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# ON NEW SUBCLASS OF MEROMORPHICALLY CONVEX FUNCTIONS WITH POSITIVE COEFFICIENTS

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**Abstract**. In this paper we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta$-neighborhoods for the class $\sigma_p(\alpha)$.

## 1 Introduction

Let $A$ denote the class of analytic functions $f$ defined on the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Denote by $S$, the subclass of $A$ consisting of functions $f(z)$ that are univalent in $E$. A function $f(z)$ belonging to $A$ is said to be starlike of order $\alpha$ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in E), \tag{1.2}$$

for some $\alpha$, $(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $A$ consisting of functions which are starlike of order $\alpha$ in $E$.

A function $f(z)$ belonging to $A$ is said to be a convex of order $\alpha$ if it satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in E), \tag{1.3}$$

for some $\alpha, (0 \le \alpha < 1)$. We denote this class with $K(\alpha)$ the subclass of $A$ consisting of functions which are convex of order $\alpha$ in $E$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $E$ respectively.

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Also, denote by $T$ the subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \ (z \in E) \tag{1.4}$$

and let $T^*(\alpha) = T \cap S^*(\alpha)$, $C(\alpha) = T \cap K(\alpha)$. The classes $T^*(\alpha)$ and $C(\alpha)$ possess some interesting properties and have been extensively studied by Silverman [18] and others.

Following Goodman [7, 8], Ronning [14, 15] introduced and studied the following subclasses

(i) A function $f \in A$ is said to be in the class $S_p(\alpha,\beta)$ of uniformly $\beta-$starlike functions if it satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in E \tag{1.5}$$

$-1 < \alpha \leq 1$ and $\beta \geq 0$.

(ii) A function $f \in A$ is said to be in the class $UCV(\alpha,\beta)$ of uniformly $\beta-$convex functions if it satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in E \tag{1.6}$$

$-1 < \alpha \leq 1$ and $\beta \geq 0$.

Indeed it follows from (1.6) and (1.5) that

$$f \in UCV(\alpha,\beta) \Leftrightarrow zf' \in SP(\alpha,\beta). \tag{1.7}$$

Further Ahuja et al. [1], Bharathi et al. [4], Murugusundaramoorthy and Magesh [12] and others have studied and investigated interesting properties for the classes $UCV(\alpha,\beta)$ and $SP(\alpha,\beta)$.

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \tag{1.8}$$

which are regular in domain $E = \{z : 0 < z < 1\}$ with a simple pole at the origin with residue 1 there.

Let $\sum_s$, $\sum^*(\alpha)$ and $\sum_k(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\sum$ that are univalent, meromorphically starlike of order $\alpha$ and meromorphically convex of order $\alpha$ respectively. Analytically $f(z)$ of the form (1.8) is in $\sum^*(\alpha)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in E. \tag{1.9}$$

Similarly, $f \in \sum_k$ *(α)* if and only if, $f$(z) is of the form (1.8) and satisfies

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, z \in E \tag{1.10}$$

and similar classes of meromorphically univalent functions have been extensively studied by Pommerenke [13], Clunie [5], Royster [16] and others [2, 3, 10, 11, 19].

Since, to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of $\sum_s$ that has properties analogous to those of $T^*$ *(α)*. Juneja and Reddy [9] introduced the class $\sum_p$of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \quad a_m \geq 0, \tag{1.11}$$

$$\Sigma_p^*(\alpha) = \Sigma_p \cap \Sigma^*(\alpha).$$

For functions $f(z)$ in the class $\sum_p$, we define a linear operator $D^n$ by the following form

$$\begin{aligned}
D^0 f(z) &= f(z) \\
D^1 f(z) &= \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \cdots \quad = \frac{(z^2 f(z))'}{z} \\
D^2 f(z) &= D(D^1 f(z)) \\
&\vdots
\end{aligned}$$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m = \frac{(z^2 D^{n-1} f(z))'}{z}, \text{ for } n = 1, 2, \cdots \tag{1.12}$$

Now, we define a new subclass $\sigma_p(\alpha)$ of $\sum_p$.

