Theory and Applications of Categories, Vol. 21, No. 8, 2008, pp. 152–171.

APPROXIMATE MAL'TSEV OPERATIONS

Dedicated to Walter Tholen on the occasion of his 60th birthday

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ABSTRACT. Let X and A be sets and $\alpha : X \to A$ a map between them. We call a map $\mu : X \times X \times X \to A$ an *approximate Mal'tsev operation with approximation* α, if it satisfies $\mu(x, y, y) = \alpha(x) = \mu(y, y, x)$ for all $x, y \in X$. Note that if $A = X$ and the approximation α is an identity map, then μ becomes an ordinary Mal'tsev operation. We prove the following two characterization theorems: a category $\mathbb{X}$ is a Mal'tsev category if and only if in the functor category $\mathbf{Set}^{\mathbb{X}^{\mathrm{op}}\times\mathbb{X}}$ there exists an internal approximate Mal'tsev operation $\hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \to A$ whose approximation α satisfies a suitable condition; a regular category $\mathbb{X}$ with finite coproducts is a Mal'tsev category, if and only if in the functor category $\mathbb{X}^{\mathbb{X}}$ there exists an internal approximate Mal'tsev co-operation $A \to 1_{\mathbb{X}} + 1_{\mathbb{X}} + 1_{\mathbb{X}}$ whose approximation α is a natural transformation with every component a regular epimorphism in $\mathbb{X}$. Note that in both of these characterization theorems, if require further the approximation α to be an identity morphism, then the conditions there involving α become equivalent to $\mathbb{X}$ being a naturally Mal'tsev category.

1. Introduction

A *Mal'tsev category* is usually defined as a category $\mathbb{X}$ with finite limits such that

(C1) every reflexive internal relation in $\mathbb{X}$ is an equivalence relation,

although its original definition in [3] required in addition $\mathbb{X}$ to be exact in the sense of M. Barr [1]. The roots of this concept go back to the work [14] of A. I. Mal'tsev, where it was shown that for a variety $\mathbb{X}$ of universal algebras,

(C2) the composition of congruences on any object in $\mathbb{X}$ is commutative

if and only if

(C3) the algebraic theory of $\mathbb{X}$ contains a ternary term μ satisfying the term equations

$$\mu(x, y, y) = x = \mu(y, y, x). \tag{1}$$

The second author acknowledges the support of South African Claude Leon Foundation, INTAS (06-1000017-8609), and Georgian National Science Foundation (GNSF/ST06/3-004).

Received by the editors 2008-02-02 and, in revised form, 2008-11-11.

Published on 2008-11-14 in the Tholen Festschrift.

2000 Mathematics Subject Classification: 18C99, 08B05.

Key words and phrases: Mal'tsev category, Mal'tsev operation, unital category, strongly unital category, subtractive category.

Such varieties were later called *Mal'tsev varieties* (e.g. see [17]). In [6] G. D. Findlay showed that for a variety $\mathbb{X}$, (C3) implies (C1), and the converse implication was obtained by H. Werner in [19]. The following similar "relational reformulation" of (C3) is due to J. Lambek [11]:

(C4) every internal relation R in $\mathbb{X}$ is *difunctional*, i.e. it satisfies

$$(x_1 R y_2 \ \wedge \ x_2 R y_2 \ \wedge \ x_2 R y_1) \ \Rightarrow \ x_1 R y_1.$$

An advantage of conditions (C1) and (C4), to (C2) and (C3), is that (C1) and (C4) can be formulated in *any* category; indeed, an internal relation in a category is simply a diagram

$$X \xleftarrow{r_1} R \xrightarrow{r_2} Y$$

where r_1 and r_2 are *jointly monomorphic*, i.e. for any two parallel morphisms f, g with codomain R, we have

$$(r_1 f = r_1 g \ \wedge \ r_2 f = r_2 g) \ \Rightarrow \ f = g$$

(if the category has products, then this, of course, amounts to saying that the induced morphism $(r_1, r_2) : R \to X \times Y$ is a monomorphism). Then "reflexivity", "symmetry", "transitivity" and "difunctionality" of R can be expressed using the Yoneda embedding; that is, we say that R is reflexive, symmetric, transitive or difunctional (assuming $X = Y$ for the first three cases), if for every object S in the given category, the relation

$$\hom(S, X) \xleftarrow{\hom(1_S, r_1)} \hom(S, R) \xrightarrow{\hom(1_S, r_2)} \hom(S, Y)$$

between sets is, respectively, reflexive, symmetric, transitive or difunctional in the usual sense. So, (C1) and (C4) make sense when $\mathbb{X}$ is an arbitrary category. In [4] the equivalence of (C1) and (C4) was shown for arbitrary categories with finite limits (for general categories we only have (C4)$\Rightarrow$(C1) — an observation which is clear but never explicitly written, as far as we know). There are also many other relational conditions equivalent to (C1) and (C4) for finitely complete categories (see [8] for a unified exposition of many of those relational conditions).

Condition (C2) is also *categorical*, but it can be formulated only in special categories such as exact categories.

For exact categories, the equivalence of (C1), (C2) and (C4) was obtained by J. Meisen in [15]; in [5] T. Fay generalized this result to categories equipped with a suitable factorization system, and two decades later in [3] the same was proved in the more special case of regular categories.

Condition (C3) cannot be formulated even for exact categories, since we do not have "terms" there. However, we can still give a "categorical form" to (C3) by using F. W. Lawvere's observation [12] that an n-ary term in the algebraic theory of a variety $\mathbb{X}$ is the same as an internal n-ary operation $U^n \to U$ in the functor category $\mathbf{Set}^{\mathbb{X}}$, where U is the forgetful functor $U : \mathbb{X} \to \mathbf{Set}$. Then (C3) can be rephrased as follows:

(C3′) The object U in the functor category $\mathbf{Set}^{\mathbb{X}}$ admits an internal *Mal'tsev operation*, i.e. a ternary operation μ satisfying (1).

Now if we take $\mathbb{X}$ to be an abstract category, then the above condition makes sense as long as we have a specified functor $U : \mathbb{X} \to \mathbf{Set}$. In [16] M. C. Pedicchio showed the equivalence of (C1) and (C3′) in the case when U is a monadic functor.

