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FIXED POINTS OF MULTIVALUED MAPPINGS IN METRIC SPACES

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Abstract. Admissibility of mappings are introduced to create conditions to minimally restrict various contractive conditions on pairs of points from a metric space in order to ensure fixed point property of the respective contractions. In the present work we define new admissibility conditions and control functions to obtain certain multivalued fixed point theorems. The corresponding single valued case is discussed. We define four weak contraction mappings of which two are multivalued and two are single valued. The results are without any assumption of continuity. There is an illustrative example.

1 Introduction and Definitions

It is observed that the conventional proofs of many fixed point theorems of contractive mappings in metric spaces including that of the famous Banach's contraction mapping principle do not utilize the contraction condition for every pair of points from a metric space. In many of these results it has been possible to restrict the contraction condition to certain sets of the pairs of points for which the conclusions of these theorems are unaltered. Admissibility conditions on functions are introduced for that purpose. It was first introduced by Samet et al. [26] and was further generalized for different types of mappings. References [1, 5, 8, 17, 20] are some of the results from this line of research in metric fixed point theory. It is an alternative to the approach of introducing partial order in metric spaces which also serves the same purpose. Fixed point theory in partially ordered metric spaces has a vast literature for which [7, 16, 22, 23, 24] are some recent references.

We consider multivalued operators in our results. It was through the work of Nadler [21] that the fixed point theory got extended to the domain of set valued analysis. Today it has a vast literature and is a subject in its own right. Some recent references on this topic are [8, 9, 13, 14, 15, 27].

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In the following we note some mathematical concepts which form the background of the present work.

Let $(X,\ d)$ be a metric space. We consider the following classes of subsets of the metric space X :

$$N(X) = \{A : A \text{ is a nonempty subset of } X\},$$
$$B(X) = \{A : A \text{ is a nonempty bounded subset of } X\}.$$

For $A,\ B \in B(X)$, the functions D and δ are defined as follows :

$$D(A,\ B) = \inf\ \{d(a,\ b) : a \in A,\ b \in B\},$$
$$\delta(A,\ B) = \sup\ \{d(a,\ b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. Obviously, $D(A,\ B) \leq \delta(A,\ B)$. For all $A,\ B,\ C \in B(X)$, the definition of $\delta(A,\ B)$ yields the following:

$$\delta(A,\ B) = \delta(B,\ A);\quad \delta(A,\ B) \leq \delta(A,\ C) + \delta(C,\ B);$$
$$\delta(A,\ B) = 0 \text{ if and only if } A = B = \{a\};\quad \delta(A,\ A) = \text{diam } A[12].$$

There are several fixed point results which have utilized δ-function [2, 3, 12, 18, 28].

Definition 1. Let X be a nonempty set and $T : X \to X$. A point $x \in X$ is said to be a fixed point of T if $x = Tx$.

Definition 2. Let X be a nonempty set and $T : X \to N(X)$ be a multivalued mapping. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$.

In [26] Samet et al. first introduced the concept of α-admissible mappings and utilized these mappings to prove some fixed point results in metric spaces.

Definition 3 ([26])**.** Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. The mapping T is α-admissible if for $x,\ y \in X$,

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

In a separate vein the following definition was introduced in [1].

Definition 4 ([1])**.** Let X be a nonempty set and $T : X \to X$. Let $\alpha,\ \beta : X \to [0,\infty)$ be two mappings. We say that T is a cyclic (α,β)-admissible mapping if for $x,\ y \in X$,

$$\alpha(x) \geq 1 \quad \Longrightarrow \quad \beta(Tx) \geq 1 \quad \text{and} \quad \beta(y) \geq 1 \quad \Longrightarrow \quad \alpha(Ty) \geq 1.$$

In the following we define cyclic (α,β)-admissibility for multivalued mappings. We define the following notions for our purpose based on which we prove our theorems.