**Definition 1.** *For $-1 \leq \alpha < 1$, we let $\sigma_p(\alpha)$ be the subclass of $\sum_p$ consisting of the form* (1.11) *and satisfying the analytic criterion*

$$-Re\left\{\frac{z(D^n f(z))'}{D^n f(z)} + \alpha\right\} > \left|\frac{z(D^n f(z))'}{D^n f(z)} + 1\right|, \tag{1.13}$$

$D^n f(z)$ *is given by* (1.12) *.*

The main object of this paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$ neighborhoods for the class $\sigma_p(\alpha)$.

## 2 Coefficient inequality

In this section we obtain the coefficient bounds of function $f(z)$ for the class $\sigma_p(\alpha)$.

**Theorem 2.** *A function $f(z)$ of the form (1.11) is in $\sigma_p(\alpha)$ if*

$$\sum_{m=1}^{\infty}(m+2)^n[(2m+3)-\alpha]\ |a_m| \ \leq \ (1-\alpha), \quad -1\leq\alpha<1. \tag{2.1}$$

*Proof.* It suffices to show that

$$\left|\frac{z(D^nf(z))'}{D^nf(z)}+1\right|+Re\left\{\frac{z(D^nf(z))'}{D^nf(z)}+1\right\}\leq(1-\alpha).$$

We have

$$\left|\frac{z(D^nf(z))'}{D^nf(z)}+1\right|+Re\left\{\frac{z(D^nf(z))'}{D^nf(z)}+1\right\}\leq \qquad 2\left|\frac{z(D^nf(z))'}{D^nf(z)}+1\right|$$

$$\leq \frac{2\sum\limits_{m=1}^{\infty}[(m+2)]^n(m+1)|a_m||z^m|}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}[(m+2)]^n|a_m||z^m|}$$

Letting $z\to1$ along the real axis, we obtain

$$\frac{2\sum\limits_{m=1}^{\infty}[(m+2)]^n(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}[(m+2)]^n|a_m|}.$$

The above expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}[(m+2)]^n[2m+3]\ |a_m| \ \leq \ (1-\alpha).$$

Hence the theorem is completed. □

**Corollary 3.** *Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha)$. Then*

$$a_m\leq\frac{(1-\alpha)}{\sum\limits_{m=1}^{\infty}(2m+3)^n[2m+3-\alpha]}, \quad (m\geq1).$$

*Equality holds for the function of the form*

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]}\ z^m.$$

# 3 Distortion Theorems

In this section we obtain the sharp for the Distortion theorems of the form (1.11).

**Theorem 4.** *Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha)$. Then for* $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{(1-\alpha)}{3^n[5-\alpha]}\, r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)}{3^n[5-\alpha]} r \tag{3.1}$$

*with equality for the function*

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n[5-\alpha]}\, z, \ \ at \ \ z = r, ir. \tag{3.2}$$

*Proof.* Suppose $f(z)$ is in $\sigma_p(\alpha)$. In view of Theorem 2, we have
$3^n[5-\alpha] \sum\limits_{m=1}^{\infty} a_m \leq \sum\limits_{m=1}^{\infty} (m+2)^n[2m+3-\alpha] \leq (1-\alpha)$
which evidently yields $\sum\limits_{m=1}^{\infty} a_m \leq \frac{1-\alpha}{3^n[5-\alpha]}$.
Consequently, we obtain

$$|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \leq \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m \leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m \leq \frac{1}{r} + \frac{1-\alpha}{3^n[5-\alpha]}\, r.$$

Also

$$|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m \geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \geq \frac{1}{r} - \frac{1-\alpha}{3^n[5-\alpha]}\, r.$$

Hence the results (3.1) follow. □

**Theorem 5.** *Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha)$. Then for* $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{1-\alpha}{3^n[5-\alpha]} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{3^n[5-\alpha]}.$$

*The result is sharp, the extremal function being of the form (3.2).*

*Proof.* From Theorem 2, we have
$3^n[5-\alpha] \sum\limits_{m=1}^{\infty} m a_m \leq \sum\limits_{m=1}^{\infty} (m+2)^n[2m+3-\alpha] \leq (1-\alpha)$
which evidently yields $\sum\limits_{m=1}^{\infty} m a_m \leq \frac{1-\alpha}{3^n[5-\alpha]}$.
Consequently, we obtain

$$|f'(z)| \leq \left|\frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}\right| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \leq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n[5-\alpha]}.$$

Also,

$$|f'(z)| \geq \left| \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1} \right| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m \geq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n[5-\alpha]}.$$

This completes the proof. □

# 4 Class preserving integral operators

In this section we consider the class preserving integral operator of the form (1.11) .