In the present paper we add two new conditions to the list of conditions above, which are analogous to (C3′) but they have the advantage that the functor U is determined uniquely by $\mathbb{X}$, so that those conditions are on a category $\mathbb{X}$, rather than a pair $(\mathbb{X}, U)$, just as conditions (C1), (C2) and (C4). The main "trick" is to replace Mal'tsev operations with "approximate Mal'tsev operations":

Let X, A be sets and consider a map $\mu : X \times X \times X \to A$ and a map $\alpha : X \to A$. We say that μ is an *approximate Mal'tsev operation on X with approximation α*, if

$$\mu(x, y, y) = \alpha(x) = \mu(y, y, x) \tag{2}$$

holds true for all $x, y \in X$. In particular, an approximate Mal'tsev operation with approximation 1_X is a Mal'tsev operation in the usual sense. Our first new condition states that

(C5) the object $\hom_{\mathbb{X}}$ in the functor category $\mathbf{Set}^{\mathbb{X}\times\mathbb{X}^{\mathrm{op}}}$ admits an internal approximate Mal'tsev operation $\mu : \hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \to A$ whose approximation $\alpha : \hom \to A$ satisfies a suitable condition — the condition (*) in Section 3 below.

We show that (C4) and (C5) are equivalent to each other for *any* category $\mathbb{X}$ (Theorem 3.2). The important aspect of this result is that the condition (*) itself is not related to the Mal'tsev property (see Section 5).

Notice that if we require α in (C5) to be an identity natural transformation (in which case the condition (*) would be satisfied trivially), then we get a condition which is equivalent, in the presence of binary products, to the following condition:

(C6) The identity functor $1_{\mathbb{X}}$ admits an internal Mal'tsev operation μ in the functor category $\mathbb{X}^{\mathbb{X}}$.

Categories satisfying (C6) were called *naturally Mal'tsev categories* by P. T. Johnstone in [9]. Note however that condition (C6) is much stronger than conditions (C1)-(C5), even in the case of varieties. Indeed, for instance, a pointed variety (that is, a variety which admits unique constant) is a naturally Mal'tsev variety if and only if it is *abelian*, i.e. it is the variety of R-modules for some ring R. Thus, for instance, the variety of groups is a Mal'tsev variety, but it is not a naturally Mal'tsev variety.

Our second condition is on a category $\mathbb{X}$ with binary coproducts, and it states that

(C7) the identity functor $1_{\mathbb{X}}$ admits an internal approximate Mal'tsev co-operation $\mu : A \to 1_{\mathbb{X}} + 1_{\mathbb{X}} + 1_{\mathbb{X}}$ in the functor category $\mathbb{X}^{\mathbb{X}}$ whose approximation $\alpha : A \to 1_{\mathbb{X}}$ is a natural transformation with every component a strong epimorphism in $\mathbb{X}$.

This condition turns out to be equivalent to (C5) with the following additional requirement on A in (C5): *for every object X in $\mathbb{X}$ the functor $A(X,-) : \mathbb{X} \to$ **Set** is representable* (see Proposition 6.8 below).

We show that (C7) is equivalent to (C4) (and hence to (C1), (C2) and (C5)) for any regular category $\mathbb{X}$ with binary coproducts (Theorem 4.2). And again, if α in (C7) is an identity morphism, then (under the presence of binary products in $\mathbb{X}$) we obtain the condition (C6) (since when $\mathbb{X}$ has binary products and coproducts, the condition (C6) is self-dual) which defines a naturally Mal'tsev category.

2. Preliminaries

We will work in a category $\mathbb{C}$ with binary products and finite colimits. For two objects X and Y in $\mathbb{C}$, by $X \times Y$ and $X + Y$ we denote the product and coproduct, respectively, of X and Y. If $X = Y$ then we also write $X^2 = X \times X$ and $2X = X + X$; more generally, for a natural number n, we write

$$X^n = \underbrace{X \times X \times ... \times X}_{n} \quad \text{and} \quad nX = \underbrace{X + X + ... + X}_{n}.$$

For an n-tuple $X_1, ..., X_n$ of objects in $\mathbb{C}$, and for $i \in \{1, ..., n\}$, we write π_i and ι_i for the i-th product projection $X_1 \times ... \times X_n \to X_i$ and i-th coproduct injection $X_i \to X_1 + ... + X_n$, respectively.

2.1. Definition. *Let $\mu : X^3 \to A$ and $\alpha : X \to A$ be morphisms in $\mathbb{C}$. We say that μ is an approximate Mal'tsev operation with approximation α if the diagram*

$$\begin{tikzcd} X^3 \arrow[r, "\mu"] & A \\ X^2 + X^2 \arrow[u, "\begin{pmatrix} \pi_1 & \pi_2 & \pi_2 \\ \pi_2 & \pi_2 & \pi_1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \pi_1 \\ \pi_1 \end{pmatrix}"'] & X \arrow[u, "\alpha"'] \end{tikzcd} \tag{3}$$

commutes. Note that in this case $\alpha = \mu(1_X, 1_X, 1_X)$, so that α is uniquely determined by μ. We say that μ is a universal approximate Mal'tsev operation when the diagram (3) is a pushout. This is the case if and only if μ is the following coequalizer:

$$\begin{tikzcd} X^2 \arrow[r, shift left=2, "(\pi_1,\pi_2,\pi_2)"] \arrow[r, "(\pi_1,\pi_1,\pi_1)" description] \arrow[r, shift right=2, "(\pi_2,\pi_2,\pi_1)"'] & X^3 \arrow[r, "\mu"] & A \end{tikzcd}$$

If $A = X$ and $\alpha = 1_X$ then μ is said to be a Mal'tsev operation on X.

We will also consider approximate Mal'tsev operations in a category $\mathbb{C}$ with binary products, which may not have finite colimits (and in particular, binary coproducts). In this case, instead of requiring the diagram (3) to commute, we should require the equalities $\mu(\pi_1, \pi_2, \pi_2) = \alpha\pi_1 = \mu(\pi_2, \pi_2, \pi_1)$ to hold, i.e. the commutativity of the two diagrams

$$\begin{tikzcd} X^3 \arrow[r, "\mu"] & A \\ X^2 \arrow[u, "(\pi_1,\pi_2,\pi_2)"] \arrow[r, "\pi_1"'] & X \arrow[u, "\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} X^3 \arrow[r, "\mu"] & A \\ X^2 \arrow[u, "(\pi_2,\pi_2,\pi_1)"] \arrow[r, "\pi_1"'] & X \arrow[u, "\alpha"'] \end{tikzcd}$$

2.2. Observation. *A morphism $\mu : X^3 \to A$ in $\mathbb{C}$ is an approximate Mal'tsev operation with approximation $\alpha : X \to A$ if and only if for any object C in $\mathbb{C}$ and for any two morphisms $x, y : C \to X$ we have $\mu(x, y, y) = \alpha x = \mu(y, y, x)$.*

Note that there is a bijection between Mal'tsev operations on X and left inverses of the approximation α of the universal approximate Mal'tsev operation on X. In particular, this gives that α is a split monomorphism if and only if there exists a Mal'tsev operation on X. When $\mathbb{C} = \mathbf{Set}$, every X has a Mal'tsev operation and hence α is always an injective map.