Definition 5. Let X be a nonempty set and $T : X \to N(X)$ be a multivalued mapping. Let $\alpha,\ \beta : X \to [0, \infty)$. We say that T is a cyclic multivalued (α, β)-admissible mapping if for $x,\ y \in X$,

$$\alpha(x) \geq 1 \implies \beta(u) \geq 1 \text{ for all } u \in Tx \text{ and } \beta(y) \geq 1 \implies \alpha(v) \geq 1 \text{ for all } v \in Ty.$$

In our results we will use the following classes of functions.
Let Ψ denote the set of all functions $\psi\colon [0, \infty) \to [0,\ \infty)$ such that ψ is continuous, monotone nondecreasing and $\psi(t) = 0$ if and only if $t = 0$, and Θ denote the set of all functions $\theta\colon [0, \infty)^2 \to [0, \infty)$ such that for any sequence $\{(x_n, y_n)\}$ in $[0, \infty)^2$ with $(x_n, y_n) \to (u, v) \neq (0, 0)$, $\underline{\lim}\, \theta(x_n, y_n) > 0$.

Definition 6. Let (X, d) be a metric space and $T : X \to X$. Let $\alpha,\ \beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$. We say that T is a $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping if for $x,\ y \in X$, $\alpha(x)\beta(y) \geq 1 \implies$

$$\psi(d(Tx,\ Ty)) \leq \psi\Big(\frac{1}{2}\ [d(x,\ Tx) + d(y,\ Ty)]\Big) - \theta(d(x,\ Tx),\ d(y,\ Ty)).$$

Definition 7. Let (X, d) be a metric space and $T : X \to X$. Let $\alpha,\ \beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$. We say that T is a $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping if for $x,\ y \in X$, $\alpha(x)\beta(y) \geq 1 \implies$

$$\psi(d(Tx,\ Ty)) \leq \psi\Big(\frac{1}{2}\ [d(x,\ Ty) + d(y,\ Tx)]\Big) - \theta(d(x,\ Ty),\ d(y,\ Tx)).$$

Definition 8. Let (X, d) be a metric space and $T : X \to N(X)$ be a multivalued mapping. Let $\alpha,\ \beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$. We say that T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping if for $x,\ y \in X$, $\alpha(x)\beta(y) \geq 1 \implies$

$$\psi(\delta(Tx,\ Ty)) \leq \psi\Big(\frac{1}{2}\ [D(x,\ Tx) + D(y,\ Ty)]\Big) - \theta(\delta(x,\ Tx),\ \delta(y,\ Ty)).$$

Definition 9. Let (X, d) be a metric space and $T : X \to N(X)$ be a multivalued mapping. Let $\alpha,\ \beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$. We say that T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping if for $x,\ y \in X$, $\alpha(x)\beta(y) \geq 1 \implies$

$$\psi(\delta(Tx,\ Ty)) \leq \psi\Big(\frac{1}{2}\ [D(x,\ Ty) + D(y,\ Tx)]\Big) - \theta(\delta(x,\ Ty),\ \delta(y,\ Tx)).$$

The inequalities defined above are originated from the ideas of Kannan [19] and Chatterjea [4]. Incidentally these works are early references in fixed point theorems for discontinuous functions in metric spaces. Further the inequalities are weak contraction inequalities which was first introduced in metric spaces by Rhoades [25] and has been further discussed in works like [6, 7, 10, 11, 16, 22].

Some features of the present work are the following.

- We define a new admissibility condition.
- We introduce four new contractions.
- Our main results are for multivalued mappings.
- We use control functions and weak inequalities in our theorems.
- No assumption of continuity is made.
- An illustrative example is discussed.

2 Main Results

Theorem 10. *Let (X, d) be a complete metric space and $T : X \to B(X)$ be a multivalued mapping. Suppose there exist $\theta \in \Theta$ which is nondecreasing in each coordinate, $\psi \in \Psi$ and $\alpha,\ \beta : X \to [0, \infty)$ such that T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping and also the following conditions hold.*

(i) T is a cyclic multivalued (α, β)-admissible mapping,

(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(iii) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.

Then T has a fixed point.

Proof. By condition (ii) of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Let $x_1 \in Tx_0$. By condition (i) of the theorem, $\beta(x_1) \geq 1$. Let $x_2 \in Tx_1$. By condition (i) of the theorem, $\alpha(x_2) \geq 1$. Let $x_3 \in Tx_2$. By condition (i) of the theorem, $\beta(x_3) \geq 1$. Continuing this process we obtain a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in Tx_n \ \text{ with } \ \alpha(x_{2n}) \geq 1 \ \text{ and } \ \beta(x_{2n+1}) \geq 1. \tag{2.1}$$