**Theorem 6.** *Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha)$. Then*

$$f(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0 \tag{4.1}$$

*belongs to the class $\sigma_p[\delta(\alpha, n, c)]$, where*

$$\delta(\alpha, n, c) = \frac{3^n[5-\alpha](c+2) - (1-\alpha)c}{3^n[5-\alpha](c+2) + (1-\alpha)c}. \tag{4.2}$$

*The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n[5-\alpha]} z$.*

*Proof.* Suppose $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_p(\alpha)$. We have

$f(z) = cz^{-c-1} \int\limits_0^z t^c f(t)dt = \frac{1}{z} + \sum\limits_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0.$

It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta} \frac{c}{c+m+1} a_m \leq 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_p(\alpha)$, we have

$$\sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{1-\alpha} |a_m| \leq 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\frac{(m+\delta)}{(1-\delta)} \frac{c}{(c+m+1)} \leq \frac{(m+2)^n[2m+3-\alpha]}{1-\alpha}, \text{ for each } m.$$

Solving for $\delta$, we obtain

$$\delta \leq \frac{(m+2)^n[2m+3-\alpha](c+m+1) - mc(1-\alpha)}{(m+2)^n[2m+3-\alpha](c+m+1) + c(1-\alpha)} = G(m). \tag{4.5}$$

Then $G(m+1) - G(m) > 0$, for each $m$.
Hence $G(m)$ is increasing function of $m$, since $G(1) = \frac{3^n(5-\alpha)(c+2)-c(1-\alpha)}{3^n(5-\alpha)(c+2)+c(1-\alpha)}$.
The result follows. □

**Theorem 7.** *If the function* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *is in* $\sigma_p(\alpha)$ *then* $f(z)$ *is meromorphically convex of order* $\delta$ $(0 \leq \delta < 1)$ *in* $|z| < r = r(\alpha, \delta)$, *where*

$$r(\alpha,\delta) = \inf_{n\geq 1} \left\{ \frac{(1-\delta)(m+2)^n[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}}.$$

*The result is sharp.*

*Proof.* Let $f(z)$ be in $\sigma_p(\alpha)$. Then, by Theorem 2, we have

$$\sum_{m=1}^{\infty} (m+2)^n[2m+3-\alpha]|a_m| \leq (1-\alpha). \tag{4.6}$$

It is sufficient to show that $\left|2 + \frac{zf''(z)}{f'(z)}\right| \leq (1-\delta)$ for $|z| < r = r(\alpha,\delta)$, where $r(\alpha,\delta)$ is specified in the statement of the theorem. Then

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| = \left| \frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2} + \sum\limits_{m=1}^{\infty} m a_m z^{m-1}} \right| \leq \frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m|z|^{m+1}}{1 - \sum\limits_{m=1}^{\infty} m a_m |z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty} \frac{m(m+2-\delta)}{1-\delta} a_m |z|^{m+1} \leq 1. \tag{4.7}$$

By (4.6), it follows that (4.7) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \leq \frac{(m+2)^n[2m+3-\alpha]}{1-\alpha}|a_m|, \quad m \geq 1$$

$$\text{or} \quad |z| \leq \left\{ \frac{(1-\delta)(m+2)^n[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}}. \tag{4.8}$$

Setting $|z| = r(\alpha,\delta)$ in (4.8), the result follows. The result is sharp for the function

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]} z^m, \quad m \geq 1.$$

□

## 5 Convex linear combinations and convolution properties

In this section we obtain sharp for $f(z)$ is meromorphically convex of order $\delta$ and necessary and sufficient condition for $f(z)$ is in the class $\sigma_p(\alpha)$. And also proved that convolution is in the class $\sigma_p(\alpha)$.