2.3. Lemma. *Let X be a set and μ a universal approximate Mal'tsev operation on X with approximation α. Then for any four elements $x, y, z, t \in X$ we have*

$$\mu(x, y, z) = \alpha(t) \;\Rightarrow\; (x = t \,\wedge\, y = z) \,\vee\, (x = y \,\wedge\, z = t).$$

Proof. This follows from the general fact that for a pushout

$$\begin{tikzcd} Y \arrow[r, "m"] & A \\ Z \arrow[u, "f"] \arrow[r, "g"'] & X \arrow[u, "a"'] \end{tikzcd}$$

in $\mathbf{Set}$, if a is a monomorphism then for any $y \in Y$ and $x \in X$ we have

$$m(y) = a(x) \;\Rightarrow\; \exists_{z \in Z}((f(z) = y) \wedge (g(z) = x)). \tag{4}$$

This general fact is well known in the case when f is also a monomorphism, since then it simply states that the diagram above is a pullback (e.g. see Corollary 1.28 in [10]). In our case f may not be a monomorphism, but then we can reduce this more general case to the case of a monomorphic f as follows: decompose f as $f = f'e$ with f' an injection and e a surjection. Then we can obtain the above pushout square by pasting together two pushout squares, as indicated in the diagram

$$\begin{tikzcd} Y \arrow[r, "m"] & A \\ \bullet \arrow[u, "f'"] \arrow[r] & \bullet \arrow[u, "f''"] \\ Z \arrow[u, "e"] \arrow[r, "g"'] & X \arrow[u, "e'"] \arrow[uu, bend right, "a"'] \end{tikzcd}$$

Since e' is a pushout of a surjection, it is itself surjective and since a is injective and $f''e' = a$ we obtain that e' is also injective, and hence bijective. Then, without loss of generality we can assume $e' = 1_X$, so that $f'' = a$. Now the top square in the above diagram is a pullback since it is a pushout and f' is injective. After this, the fact that e is surjective immediately gives (4) for any $y \in Y$ and $x \in X$. ∎

Suppose $\mathbb{C}$ is a functor category $\mathbb{C} = \mathbb{D}^{\mathbb{E}}$, where $\mathbb{D}$ is a category with binary products, and $\mathbb{E}$ is an arbitrary category.

2.4. Observations.

(a) A morphism $\mu : X^3 \to A$ in $\mathbb{D}^{\mathbb{E}}$ is an approximate Mal'tsev operation with approximation $\alpha : X \to A$, if and only if for each object E in $\mathbb{E}$ the E-component μ_E of μ is an approximate Mal'tsev operation in $\mathbb{D}$ with approximation the E-component α_E of α. Further, if $\mathbb{D}$ has finite colimits, then μ is universal if and only if so is every μ_E.

(b) Suppose $\mathbb{D}$ has finite colimits. If a functor $X : \mathbb{E} \to \mathbb{D}$ is given, then, taking a universal approximate Mal'tsev operation $\mu_E : (X(E))^3 \to A(E)$ in $\mathbb{D}$ for each object E in $\mathbb{E}$, results in a unique approximate Mal'tsev operation $\mu : X^3 \to A$ in $\mathbb{D}^{\mathbb{E}}$ such that each E-component of μ is μ_E, and, moreover, by (a) above, the μ so constructed is universal.

In the present paper, by a Mal'tsev category we mean a category $\mathbb{X}$ satisfying (C4) from Introduction. Note that we do not assume the existence of finite limits, as this is usually the case.[1] Rephrasing (C4) as a suitable "diagram condition", we obtain the following

[1]The authors thank the Referee for reminding us to say a few words about examples of Mal'tsev categories (in the sense of Definition 2.5) without finite limits: In any non-trivial Mal'tsev variety $\mathbb{C}$, take any full subcategory $\mathbb{X}$ which contains at least one algebra free over a non-empty set of generators, and which is not closed under certain finite limits in $\mathbb{C}$. Then, $\mathbb{X}$ is a Mal'tsev category without finite limits, since the inclusion $\mathbb{X} \hookrightarrow \mathbb{C}$ preserves internal relations and all limits which exist in $\mathbb{X}$. Thus, for instance, the categories of free/infinite groups, rings, and other similar group-like structures are examples.

2.5. Definition. *A category $\mathbb{X}$ is said to be a Mal'tsev category if in $\mathbb{X}$, for any commutative diagram of solid arrows*

$$
\begin{tikzcd}
 & & S \arrow[dl, "x_1"'] \arrow[dr, "y_2"] \arrow[dd, "z_1"] & & \\
 & X & & Y & \\
S \arrow[ur, "x_1"] \arrow[dr, "y_1"'] \arrow[rr, dotted, "z_4"] & & R \arrow[ul, "r_1"'] \arrow[ur, "r_2"] \arrow[dl, "r_2"] \arrow[dr, "r_1"'] & & S \arrow[ul, "y_2"'] \arrow[ll, "z_2"'] \arrow[dl, "x_2"] \\
 & Y & & X & \\
 & & S \arrow[ul, "y_1"] \arrow[ur, "x_2"'] \arrow[uu, "z_3"'] & &
\end{tikzcd}
\tag{5}
$$

with r_1, r_2 jointly monomorphic, there exists the dotted arrow z_4 making the diagram commute.

3. The first characterization theorem

In this section $\mathbb{C}$ is the functor category $\mathbb{C} = \mathbf{Set}^{\mathbb{X}\times\mathbb{X}^{\mathrm{op}}}$, where $\mathbb{X}$ is an arbitrary category. We are interested in approximate Mal'tsev operations $\mu : X^3 \to A$ in $\mathbb{C}$, where $X = \hom_{\mathbb{X}}$. We will consider the following condition on the approximation $\alpha : \hom_{\mathbb{X}} \to A$ of μ:

(*) In $\mathbb{X}$, for any diagram of solid arrows

$$
\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y \\
 & S \arrow[ul, "s_1"] \arrow[ur, "s_2"'] \arrow[uu, dotted, "q"] &
\end{tikzcd}
\tag{6}
$$

with r_1, r_2 jointly monomorphic, the dotted arrow q exists, making the diagram commutative, if and only if there exists an element $a \in A(S, R)$ such that $A(1_S, r_1)(a) = \alpha_{S,X}(s_1)$ and $A(1_S, r_2)(a) = \alpha_{S,Y}(s_2)$.