Since T is a cyclic multivalued (α, β)-admissible mapping and $\beta(x_0) \geq 1$, for the sequence $\{x_n\}$, we get

$$\alpha(x_{2n+1}) \geq 1 \text{ and } \beta(x_{2n}) \geq 1. \tag{2.2}$$

So from (2.1) and (2.2), we summarize that

$$x_{n+1} \in Tx_n \text{ with } \alpha(x_n) \geq 1 \text{ and } \beta(x_n) \geq 1, \text{ for all } n \geq 0. \tag{2.3}$$

Let $R_n = d(x_n,\ x_{n+1})$, for all $n \geq 0$.
As $\alpha(x_n)\,\beta(x_{n+1}) \geq 1$ and T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping, using properties of ψ and θ, we have

$$\begin{aligned} & \psi(R_{n+1}) = \\ = \ & \psi(d(x_{n+1},\ x_{n+2})) \leq \psi(\delta(Tx_n,\ Tx_{n+1})) \\ \leq \ & \psi\Big(\frac{1}{2}\,[D(x_n,\ Tx_n) + D(x_{n+1},\ Tx_{n+1})]\Big) - \theta(\delta(x_n,\ Tx_n),\ \delta(x_{n+1},\ Tx_{n+1})) \\ \leq \ & \psi\Big(\frac{1}{2}\,[d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})]\Big) - \theta(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})) \\ \leq \ & \psi\Big(\frac{1}{2}\,[R_n + R_{n+1}]\Big) - \theta(R_n,\ R_{n+1}). \end{aligned} \tag{2.4}$$

Since $\theta(R_n,\ R_{n+1}) \geq 0$, we have from (2.4) that $\psi(R_{n+1}) \leq \psi\Big(\frac{1}{2}\,[R_n + R_{n+1}]\Big)$. By monotone property of ψ, it follows that $R_{n+1} \leq \frac{1}{2}\,(R_n + R_{n+1})$, that is, $R_{n+1} \leq R_n$. Thus $\{R_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that

$$R_n = d(x_n,\ x_{n+1}) \to r \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Taking limit supremum in both sides of (2.4), using (2.5) and continuity of ψ, we have

$$\psi(r) \leq \psi(r) + \overline{\lim}\ (-\ \theta(R_n,\ R_{n+1})).$$

Since $\overline{\lim}\ (-\ \theta(R_n,\ R_{n+1})) = -\ \underline{\lim}\ \theta(R_n,\ R_{n+1})$, we have

$$\psi(r) \leq \psi(r) - \underline{\lim}\ \theta(R_n,\ R_{n+1}),$$

that is,

$$\underline{\lim}\ \theta(R_n,\ R_{n+1}) \leq 0.$$

Since $(R_n, R_{n+1}) \to (r, r)$, by properties of θ, the above is a contradiction unless $(r, r) = (0, 0)$, that is, $r = 0$. Therefore,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{2.6}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)},\ x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, then we have

$$d(x_{m(k)},\ x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{n(k)}),$$

that is,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) < \epsilon + d(x_{n(k)-1},\ x_{n(k)}).$$

Taking limit as $k \to \infty$ in the above inequality and using (2.6), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{2.7}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{m(k)+1}) + d(x_{m(k)+1},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)+1},\ x_{n(k)+1}) \leq d(x_{m(k)+1},\ x_{m(k)}) + d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Taking limit as $k \to \infty$ in the above inequalities and using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon. \tag{2.8}$$

As $\alpha(x_{m(k)})\ \beta(x_{n(k)}) \geq 1$ and T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping, using properties of ψ and θ, we have
$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\delta(Tx_{m(k)},\ Tx_{n(k)}))$
$\leq \psi\Big(\frac{1}{2}\ [D(x_{m(k)}, Tx_{m(k)}) + D(x_{n(k)}, Tx_{n(k)})]\Big) - \theta(\delta(x_{m(k)}, Tx_{m(k)}),\ \delta(x_{n(k)}, Tx_{n(k)}))$
$\leq \psi\Big(\frac{1}{2}\ [d(x_{m(k)}, x_{m(k)+1}) + d(x_{n(k)}, x_{n(k)+1})]\Big) - \theta(d(x_{m(k)}, x_{m(k)+1}),\ d(x_{n(k)}, x_{n(k)+1}))$.
Since $\theta(d(x_{m(k)}, x_{m(k)+1}),\ d(x_{n(k)}, x_{n(k)+1})) \geq 0$, we have

$$\psi(d(x_{m(k)+1},\ x_{n(k)+1})) \leq \psi\Big(\frac{1}{2}\ [d(x_{m(k)},\ x_{m(k)+1}) + d(x_{n(k)},\ x_{n(k)+1})]\Big),$$

which, by monotone property of ψ, implies that

$$d(x_{m(k)+1},\ x_{n(k)+1}) \leq \frac{1}{2}\ [d(x_{m(k)},\ x_{m(k)+1}) + d(x_{n(k)},\ x_{n(k)+1})].$$

