (1.1) has an unbounded solution $x = x(t)$ satisfying

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \text{for } t \in [0, +\infty).$$

□

4 Examples

Two examples are provided to illustrate our existence result.

Example 10. *Consider the following boundary value problem on the half-line:*

$$\begin{cases} x''(t) + \sqrt{\frac{|x(t)|}{1+2t}}\, e^{-t} = 0, & 0 < t < +\infty \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = \alpha \int_0^1 x(s)ds, \end{cases} \tag{4.1}$$

where $0 < \alpha < 2,\ \eta = 1$.
We will apply Theorem 9 to show that problem (4.1) has at least a solution. Let

$$f(t,x) = \sqrt{\frac{|x|}{1+2t}}\, e^{-t}.$$

Choose

$$\psi(x) = \sqrt{x}, \quad \varphi(t) = e^{-t}, \quad \rho \geqslant \left(\frac{1}{2-\alpha}\right)^2.$$

Then, we have
$(H1)$ *Since* $\eta = 1$ *so* $0 < \alpha < \frac{2}{\eta^2}$ *satisfies.*
$(H2)$ $f : [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ *is continuous.*
$(H3)$ $|f(t,(1+2t)x)| = \sqrt{|x|}\, e^{-t} = \psi(|x|)\varphi(t)$ *on* $[0,+\infty) \times \mathbb{R}$ *with* $\varphi \in L^1[0,+\infty)$ *and* $\psi \in C([0,+\infty),[0,+\infty))$ *is nondecreasing.*
$(H4)$ $\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^\infty \varphi(s)ds} = \frac{\rho(2-\alpha)}{\sqrt{\rho}\times 1} = \sqrt{\rho}(2-\alpha) \geqslant 1.$
Hence all conditions of Theorem 9 hold. As a consequence, problem (4.1) has at least a solution x *such that*

$$0 \leqslant \frac{|x(t)|}{1+2t} \leqslant \rho, \quad \textit{for } t \in [0,+\infty).$$

Example 11. *Consider the bvp set on the half-line:*

$$\begin{cases} x''(t) + \frac{x^2(t)}{(1+2t)^4} = 0, & 0 < t < +\infty \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = \frac{1}{2} \int_0^1 x(s)ds. \end{cases} \tag{4.2}$$

We will apply Theorem 9 to show that problem (4.2) has at least one solution. Let

$$f(t,x) = \frac{x^2}{(1+2t)^4}$$

and

$$\psi(x) = x^2, \quad \varphi(t) = \frac{1}{(1+2t)^2}, \quad \rho < 3.$$

Then, we check the hypotheses
$(H1)$ *Since* $\eta = 1$, $\alpha = \frac{1}{2}$ *so* $0 < \alpha < \frac{2}{\eta^2}$ *satisfies.*
$(H2)$ $f : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ *is continuous.*
$(H3)$ $|f(t, (1 + 2t)x)| = \frac{x^2}{(1+2t)^2} = \psi(|x|)\varphi(t)$ *on* $[0, +\infty) \times \mathbb{R}$ *with* $\varphi \in L^1[0, +\infty)$ *and* $\psi \in C([0, +\infty), [0, +\infty))$ *is nondecreasing.*
$(H4)$ $\frac{\rho(2-\alpha\eta^2)}{\psi(\rho)\int_0^\infty \varphi(s)ds} = \frac{\rho(\frac{3}{2})}{\rho^2 \times \frac{1}{2}} = \frac{3}{\rho} \geqslant 1.$
Hence all conditions of Theorem 9 hold. Then problem (4.2) has at least a solution x *such that*

$$0 \leqslant \frac{|x(t)|}{1 + 2t} \leqslant \rho, \quad \textit{for } t \in [0, +\infty).$$

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