Notice that the "only if" part in (*) is always true (indeed, for each q there we can take $a = \alpha_{S,R}(q)$).

3.1. Lemma. *In a Mal'tsev category $\mathbb{X}$, consider a diagram of solid arrows (6) with r_1, r_2 jointly monomorphic. The following conditions are equivalent to each other:*

(a) There exists the dotted arrow q making the diagram (6) commute.

(b) *There exist morphisms $z_1, z_2, z_3 : S \to R$ such that $\mu_{S,X}(r_1z_1, r_1z_2, r_1z_3) = \alpha_{S,X}(s_1)$ and $\mu_{S,Y}(r_2z_1, r_2z_2, r_2z_3) = \alpha_{S,Y}(s_2)$ where μ is the universal approximate Mal'tsev operation on* $\hom_{\mathbb{X}}$ *with approximation α.*

PROOF. (a)⇒(b) is obvious (take $z_1 = z_2 = z_3 = q$).

(b)⇒(a): Suppose we have

$$\mu_{S,X}(r_1z_1, r_1z_2, r_1z_3) = \alpha_{S,X}(s_1),$$

$$\mu_{S,Y}(r_2z_1, r_2z_2, r_2z_3) = \alpha_{S,Y}(s_2).$$

By Observations 2.4(a) we have:

- $\mu_{S,X}$ is a universal approximate Mal'tsev operation on $\hom_{\mathbb{X}}(S, X)$ with approximation $\alpha_{S,X}$,
- $\mu_{S,Y}$ is a universal approximate Mal'tsev operation on $\hom_{\mathbb{X}}(S, Y)$ with approximation $\alpha_{S,Y}$.

Now we apply Lemma 2.3 (two times) to obtain from the above equalities that one of the following four systems of equalities holds true:

$$\text{(I)}\begin{cases} r_1z_1 &= s_1, \\ r_1z_2 &= r_1z_3, \\ r_2z_3 &= s_2, \\ r_2z_1 &= r_2z_2; \end{cases} \qquad \text{(II)}\begin{cases} r_1z_1 &= s_1, \\ r_1z_2 &= r_1z_3, \\ r_2z_1 &= s_2, \\ r_2z_2 &= r_2z_3; \end{cases}$$

$$\text{(III)}\begin{cases} r_1z_3 &= s_1, \\ r_1z_1 &= r_1z_2, \\ r_2z_3 &= s_2, \\ r_2z_1 &= r_2z_2; \end{cases} \qquad \text{(IV)}\begin{cases} r_1z_3 &= s_1, \\ r_1z_1 &= r_1z_2, \\ r_2z_1 &= s_2, \\ r_2z_2 &= r_2z_3. \end{cases}$$

In each case, we obtain the desired q as follows:

(I): For $x_1 = s_1, y_1 = s_2$ form a commutative diagram (5), and then take $q = z_4$.

(II): Take $q = z_1$.

(III): Take $q = z_3$.

(IV): For $x_1 = s_2, y_1 = s_1$ form a commutative diagram (5), with the places of z_1 and z_3 interchanged, and then take $q = z_4$.

■

3.2. THEOREM. *For a category $\mathbb{X}$ the following conditions are equivalent to each other:*

(a) The approximation $\alpha : \hom_{\mathbb{X}} \to A$ of the universal approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ satisfies (*).

(b) There exists an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ whose approximation α satisfies (*).

(c) $\mathbb{X}$ is a Mal'tsev category.

PROOF. (a)⇒(b) is trivially true.

(b)⇒(c): Let μ be an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ whose approximation $\alpha : \hom_{\mathbb{X}} \to A$ satisfies (*). Consider a commutative diagram of solid arrows (5), with r_1, r_2 jointly monomorphic. We want to show that there exists the dotted arrow z_4 making the diagram (5) commute. Consider the element $a = \mu_{S,R}(z_1, z_2, z_3)$ in $A(S, R)$. We have

$$A(1_S, r_1)(a) = \mu_{S,X}(r_1 z_1, r_1 z_2, r_1 z_3) = \mu_{S,X}(x_1, x_2, x_2) = \alpha_{S,X}(x_1),$$

$$A(1_S, r_2)(a) = \mu_{S,Y}(r_2 z_1, r_2 z_2, r_2 z_3) = \mu_{S,Y}(y_2, y_2, y_1) = \alpha_{S,Y}(y_1).$$

Now, applying (*) we get the existence of the desired z_4.

(c)⇒(a): Let $\mu : \hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \times \hom_{\mathbb{X}} \to A$ be a universal approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ with approximation α. Consider a diagram (6) of solid arrows with r_1, r_2 jointly monomorphic. Suppose for an element $a \in A(S, R)$ we have $A(1_S, r_1)(a) = \alpha_{S,X}(s_1)$ and $A(1_S, r_2)(a) = \alpha_{S,Y}(s_2)$. We want to show that there exists the dotted arrow q in the diagram (6) making it commutative. The fact that the approximate Mal'tsev operation μ is universal implies that its every component is a surjective map, and in particular, $\mu_{S,R}$ is surjective. So there exist morphisms $z_1, z_2, z_3 : S \to R$ such that $\mu_{S,R}(z_1, z_2, z_3) = a$. Now, we have

$$\alpha_{S,X}(s_1) = A(1_S, r_1)(a) = \mu_{S,X}(r_1 z_1, r_1 z_2, r_1 z_3),$$

$$\alpha_{S,Y}(s_2) = A(1_S, r_2)(a) = \mu_{S,Y}(r_2 z_1, r_2 z_2, r_2 z_3).$$

Applying Lemma 3.1 we obtain the existence of the desired q. ■

4. The second characterization theorem

In this section $\mathbb{C}$ is the category $\mathbb{C} = (\mathbb{X}^{\mathbb{X}})^{\mathrm{op}}$, where $\mathbb{X}$ is a category with finite limits and binary coproducts. But we will work in the dual category $\mathbb{C}^{\mathrm{op}} = \mathbb{X}^{\mathbb{X}}$; so a *(universal) approximate Mal'tsev operation $\mu : X^3 \to A$ in $\mathbb{C}$ with approximation $\alpha : X \to A$* now becomes a *(universal) approximate Mal'tsev co-operation $\mu : A \to 3X$ (in $\mathbb{C}^{\mathrm{op}}$) with approximation $\alpha : A \to X$*. In particular, we are interested in the case when $X = 1_{\mathbb{X}}$.

4.1. Proposition. *For a finitely complete category $\mathbb{X}$ with binary coproducts the following conditions are equivalent to each other:*

(a) $\mathbb{X}$ is a Mal'tsev category.