Taking limit as $k \to \infty$ in the above inequality and using (2.6) and (2.8), we obtain $\epsilon \leq 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

As (X, d) is a complete metric space, there exists $z \in X$ such that

$$x_n \to z \quad \text{as} \quad n \to \infty. \tag{2.9}$$

By the condition (iii) of the theorem $\beta(z) \geq 1$. Now $\alpha(x_n)\ \beta(z) \geq 1$ for all $n \geq 0$. As T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping, using properties of ψ and θ, we have

$$\begin{aligned}
\psi(\delta(x_{n+1},\ Tz)) &\leq \psi(\delta(Tx_n,\ Tz)) \\
&\leq \psi\Big(\frac{1}{2}\ [D(x_n,\ Tx_n) + D(z,\ Tz)]\Big) - \theta(\delta(x_n,\ Tx_n),\ \delta(z,\ Tz)) \\
&\leq \psi\Big(\frac{1}{2}\ [d(x_n,\ x_{n+1}) + D(z,\ Tz)]\Big) - \theta(d(x_n,\ x_{n+1}),\ \delta(z,\ Tz)).
\end{aligned}$$

Since $\theta(d(x_n,\ x_{n+1}),\ \delta(z,\ Tz)) \geq 0$, we have

$$\psi(\delta(x_{n+1},\ Tz)) \leq \psi\Big(\frac{1}{2}\ [d(x_n,\ x_{n+1}) + D(z,\ Tz)]\Big),$$

which, by monotone property of ψ, implies that

$$\delta(x_{n+1},\ Tz) \leq \frac{1}{2}\ [d(x_n,\ x_{n+1}) + D(z,\ Tz)].$$

Taking limit as $n \to \infty$ in the above inequality and using (2.6) and (2.9), we have

$$\delta(z,\ Tz) \leq \frac{1}{2}\ D(z,\ Tz).$$

Since $D(z,\ Tz) \leq \delta(z,\ Tz)$, it follows from the above inequality that

$$\delta(z,\ Tz) \leq \frac{1}{2}\ \delta(z,\ Tz),$$

which is a contradiction unless $\delta(z,\ Tz) = 0$, that is, $Tz = \{z\}$, that is, $z \in Tz$. Therefore, z is a fixed point of T. □

Theorem 11. *Let (X, d) be a complete metric space $T : X \to B(X)$ be a multivalued mapping. Suppose there exist $\theta \in \Theta$ which is nondecreasing in each coordinate, $\psi \in \Psi$ and $\alpha,\ \beta : X \to [0, \infty)$ such that T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping and also the conditions (i), (ii) and (iii) of Theorem 10 hold. Then T has a fixed point.*

Proof. Following the line of proof of Theorem 10, we construct the sequence $\{x_n\}$ such that (2.3) is satisfied, that is,

$$x_{n+1} \in Tx_n \text{ with } \alpha(x_n) \geq 1 \text{ and } \beta(x_n) \geq 1, \text{ for all } n \geq 0.$$

Let $R_n = d(x_n,\ x_{n+1})$, for all $n \geq 0$.
As $\alpha(x_n)\ \beta(x_{n+1}) \geq 1$ and T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type

mapping, using properties of ψ and θ, we have

$$\begin{aligned}
&\psi(R_{n+1}) = \psi(d(x_{n+1},\ x_{n+2})) \leq \psi(\delta(Tx_n,\ Tx_{n+1})) \\
\leq\ & \psi\Big(\frac{1}{2}\ [D(x_n,\ Tx_{n+1}) + D(x_{n+1},\ Tx_n)]\Big) - \theta(\delta(x_n,\ Tx_{n+1}),\ \delta(x_{n+1},\ Tx_n)) \\
\leq\ & \psi\Big(\frac{1}{2}\ [d(x_n,\ x_{n+2}) + d(x_{n+1},\ x_{n+1})]\Big) - \theta(d(x_n,\ x_{n+2}),\ d(x_{n+1},\ x_{n+1})) \\
\leq\ & \psi\Big(\frac{1}{2}\ [d(x_n,\ x_{n+2}) + 0]\Big) - \theta(d(x_n,\ x_{n+2}),\ 0). \qquad (2.10)
\end{aligned}$$