**Theorem 8.** *Let* $f_0(z) = \frac{1}{z}$ *and* $f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]} z^m, \quad m \geq 1$. *Then* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *is in the class* $\sigma_p(\alpha)$ *if and only if it can be expressed in the form* $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$, *where* $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ *and* $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$.

*Proof.* Let $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$ with $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ and $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$. Then

$$f(z) = \omega_0 f_0(z) + \sum_{m=1}^{\infty} \omega_m f_m(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \omega_m \frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]} z^m.$$

Since

$$\sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} \omega_m \frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]} = \sum_{m=1}^{\infty} \omega_m = 1 - \omega_0 \leq 1.$$

By Theorem 2, $f(z)$ is in the class $\sigma_p(\alpha)$.

Conversely suppose that the function $f(z)$ is in the class $\sigma_p(\alpha)$. Then

$$a_m \leq \frac{(1-\alpha)}{(m+2)^n[2m+3-\alpha]} z^m, \; m \geq 1.$$

$$\omega_m = \sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} a_m \quad \text{and} \quad \omega_0 = 1 - \sum_{m=1}^{\infty} \omega_m.$$

It follows that $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$.

This completes the proof of the theorem. □

For the functions $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ belongs to $\sum\limits_p$, we denoted by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

**Theorem 9.** *If the function* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ *are in the class* $\sigma_p(\alpha)$ *then* $(f * g)(z)$ *is in the class* $\sigma_p(\alpha)$*.*

*Proof.* Suppose $f(z)$ and $g(z)$ are in $\sigma_p(\alpha)$. By Theorem 2, we have

$$\sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} a_m \leq 1$$

$$\text{and } \sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} b_m \leq 1\ .$$

Since $f(z)$ and $g(z)$ are regular are in $E$, so is $(f * g)(z)$. Further more

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} a_m b_m \\
&\leq \sum_{m=1}^{\infty} \left\{ \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} \right\}^2 a_m b_m \\
&\leq \left( \sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} a_m \right) \left( \sum_{m=1}^{\infty} \frac{(m+2)^n[2m+3-\alpha]}{(1-\alpha)} b_m \right) \\
&\leq 1.
\end{aligned}$$

Hence, by Theorem 2, $(f * g)(z)$ is in the class $\sigma_p(\alpha)$. □

## 6 Neighborhoods for the class $\sigma_p(\alpha, \gamma)$

In this section we define the $\delta-$neighborhood of a function $f(z)$ and establish a relation between $\delta-$neighborhood and $\sigma_p((\alpha, \beta, \gamma, \lambda)$ class of a function.

**Definition 10.** *A function* $f \in \sum\limits_p$ *is said to in the class* $\sigma_p(\alpha, \gamma)$ *if there exists a function* $g \in \sigma_p(\alpha)$ *such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < (1-\gamma), \quad z \in E,\ 0 \leq \gamma < 1. \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruschweyh [17], we defined the $\delta-$neighborhood of a function $f \in \sum_p$ by

$$N_\delta(f) = \left\{ g \in \sum_p \ \mid\ g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m \ :\ \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \right\}. \tag{6.2}$$

**Theorem 11.** *If $g \in \sigma_p(\alpha)$ and*

$$\gamma = 1 - \frac{\delta[5-\alpha]}{4} \tag{6.3}$$

*then $N_\delta(g) \subset \sigma_p(\alpha, \gamma)$.*

*Proof.* Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \tag{6.4}$$

which implies the coefficient of inequality $\sum\limits_{m=1}^{\infty} |a_m - b_m| \leq \delta, \quad m \in \mathbb{N}$.

Since $g \in \sigma_p(\alpha)$, we have $\sum\limits_{m=1}^{\infty} b_m = \frac{1-\alpha}{5-\alpha}$.

So that $\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta[5-\alpha]}{4} = 1 - \gamma$, provided $\gamma$ is given by (6.3).

Hence, by Definition, $f \in \sigma_p(\alpha, \gamma)$ for $\gamma$ given by (6.3), which completes the proof of theorem. □

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