(b) In $\mathbb{X}$, for any diagram of solid arrows

$$\begin{tikzcd}
 & & S \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] \arrow[dd, "k_1"] & & \\
 & 2S & & 2S & \\
S \arrow[ur, "\iota_1"] \arrow[dr, "\iota_1"'] \arrow[rr, dotted, "k_4"] & & R' \arrow[ul, "r'_1"'] \arrow[ur, "r'_2"] \arrow[dl, "r'_2"] \arrow[dr, "r'_1"'] & & S \arrow[ul, "\iota_2"'] \arrow[dl, "\iota_2"] \arrow[ll, "k_2"'] \\
 & 2S & & 2S & \\
 & & S \arrow[ul, "\iota_1"] \arrow[ur, "\iota_2"'] \arrow[uu, "k_3"'] & &
\end{tikzcd} \tag{7}$$

with r'_1, r'_2 jointly monomorphic, there exists the dotted arrow k_4 making the diagram (7) commute.

Proof. (a)⇒(b): The diagram (7) is a special case of the diagram (5).

(b)⇒(a): Consider a diagram of solid arrows (5). Take the pullback

$$\begin{tikzcd}
R' \arrow[r, "p"] \arrow[d, "r'=(r'_1,r'_2)"'] & R \arrow[d, "r=(r_1,r_2)"] \\
2S \times 2S \arrow[r, "\begin{pmatrix}x_1\\x_2\end{pmatrix}\times\begin{pmatrix}y_1\\y_2\end{pmatrix}"'] & X \times Y
\end{tikzcd} \tag{8}$$

Commutativity of the diagrams

$$\begin{tikzcd}
S \arrow[r, "z_1"] \arrow[d, "(\iota_1,\iota_2)"'] & R \arrow[d, "r"] \\
2S \times 2S \arrow[r, "\begin{pmatrix}x_1\\x_2\end{pmatrix}\times\begin{pmatrix}y_1\\y_2\end{pmatrix}"'] & X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "z_2"] \arrow[d, "(\iota_2,\iota_2)"'] & R \arrow[d, "r"] \\
2S \times 2S \arrow[r, "\begin{pmatrix}x_1\\x_2\end{pmatrix}\times\begin{pmatrix}y_1\\y_2\end{pmatrix}"'] & X \times Y
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[r, "z_3"] \arrow[d, "{(\iota_2,\iota_1)}"'] & R \arrow[d, "r"] \\
2S \times 2S \arrow[r, "\binom{x_1}{x_2}\times\binom{y_1}{y_2}"'] & X \times Y
\end{tikzcd}$$

implies the existence of morphisms $k_1, k_2, k_3 : S \to R'$ such that the diagram of solid arrows (7) commutes. Now, take $z_4 = pk_4$, where k_4 is the dotted arrow in (7), and p is taken from the pullback (8). ■

4.2. Theorem. *For a regular category $\mathbb{X}$ with binary coproducts the following conditions are equivalent to each other:*

(a) The approximation $\alpha : A \to 1_{\mathbb{X}}$ of the universal approximate Mal'tsev co-operation on $1_{\mathbb{X}}$ is a natural transformation with every component a regular epimorphism in $\mathbb{X}$.

(b) There exists an approximate Mal'tsev co-operation on $1_{\mathbb{X}}$ whose approximation α has every component a regular epimorphism in $\mathbb{X}$.

(c) $\mathbb{X}$ is a Mal'tsev category.

Proof. (a)⇒(b) is trivially true.

(b)⇒(c): Suppose (b) is satisfied. We show that then 4.1(b) is satisfied. Consider a diagram (7) as in 4.1(b). Let $\mu : A \to 1_{\mathbb{X}} + 1_{\mathbb{X}} + 1_{\mathbb{X}}$ be an approximate Mal'tsev co-operation with approximation α whose every component is a regular epimorphism. Now, in the diagram

$$\begin{tikzcd}
A(S) \arrow[r, "\mu_S"] \arrow[d, "\alpha_S"'] & 3S \arrow[d, "\begin{pmatrix}\iota_1 & \iota_2\\ \iota_2 & \iota_2\\ \iota_2 & \iota_1\end{pmatrix}"'] \arrow[dr, "\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}"] & \\
 & & R' \arrow[dl, "r'=(r_1',r_2')"] \\
S \arrow[r, "{(\iota_1,\iota_1)}"'] & 2S \times 2S &
\end{tikzcd} \tag{9}$$

the rectangle commutes because μ_S is an approximate Mal'tsev co-operation with approximation α_S, and the triangle commutes because the solid arrow part in (7) commutes. Since α_S is a regular epimorphism, it is also a strong epimorphism, and hence there exists $f : S \to R'$ with $r'f = (\iota_1, \iota_1)$. We then take $k_4 = f$ in (7).

(c)⇒(a): Let μ be the universal approximate Mal'tsev operation on $1_{\mathbb{X}}$ with approximation $\alpha : A \to 1_{\mathbb{X}}$. Then for any object S in $\mathbb{X}$ the rectangle in (9) is a pullback. Form the commutative diagram (9) by taking the arrows $3S \to R'$ and $R' \to 2S \times 2S$ in it to be a regular epimorphism and monomorphism decomposition of the arrow $3S \to 2S \times 2S$. Then we get a commutative diagram of solid arrows (7). If $\mathbb{X}$ is a Mal'tsev category, then we get an arrow $k_4 : S \to R'$ with $r'k_4 = (\iota_1, \iota_1)$. Now, the fact that the rectangle in (9) is a pullback implies that the rectangle

$$\begin{tikzcd} A(S) \arrow[r, "\mu_S"] \arrow[d, "\alpha_S"'] & 3S \arrow[d, "\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}"] \\ S \arrow[r, "k_4"'] & R' \end{tikzcd}$$

is a pullback. Since the morphism $3S \to R'$ here is a regular epimorphism, we conclude that α_S is also a regular epimorphism. ∎

5. Beyond Mal'tsev categories

The class of Mal'tsev categories, as well as the classes of *unital* and *strongly unital categories* in the sense of [2], and the class of *subtractive categories*[2] in the sense of [7], are particular instances of a class of *categories with closed relations* in the sense of [8]. All results obtained in this paper (including the Appendix) can be straightforwardly extended to those general classes of categories. In particular, this means that analogous results also hold true for unital, strongly unital and subtractive categories. For them, the condition (*) remains the same; however, instead of the diagram (3) we should work with the diagrams displayed in the following table:

Unital categories	$\begin{tikzcd} X^2 \arrow[r] & A \\ X+X \arrow[u, "\begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1_X \\ 1_X \end{pmatrix}"'] & X \arrow[u] \end{tikzcd}$	$\mu(x,0)=x=\mu(0,x)$
Strongly unital categories	$\begin{tikzcd} X^3 \arrow[r] & A \\ X^2+X^2 \arrow[u, "\begin{pmatrix} \pi_1 & 0 & 0 \\ \pi_2 & \pi_2 & \pi_1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \pi_1 \\ \pi_1 \end{pmatrix}"'] & X \arrow[u] \end{tikzcd}$	$\mu(x,0,0)=x=\mu(y,y,x)$
Subtractive categories	$\begin{tikzcd} X^2 \arrow[r] & A \\ X+X \arrow[u, "\begin{pmatrix} 1_X & 0 \\ 1_X & 1_X \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1_X \\ 0 \end{pmatrix}"'] & X \arrow[u] \end{tikzcd}$	$\mu(x,0)=x,\ \mu(x,x)=0$

These diagrams, in each case, naturally correspond to the term identities (displayed right-hand side in the table) that characterize the corresponding varieties of universal algebras.