Since $\theta(d(x_n,\ x_{n+2}),\ 0) \geq 0$, we get

$$\begin{aligned}
\psi(R_{n+1}) \leq \psi\Big(\frac{1}{2}\ d(x_n,\ x_{n+2})\Big) &\leq \psi\Big(\frac{1}{2}\ [d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})]\Big) \\
&= \psi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big). \qquad (2.11)
\end{aligned}$$

Using monotone property of ψ, we have $R_{n+1} \leq \frac{1}{2}\ (R_n + R_{n+1})$, that is, $R_{n+1} \leq R_n$. Thus $\{R_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that (2.5) is satisfied, that is,

$$R_n = d(x_n,\ x_{n+1}) \to r \quad \text{as} \quad n \to \infty.$$

Using monotone property of ψ, we have from (2.11) that

$$R_{n+1} \leq \frac{1}{2}\ d(x_n,\ x_{n+2}) \leq \frac{1}{2}\ (R_n + R_{n+1}).$$

Taking $n \to \infty$ in the above inequality and using (2.5), we have

$$r \leq \lim_{n\to\infty} \frac{1}{2}\ d(x_n,\ x_{n+2}) \leq \frac{1}{2}\ (r + r) = r.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{2}\ d(x_n,\ x_{n+2}) = r, \text{ that is, } \lim_{n\to\infty}\ d(x_n,\ x_{n+2}) = 2r. \qquad (2.12)$$

Taking limit supremum on both sides of (2.10), using (2.5), (2.12) and continuity of ψ, we have

$$\psi(r) \leq \psi(r) + \overline{\lim}\ (-\ \theta(d(x_n, x_{n+2}),\ 0)).$$

Since $\overline{\lim}\ (-\ \theta(d(x_n,\ x_{n+2}),\ 0)) = -\ \underline{\lim}\ \theta(d(x_n, x_{n+2}),\ 0)$, we have

$$\psi(r) \leq \psi(r) - \underline{\lim}\ \theta(d(x_n, x_{n+2}),\ 0),$$

that is,

$$\underline{\lim}\ \theta(d(x_n, x_{n+2}),\ 0) \leq 0.$$

Since $(d(x_n, x_{n+2}),\ 0) \to (2r, 0)$, by properties of θ, the above is a contradiction unless $(2r, 0) = (0, 0)$, that is, $r = 0$. Therefore, we have (2.6), that is,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0.$$

Again

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.13}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, arguing similarly as in the proof of Theorem 10, we have an $\epsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$, $d(x_{m(k)},\ x_{n(k)}) \geq \epsilon$ and (2.7), (2.8) are satisfied, that is,

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon \quad \text{and} \quad \lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon.$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \leq d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \to \infty$ in the above inequalities and using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{2.14}$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)+1}) = \epsilon. \tag{2.15}$$

As $\alpha(x_{m(k)})\ \beta(x_{n(k)}) \geq 1$ and T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping, using properties of ψ and θ, we have

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\delta(Tx_{m(k)},\ Tx_{n(k)}))$$
$$\leq \psi\Big(\frac{1}{2}\,[D(x_{m(k)}, Tx_{n(k)}) + D(x_{n(k)}, Tx_{m(k)})]\Big) - \theta(\delta(x_{m(k)}, Tx_{n(k)}),\ \delta(x_{n(k)}, Tx_{m(k)}))$$
$$\leq \psi\Big(\frac{1}{2}\,[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})]\Big) - \theta(d(x_{m(k)}, x_{n(k)+1}),\ d(x_{n(k)}, x_{m(k)+1})).$$

Taking limit supremum in the above inequality, using (2.8), (2.14), (2.15) and continuity of ψ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) + \overline{\lim}\ (-\theta(d(x_{m(k)},\ x_{n(k)+1}),\ d(x_{n(k)},\ x_{m(k)+1}))).$$