---

[2]The notion of a subtractive category introduced in [7] is a pointed categorical version of the notion of a subtractive variety due to A. Ursini [18].

Note also that when we replace the diagram (3) with the diagrams in the above table, we have to replace difunctionality in (C4) with *M-closedness* in the sense of [8], where M is the extended matrix associated with the corresponding term identities (see [8]). Note: unital, strongly unital and subtractive categories are all pointed categories, and the morphisms 0 in the diagrams in the table represent zero morphisms in a pointed category; a variety is a pointed category if and only if its theory contains a unique constant — the symbol 0 in the term equalities given in the third column represents this unique constant.

6. Appendix

In this more technical section we investigate how much (C5) is related to (C7). In particular (C7) turns out to be equivalent to (C5) when for any object X, $A(X, -)$ is representable.

Let $\mathbb{D}$ and $\mathbb{E}$ be categories. Let n be a natural number $n \geqslant 1$ and suppose that $\mathbb{D}$ has n-fold products. Then for any two functors $U, V : \mathbb{E} \to \mathbb{D}$ there is a map

$$\Phi_n : \mathrm{Nat}(U^n, V) \to \mathrm{Nat}((\hom_{\mathbb{E}})^n, \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times V)), \tag{10}$$

which assigns to each natural transformation $\nu : U^n \to V$ the natural transformation

$$\Phi_n(\nu) : (\hom_{\mathbb{E}})^n \to \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times V),$$

$$(\Phi_n(\nu))_{X,Y} : (f_1, ..., f_n : X \to Y) \mapsto (\nu_Y(U(f_1), ..., U(f_n)) : U(X) \to V(Y)).$$

If, further, $\mathbb{E}$ has n-fold products and U preserves them, then Φ_n is a bijection whose inverse $(\Phi_n)^{-1}$ is constructed as described below:

Suppose $\mathbb{E}$ has n-fold products. Note: to say that U preserves n-fold products is the same as to say that for each object Y in $\mathbb{E}$, the canonical morphism

$$(U(\pi_1), ..., U(\pi_n)) : U(Y^n) \to (U(Y))^n$$

(where π_i denotes i-th product projection $\pi_i : Y^n \to Y$) is an isomorphism. For each natural transformation $\mu : (\hom_{\mathbb{E}})^n \to \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times V)$ and for each object Y in $\mathbb{E}$, the morphism

$$((\Phi_n)^{-1}(\mu))_Y : (U(Y))^3 \to V(Y)$$

is defined as the unique morphism making the triangle

$$\begin{tikzcd}
(U(Y))^n \arrow[r, "((\Phi_n)^{-1}(\mu))_Y"] & V(Y) \\
U(Y^n) \arrow[u, "{(U(\pi_1),...,U(\pi_n))}"] \arrow[ur, "{\mu_{Y^n,Y}(\pi_1,...,\pi_n)}"'] &
\end{tikzcd}$$

commute.

6.1. Remark. *The bijection $(\Phi_n)^{-1}$ can be obtained from the bijection in Yoneda Lemma (e.g. see [13]). Let A be an arbitrary functor $A : \mathbb{E}^{\mathrm{op}} \times \mathbb{E} \to \mathbf{Set}$. By the Yoneda Lemma for $\mathbb{E}^{\mathrm{op}}$, we have a bijection*

$$\mathrm{Nat}(\hom_{\mathbb{E}}(-, E), K) \to K(E), \quad E \in \mathbb{E}^{\mathrm{op}}, K \in \mathbf{Set}^{\mathbb{E}^{\mathrm{op}}},$$

which is natural in E and in K. This gives a bijection

$$\mathrm{Nat}(\hom_{\mathbb{E}}(-, E^n), A(-, E')) \to A(E^n, E'), \quad E \in \mathbb{E}^{\mathrm{op}}, E' \in \mathbb{E},$$

natural in E and E', where the product E^n is taken in $\mathbb{E}$. This itself gives a bijection

$$\int_E \mathrm{Nat}(\hom_{\mathbb{E}}(-, E^n), A(-, E)) \to \int_E A(E^n, E).$$

Note that we also have a bijection

$$\mathrm{Nat}((\hom_{\mathbb{E}})^n, A) \to \int_E \mathrm{Nat}(\hom_{\mathbb{E}}(-, E^n), A(-, E)).$$

Composing these two bijections we obtain a bijection

$$\mathrm{Nat}((\hom_{\mathbb{E}})^n, A) \to \int_E A(E^n, E).$$

Now take $A = \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times V)$. Then

$$\int_E A(E^n, E) = \mathrm{Nat}(U((1_{\mathbb{E}})^n), V),$$

so the last bijection above in this special case becomes a bijection

$$\mathrm{Nat}((\hom_{\mathbb{E}})^n, \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times V)) \to \mathrm{Nat}(U((1_{\mathbb{E}})^n), V).$$

Since U preserves n-fold products, there is a canonical isomorphism $U((1_{\mathbb{E}})^n) \to U^n$ which induces a bijection

$$\mathrm{Nat}(U((1_{\mathbb{E}})^n), V) \to \mathrm{Nat}(U^n, V).$$

Composing the last two bijections above we obtain precisely $(\Phi_n)^{-1}$.

6.2. Proposition. *If $\mathbb{D}$ has binary products then for any natural transformations $\omega : U^3 \to V$ and $\nu : U \to V$, the condition (a) below implies (b). If, further, $\mathbb{E}$ has binary products and U preserves them, then (a) is equivalent to (b).*

(a) ω is an approximate Mal'tsev operation on U with approximation ν.

(b) $\Phi_3(\omega)$ is an approximate Mal'tsev operation on $\hom_{\mathbb{E}}$ with approximation $\Phi_1(\nu)$.

Proof. (a)⇒(b): Suppose ω is an approximate Mal'tsev operation on U with approximation ν. To see that $\Phi_3(\omega)$ is an approximate Mal'tsev operation on $\hom_{\mathbb{E}}$ with approximation $\Phi_1(\nu)$, we take any two objects X and Y in $\mathbb{E}$ and show that $(\Phi_3(\omega))_{X,Y}$ is an approximate Mal'tsev operation on $\hom_{\mathbb{E}}(X,Y)$ with approximation $(\Phi_1(\nu))_{X,Y}$. Take any two elements $f, g \in \hom_{\mathbb{E}}(X,Y)$. We must show

$$(\Phi_3(\omega))_{X,Y}(f,g,g) = (\Phi_1(\nu))_{X,Y}(f) = (\Phi_3(\omega))_{X,Y}(g,g,f).$$

But this follows straightforwardly from the constructions of $\Phi_3(\omega)$ and $\Phi_1(\nu)$ and the fact that ω_Y is an approximate Mal'tsev operation on $U(Y)$ with approximation ν_Y.

Next, assuming that $\mathbb{E}$ has binary products and U preserves them, we prove (b)⇒(a): Suppose $\Phi_3(\omega)$ is an approximate Mal'tsev operation on $\hom_{\mathbb{E}}$ with approximation $\Phi_1(\nu)$. Then, we have to show that for each object Y in $\mathbb{E}$, the morphisms ω_Y and ν_Y satisfy $\omega_Y(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_2) = \nu_Y \overline{\pi}_1 = \omega_Y(\overline{\pi}_2, \overline{\pi}_2, \overline{\pi}_1)$, where $\overline{\pi}_1$ and $\overline{\pi}_2$ denote product projections $(U(Y))^2 \to U(Y)$. To show $\omega_Y(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_2) = \nu_Y \overline{\pi}_1$, consider the diagram

$$\begin{tikzcd}
(U(Y))^3 \arrow[rr, "\omega_Y"] & & V(Y) \\
& U(Y^2) \arrow[ul, "(U(\pi_1),U(\pi_2),U(\pi_2))"'] \arrow[ur, "(\Phi_3(\omega))_{Y^2,Y}(\pi_1,\pi_2,\pi_2)=(\Phi_1(\nu))_{Y^2,Y}(\pi_1)"'] \arrow[dl, "(U(\pi_1),U(\pi_2))"] \arrow[dr, "U(\pi_1)"] & \\
(U(Y))^2 \arrow[uu, "(\overline{\pi}_1,\overline{\pi}_2,\overline{\pi}_2)"] \arrow[rr, "\overline{\pi}_1"'] & & U(Y) \arrow[uu, "\nu_Y"']
\end{tikzcd}$$

Since every triangle in this diagram commutes, and since $(U(\pi_1), U(\pi_2))$ is an isomorphism, we obtain that the rectangle commutes. The proof of the second equality

$$\omega_Y(\overline{\pi}_2, \overline{\pi}_2, \overline{\pi}_1) = \nu_Y \overline{\pi}_1$$

is analogous. ■

We now consider a *dual setting*, i.e. when

$$\mathbb{D} = \mathbb{W}^{\mathrm{op}}, \ \ \mathbb{E} = \mathbb{X}^{\mathrm{op}}, \ \ U = G^{\mathrm{op}}, \ \ V = F^{\mathrm{op}},$$

where F, G are functors $F, G : \mathbb{X} \to \mathbb{W}$ between categories $\mathbb{X}$ and $\mathbb{W}$. Then, for each n, the map Φ_n gives rise to the map

$$\Psi_n : \mathrm{Nat}(F, nG) \to \mathrm{Nat}((\hom_{\mathbb{X}})^n, \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G)), \tag{11}$$

which assigns to each natural transformation $\nu : F \to nG$ the natural transformation

$$\Psi_n(\nu) : (\hom_{\mathbb{X}})^n \to \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G),$$

$$(\Psi_n(\nu))_{X,Y} : (f_1, ..., f_n : X \to Y) \mapsto (\begin{pmatrix} G(f_1) \\ \vdots \\ G(f_n) \end{pmatrix} \nu_X : F(X) \to G(Y)).$$

In particular, Ψ_n is obtained from Φ_n as the composite indicated in the diagram

$$\begin{tikzcd} \mathrm{Nat}((G^{\mathrm{op}})^n, F^{\mathrm{op}}) \arrow[r, "\Phi_n"] & \mathrm{Nat}((\hom_{\mathbb{X}^{\mathrm{op}}})^n, \hom_{\mathbb{W}^{\mathrm{op}}} \circ (G \times F^{\mathrm{op}})) \arrow[d, "\Sigma_n"] \\ \mathrm{Nat}(F, nG) \arrow[u, "\Theta_n"] \arrow[r, dotted, "\Psi_n"'] & \mathrm{Nat}((\hom_{\mathbb{X}})^n, \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G)) \end{tikzcd}$$

where Θ_n and Σ_n are the obvious bijections.

In this dual setting we have, dually: the map Ψ_n is defined when n-fold coproducts exist in $\mathbb{W}$, and it is a bijection when $\mathbb{X}$ also has n-fold coproducts and G preserves them.

Consider natural transformations

$$\alpha : \hom_{\mathbb{X}} \to \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G), \quad \mu : (\hom_{\mathbb{X}})^n \to \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G).$$

It is easy to see that μ is an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ with approximation α if and only if $\Sigma_n^{-1}(\mu)$ is an approximate Mal'tsev operation on $\hom_{\mathbb{X}^{\mathrm{op}}}$ with approximation $\Sigma_1^{-1}(\alpha)$. So from Proposition 6.2 we get:

6.3. PROPOSITION. [A dual of Proposition 6.2] *If $\mathbb{W}$ has binary coproducts then for any natural transformations $\omega : F \to 3G$ and $\nu : F \to G$, the condition (a) below implies (b). If, further, $\mathbb{X}$ has binary coproducts and G preserves them, then (a) is equivalent to (b).*

(a) ω is an approximate Mal'tsev co-operation on G with approximation ν.

(b) $\Psi_3(\omega)$ is an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ with approximation $\Psi_1(\nu)$.