Since

$$\overline{\lim}\ (-\theta(d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1}))) =$$
$$-\ \underline{\lim}\ \theta(d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})),$$

we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \underline{\lim}\ \theta(d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})),$$

that is,

$$\underline{\lim}\ \theta(d(x_{m(k)},\ x_{n(k)+1}),\ d(x_{n(k)},\ x_{m(k)+1})) \leq 0.$$

Since $(d(x_{m(k)},\ x_{n(k)+1}), d(x_{n(k)},\ x_{m(k)+1})) \to (\epsilon, \epsilon) \neq (0, 0)$, by properties of θ, the above is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. As (X, d) is a complete metric space, there exist $z \in X$ such that (2.9) is satisfied, that is, $x_n \to z$ as $n \to \infty$. By the condition (iii) of the theorem $\beta(z) \geq 1$. Now $\alpha(x_n)\ \beta(z) \geq 1$. As T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping, using properties of ψ and θ, we have

$$\begin{aligned}
\psi(\delta(x_{n+1}, Tz)) &\leq \psi(\delta(Tx_n, Tz)) \\
&\leq \psi\Big(\frac{1}{2}\ [D(x_n,\ Tz) + D(z,\ Tx_n)]\Big) - \theta(\delta(x_n,\ Tz), \delta(z,\ Tx_n)) \\
&\leq \psi\Big(\frac{1}{2}\ [D(x_n,\ Tz) + d(z,\ x_{n+1})]\Big) - \theta(\delta(x_n,\ Tz), d(z,\ x_{n+1})).
\end{aligned}$$

Since $\theta(\delta(x_n,\ Tz), d(z,\ x_{n+1})) \geq 0$, we have

$$\psi(\delta(x_{n+1},\ Tz)) \leq \psi\Big(\frac{1}{2}\ [D(x_n,\ Tz) + d(z,\ x_{n+1})]\Big),$$

which, by monotone property of ψ, implies that

$$\delta(x_{n+1},\ Tz) \leq \frac{1}{2}\ [D(x_n,\ Tz) + d(z,\ x_{n+1})].$$

Taking limit as $n \to \infty$ in the above inequality and using (2.9), we have

$$\delta(z,\ Tz) \leq \frac{1}{2}\ D(z,\ Tz).$$

Similarly as in the proof of Theorem 10, we can show that $\delta(z,\ Tz) = 0$, that is, $Tz = \{z\}$, that is, $z \in Tz$. Therefore, z is a fixed point of T. □

Example 12. Let $X = [0, \infty)$ be equipped with usual metric d. Then (X, d) is a complete metric space. Let $T : X \to B(X)$ be defined as follows:

$$Tx = \begin{cases} \{\frac{x}{16}\}, & \text{if} \quad 0 \leq x \leq 1. \\ \Big[x + \frac{1}{x} - \frac{1}{n},\ n\Big], & \text{if} \quad n-1 \leq x \leq n \text{ with } n \geq 2. \end{cases}$$

Let $\alpha,\ \beta : X \to [0, \infty)$ be respectively defined as follows:

$$\alpha(x) = \begin{cases} e^x, & \text{if} \quad 0 \leq x \leq 1 \\ \dfrac{1}{4}, & \text{if} \quad x > 1, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} x+1, & \text{if} \quad 0 \leq x \leq 1 \\ 0, & \text{if} \quad x > 1. \end{cases}$$

Let ψ: $[0,\infty) \to [0,\infty)$ and $\theta : [0,\infty)^2 \to [0,\infty)$ be respectively defined as follows:

$$\psi(t) = \begin{cases} \frac{t}{2}, & \text{if } \ 0 \leq t \leq 1 \\ \frac{1}{2}, & \text{if } \ t > 1. \end{cases} \qquad \text{and} \ \ \theta(x,y) = \frac{x+y}{8}, \text{ for } x,\ y \geq 0.$$

Clearly, $\psi \in \Psi$ and $\theta \in \Theta$.

(i) Suppose that $x \in X$ and $\alpha(x) \geq 1$. Then $x \in [0,\ 1]$ and $Tx \subseteq [0,\ 1]$. It follows that $\beta(u) \geq 1$ for all $u \in Tx$. Similarly, if $y \in X$ and $\beta(y) \geq 1$, it can be shown that $\alpha(v) \geq 1$ for all $v \in Ty$. Therefore, T is a cyclic (α,β)-admissible mapping.

(ii) $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for every $x \in [0,\ 1]$.

(iii) Suppose that $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all n. Then $\{x_n\}$ is a sequence in $[0,\ 1]$ and also $x \in [0,\ 1]$. Then it follows that $\beta(x) \geq 1$.

(iv) Let $x, y \in X$ with $\alpha(x)\ \beta(y) \geq 1$. Now, $\alpha(x)\ \beta(y) \geq 1$ implies that $x, y \in [0,\ 1]$. Therefore, it is required to verify the inequalities of Theorems 10 and 11 only for $x, y \in [0,\ 1]$. With out loss of generality, we suppose $x, y \in [0,\ 1]$ with $x \geq y$ [calculation is similar for $x \leq y$]. Then

$\psi(\delta(Tx,Ty)) = \psi(\frac{|x-y|}{16}) = \frac{|x-y|}{32} = \frac{x-y}{32}$,

$\psi(\frac{1}{2}\ [D(x,\ Tx) + D(y,\ Ty)]) = \psi(\frac{x-\frac{x}{16}+y-\frac{y}{16}}{2}) = \frac{x+y}{4} - \frac{x+y}{64} = \frac{15(x+y)}{64}$,

$\psi(\frac{1}{2}\ [D(x,\ Ty) + D(y,\ Tx)]) = \psi\Big(\frac{|x-\frac{y}{16}|+|y-\frac{x}{16}|}{2}\Big) = \frac{x-\frac{y}{16}+|y-\frac{x}{16}|}{4}$,

$\theta(\delta(x,\ Tx), \delta(y,\ Ty)) = \theta(x - \frac{x}{16},\ y - \frac{y}{16}) = \frac{x-\frac{x}{16}+y-\frac{y}{16}}{8} = \frac{15(x+y)}{128}$,

$\theta(\delta(x,\ Ty), \delta(y,\ Tx)) = \theta(|\ x - \frac{y}{16}\ |,\ |\ y - \frac{x}{16}\ |) = \frac{x-\frac{y}{16}+|y-\frac{x}{16}|}{8}$,

Now, $\psi(\delta(Tx,Ty)) = \frac{x-y}{32} \leq \frac{x+y}{32} \leq \frac{15(x+y)}{128} = \frac{15(x+y)}{64} - \frac{15(x+y)}{128} = \psi(\frac{1}{2}\ [D(x,\ Tx) + D(y,\ Ty)]) - \theta(\delta(x,\ Tx), \delta(y,\ Ty))$. Therefore, the inequality of Theorem 10 is satisfied for all $x, y \in X$ with $\alpha(x)\ \beta(y) \geq 1$ and hence T is a multivalued $(\alpha,\beta,\psi,\theta)$-weak Kannan type mapping.

Again,

$$\psi\Big(\frac{1}{2}\ [D(x,\ Ty) + D(y,\ Tx)]\Big) = \begin{cases} \frac{15}{64}(x+y), & \text{if } \ y \geq \frac{x}{16} \\ \frac{17(x-y)}{64}, & \text{if } \ y < \frac{x}{16}, \end{cases}$$

$$\theta(\delta(x,\ Ty), \delta(y,\ Tx)) = \begin{cases} \frac{15(x+y)}{128}, & \text{if } \ y \geq \frac{x}{16} \\ \frac{17(x-y)}{128}, & \text{if } \ y < \frac{x}{16}, \end{cases}$$

If $y \geq \frac{x}{16}$, then $\psi(\delta(Tx,Ty)) = \frac{x-y}{32} \leq \frac{x+y}{32} \leq \frac{15(x+y)}{128} = \frac{15(x+y)}{64} - \frac{15(x+y)}{128} = \psi(\frac{1}{2}\ [D(x,Ty) + D(y,Tx)]) - \theta(\delta(x,Ty), \delta(y,Tx))$. If $y < \frac{x}{16}$, then $\psi(\delta(Tx,Ty)) = \frac{x-y}{32} \leq \frac{17(x-y)}{128} = \frac{17(x-y)}{64} - \frac{17(x-y)}{128} = \psi(\frac{1}{2}\ [D(x,Ty) + D(y,Tx)]) - \theta(\delta(x,Ty), \delta(y,Tx))$.