Note that for $n = 1$ the map Ψ_1 is the same as the map Φ_1 in (10) where this time $U = F$ and $V = G$. That is, we have:

$$\Psi_1 = \Phi_1 : \mathrm{Nat}(F, G) \to \mathrm{Nat}(\hom_{\mathbb{X}}, \hom_{\mathbb{W}} \circ (F^{\mathrm{op}} \times G)).$$

In this case we adopt the following special notation: for a natural transformation $\nu : F \to G$ we denote

$$\widetilde{\nu} = \Psi_1(\nu) = \Phi_1(\nu).$$

We can now make the following

6.4. CONCLUSION. *Let $\mathbb{X}$ and $\mathbb{D}$ be categories, and let U, G be functors $U, G : \mathbb{X} \to \mathbb{D}$. The map*

$$\mathrm{Nat}(U, G) \to \mathrm{Nat}(\hom_{\mathbb{X}}, \hom_{\mathbb{D}} \circ (U^{\mathrm{op}} \times G)), \quad \nu \mapsto \widetilde{\nu},$$

is a bijection. Further, for each $\nu \in \mathrm{Nat}(U, G)$ we have:

(a) *If $\mathbb{X}$ and $\mathbb{D}$ have binary products and U preserves binary products, then there is a bijection between approximate Mal'tsev operations on U with approximation ν, and approximate Mal'tsev operations on* $\hom_{\mathbb{X}}$ *with approximation $\widetilde{\nu}$.*

(b) *If $\mathbb{X}$ and $\mathbb{D}$ have binary coproducts and G preserves binary coproducts, then there is a bijection between approximate Mal'tsev co-operations on G with approximation ν, and approximate Mal'tsev operations on* $\hom_{\mathbb{X}}$ *with approximation $\widetilde{\nu}$.*

Let $\mathbb{X}$ and $\mathbb{D}$ be categories and let ν be a natural transformation $\nu : U \to G$ between functors $U, G : \mathbb{X} \to \mathbb{D}$. The condition (*) for $\alpha = \widetilde{\nu}$ becomes:

(*′) In $\mathbb{X}$, for any diagram (6) of solid arrows with r_1, r_2 jointly monomorphic, the dotted arrow q exists, making the diagram (6) commute in $\mathbb{X}$, if (and only if) there exists the dotted arrow a in the diagram

$$\begin{tikzcd}
 & G(R) \arrow[dl, "G(r_1)"'] \arrow[dr, "G(r_2)"] & \\
G(X) & & G(Y) \\
U(X) \arrow[u, "\nu_X"] & U(S) \arrow[l, "U(s_1)"] \arrow[r, "U(s_2)"'] \arrow[uu, dotted, "a"] & U(Y) \arrow[u, "\nu_Y"']
\end{tikzcd} \tag{12}$$

making the diagram (12) commute in $\mathbb{D}$.

Note that by naturality of ν, commutativity of (12) is equivalent to commutativity of the following diagram:

$$\begin{tikzcd}
 & G(R) \arrow[dl, "G(r_1)"'] \arrow[dr, "G(r_2)"] & \\
G(X) & & G(Y) \\
G(S) \arrow[u, "G(s_1)"] & U(S) \arrow[l, "\nu_S"] \arrow[r, "\nu_S"'] \arrow[uu, dotted, "a"] & G(S) \arrow[u, "G(s_2)"']
\end{tikzcd} \tag{13}$$

If $\mathbb{D}$ has binary products, then we can further replace the above diagram with the following one:

$$\begin{tikzcd}
U(S) \arrow[r, dotted, "a"] \arrow[d, "\nu_S"'] & G(R) \arrow[d, "(G(r_1),G(r_2))"] \\
G(S) \arrow[r, "(G(s_1),G(s_2))"'] & G(X) \times G(Y)
\end{tikzcd} \tag{14}$$

After these observations we easily get the following two lemmas:

6.5. Lemma. *Suppose both $\mathbb{X}$ and $\mathbb{D}$ have finite limits and G preserves finite limits and reflects isomorphisms. If every component of ν is a strong epimorphism in $\mathbb{D}$, then $\alpha = \widetilde{\nu}$ satisfies (*). In particular, $\alpha = \widetilde{\nu}$ satisfies (*) when $U = G$ and $\nu = 1_U$.*

6.6. Lemma. *Suppose $\mathbb{X} = \mathbb{D}$ and $\mathbb{X}$ has finite products, and $G = 1_{\mathbb{X}}$. Then, $\alpha = \widetilde{\nu}$ satisfies (*) if and only if every component of ν is a strong epimorphism.*

Lemma 6.5 allows to deduce the following known result from Theorem 3.2:

6.7. Corollary. [5], [16] *Let $\mathbb{X}$ and $\mathbb{D}$ be categories having finite limits and U a functor $U : \mathbb{X} \to \mathbb{D}$ which preserves finite limits and reflects isomorphisms. If there exists a Mal'tsev operation on U then $\mathbb{X}$ is a Mal'tsev category.*

Proof. According to Conclusion 6.4(a) there exists an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ with approximation $\widetilde{1_U}$. By Lemma 6.5, $\alpha = \widetilde{1_U}$ satisfies (*). Therefore, by Theorem 3.2, $\mathbb{X}$ is a Mal'tsev category. ■

6.8. Proposition. *For a category $\mathbb{X}$ with binary coproducts the following conditions are equivalent to each other:*

(a) There exists an approximate Mal'tsev operation on $\hom_{\mathbb{X}}$ *whose approximation α : $\hom_{\mathbb{X}} \to A$ satisfies (*) and for each object X in $\mathbb{X}$ the functor $A(X, -) : \mathbb{X} \to$* **Set** *is representable.*

(b) There exists an approximate Mal'tsev co-operation μ on $1_{\mathbb{X}}$ such that every component of the approximation of μ is a strong epimorphism.

Proof. For a functor $A : \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \to$ **Set** the following conditions are equivalent to each other:

- For every object X in $\mathbb{X}$ the functor $A(X, -) : \mathbb{X} \to$ **Set** is representable.
- A is naturally isomorphic to a functor of the form $\hom_{\mathbb{X}} \circ (U^{\mathrm{op}} \times 1_{\mathbb{X}})$, for some functor $U : \mathbb{X} \to \mathbb{X}$.

After this observation, the equivalence of (a) and (b) follows easily from Conclusion 6.4 and Lemma 6.6. ■

6.9. Remark. *In a regular category, regular epimorphisms coincide with strong epimorphisms. So when $\mathbb{X}$ is regular and has binary coproducts, the conditions 4.2(b) and 6.8(b) are identical, and from Theorem 4.2 and Proposition 6.8 we get: $\mathbb{X}$ is a Mal'tsev category if and only if it satisfies 6.8(a).*

6.10. Remark. *Recall that to prove (b)⇒(c) in Theorem 4.2 we used Proposition 4.1. Alternatively, (b)⇒(c) in Theorem 4.2 follows from (b)⇒(a) of Proposition 6.8 and (b)⇒(c) of Theorem 3.2.*

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