Therefore, the inequality of Theorem 11 is satisfied for all $x, y \in X$ with $\alpha(x)\, \beta(y) \geq 1$ and hence T is a multivalued $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping.

Therefore, all the conditions of Theorems 10 and 11 are satisfied and the set of fixed points of T is {0, 2, 3, 4,...n,...}.

In Theorems 10 and 11, considering $\alpha(x) = \beta(x) = 1$ for all $x \in X$, we have the following corollaries.

Corollary 13. *Let (X, d) be a complete metric space and $T : X \to B(X)$ be a multivalued mapping. Suppose there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi(\delta(Tx, Ty)) \leq \psi\Big(\frac{1}{2}\,[D(x, Tx) + D(y, Ty)]\Big) - \theta(\delta(x, Tx), \delta(y, Ty)), \text{ for all } x, y \in X.$$

Then T has a fixed point.

Corollary 14. *Let (X, d) be a complete metric space and $T : X \to B(X)$ be a multivalued mapping. Suppose there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi(\delta(Tx, Ty)) \leq \psi\Big(\frac{1}{2}\,[D(x, Ty) + D(y, Tx)]\Big) - \theta(\delta(x, Ty), \delta(y, Tx)), \text{ for all } x, y \in X.$$

Then T has a fixed point.

3 Consequences in singlevalued cases

In this section we obtain some consequences of the corresponding results of Section 2 in the cases of singlevalued mappings. For the following results the monotone property of θ is not necessary.

Theorem 15. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exist $\alpha,\ \beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that T is a $(\alpha, \beta, \psi, \theta)$-weak Kannan type mapping and also the following conditions hold.*

- *(i) T is a cyclic (α, β)-admissible mapping,*
- *(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
- *(iii) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.*

Then T has a fixed point.

Proof. We know that for every $x \in X$, $\{x\} \in B(X)$. We define a mapping $S : X \to B(X)$ as $Sx = \{Tx\}$, for $x \in X$. Then all the conditions of the theorem reduce to the conditions of Theorem 10 and hence by application of Theorem 10, there exists $u \in X$ such that $\{u\} = Su$. By the definition of S, we have $Su = \{Tu\}$. Hence $u = Tu$, that is, u is a fixed point of T. □

Theorem 16. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exist α, $\beta : X \to [0, \infty)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that T is a $(\alpha, \beta, \psi, \theta)$-weak Chatterjea type mapping and also the conditions (i), (ii) and (iii) of Theorem 15 hold. Then T has a fixed point.*

Proof. Arguing similarly as in the proof of Theorem 15 and by an application of Theorem 11, we have the required proof. □

In Theorems 15 and 16, considering $\alpha(x) = \beta(x) = 1$ for all $x \in X$, we have the following corollaries.

Corollary 17. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi(d(Tx, Ty)) \leq \psi\Big(\frac{1}{2}\ [d(x, Tx) + d(y, Ty)]\Big) - \theta(d(x, Tx), d(y, Ty)),\ \textit{for all } x, y \in X.$$

Then T has a fixed point.

Corollary 18. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi(d(Tx, Ty)) \leq \psi\Big(\frac{1}{2}\ [d(x, Ty) + d(y, Tx)]\Big) - \theta(d(x, Ty), d(y, Tx)),\ \textit{for all } x, y \in X.$$

Then T has a fixed point.

In Theorems 15 and 16, considering $\alpha(x) = \beta(x) = 1$ for all $x \in X$, $\psi(t) = t$ for all $t \in [0, \infty)$ and $\theta(u, v) = (1-k)\dfrac{u+v}{2}$ for all $(u, v) \in [0, \infty)^2$, where $k \in [0, 1)$, we have respectively Kannan [19] and Chatterjea [4] fixed point theorems.

Corollary 19. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exists k with $0 \leq k < 1$ such that*

$$d(Tx, Ty) \leq \frac{k}{2}\ [d(x,\ Tx) + d(y,\ Ty)],\ \textit{for all } x, y \in X.$$

Then T has a fixed point.

Corollary 20. *Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose there exists k with $0 \leq k < 1$ such that*

$$d(Tx,\ Ty) \leq \frac{k}{2}\ [d(x,\ Ty) + d(y,\ Tx)],\ \textit{for all } x, y \in X.$$

Then T has a fixed point.

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