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INTERNAL CROSSED MODULES AND PEIFFER CONDITION

Dedicated to Dominique Bourn on the occasion of his 60th birthday

SANDRA MANTOVANI, GIUSEPPE METERE

Abstract. In this paper we show that in a homological category in the sense of F. Borceux and D. Bourn, the notion of an internal precrossed module corresponding to a star-multiplicative graph, in the sense of G. Janelidze, can be obtained by directly internalizing the usual axioms of a crossed module, via equivariance. We then exhibit some sufficient conditions on a homological category under which this notion coincides with the notion of an internal crossed module due to G. Janelidze. We show that this is the case for any category of distributive Ω_2-groups, in particular for the categories of groups with operations in the sense of G. Orzech.

1. Introduction

Let us consider two groups $(B, \cdot, 1)$ and $(X, +, 0)$, with B acting on X on the left, and a group homomorphism $\partial : X \to B$.

In this paper we will be concerned with the following two axioms:

$$\begin{array}{lc} (PCM) & \partial(b \cdot x) = b\partial(x)b^{-1} \\ & \text{for } b \in B, x \in X, \\ & \\ (PFF) & \partial(x_1) \cdot x_2 = x_1 + x_2 - x_1 \\ & \text{for } x_1, x_2 \in X, \end{array} \qquad \begin{tikzcd} X \times X \arrow[r, "\chi_X"] \arrow[d, "\partial \times 1_X"'] \arrow[dr, phantom, "(PFF)"] & X \arrow[d, "1_X"] \\ B \times X \arrow[r, "\cdot"] \arrow[d, "1_B \times \partial"'] \arrow[dr, phantom, "(PCM)"] & X \arrow[d, "\partial"] \\ B \times B \arrow[r, "\chi_B"'] & B. \end{tikzcd} \tag{1}$$

Axiom (PCM) gives to the triple (B, X, ∂) a *precrossed module* structure; precrossed modules satisfying *axiom* (PFF), i.e. the so called *Peiffer identity*, are named *crossed module*.

The notions of precrossed module and of crossed module have been introduced since the pioneering work of J. H. C. Whitehead [Whi49], whose terminology is borrowed. Thereafter, they have proved to be a useful technical tool in many areas of mathematical research, including homotopy theory [Lod82] and homotopical algebra [CE89].

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Nevertheless its definition is technical, and certainly not very enlightening. Moreover diagrams above do not *live* in the category of groups: horizontal arrows displayed there are not group homomorphisms, but set-theoretical functions satisfying group action axioms.

In order to generalize crossed modules of groups to the context of semiabelian categories [JMT02], G. Janelidze [Jan03] describes such structures from the viewpoint of the categorical equivalence between split epimorphisms (with chosen splitting) and internal actions in the category of groups.

In fact a split epimorphism can be given the richer structure of a reflexive graph by adding a second epimorphism which is splitted by the same monomorphism. A precrossed module is precisely the corresponding structure for group actions. Moreover, in the category of groups as in any Mal'cev category, for a reflexive graph being a category is equivalent to being a groupoid, and this is a property of the reflexive graph. The corresponding property for precrossed modules is exactly what deserves to be called a crossed module. These correspondences are summarized in the diagram below:

$$\begin{tikzcd}
\text{Internal categories in } \mathbf{Groups} \arrow[r, "\sim"] \arrow[d, hook] & \text{Crossed modules} \arrow[d, hook] \\
\text{Reflexive graphs in } \mathbf{Groups} \arrow[r, "\sim"] \arrow[d] & \text{Precrossed modules} \arrow[d] \\
\text{Split epimorphisms in } \mathbf{Groups} \arrow[r, "\sim"] & \text{Actions of groups.}
\end{tikzcd}$$

All this is set–theoretical, but it can be also described in terms of internal category theory, so that "...the passage from internal categories to crossed modules becomes a purely categorical procedure rather than an algebraic translation of a categorical notion" [Jan03]. The internal description of the issues recalled above is obtained by using a notion of *internal action*, suitable for semiabelian categories, introduced in [BJ98].

It is tempting to *typographically* turn diagrams (1) into internal ones, and to investigate their meaning. They would look as follows:

$$\begin{tikzcd}
X\flat X \arrow[r, "\chi_X"] \arrow[d, "\partial\flat 1_X"'] \arrow[dr, phantom, "(PFF)"] & X \arrow[d, dash, "1_X"] \\
B\flat X \arrow[r, "\cdot"] \arrow[d, "1_B\flat\partial"'] \arrow[dr, phantom, "(PCM)"] & X \arrow[d, "\partial"] \\
B\flat B \arrow[r, "\chi_B"'] & B.
\end{tikzcd} \tag{2}$$

Conceptually this process amounts to explore to which extent these structures can be defined in terms of equivariance. This is done by what we have called the *Transalation Lemma* (see Lemma 2.3), which allows to express the same axioms both in an equivariant and in an internal-graph form. The result is straightforward for precrossed modules, thus giving a structure that can be expressed by an axiom as (PCM) for groups. Differently, Peiffer identity does not identify crossed modules among precrossed modules. We exhibit an example of a precrossed module satisfying Peiffer identity without supporting any internal category structure (see 7.5). The purpose of our work is to clarify the relations among these situations in different contexts, and partially answer to a question that concludes Janelidze's paper [Jan03].

Back–tracing Peiffer precrossed modules along the categorical equivalence described above using the Translation Lemma, we obtain what we call Peiffer (reflexive) graphs. We show that, in the homological context, this notion, the notion of star-multiplicative graph introduced in [Jan03] and the one of star-divisible graph introduced below are all equivalent, and not sufficient, to detect internal categories (that are the same as groupoids, as in any Mal'cev category).

Our approach to Peiffer Condition emphasizes a connection with categorical commutator theory, since the partial division morphism of a Peiffer graph is actually a cooperator ([Bou02]). This allows us to find necessary and sufficient conditions for a Peiffer graph to be an internal groupoid in homological categories where the Huq commutator is equivalent to Smith commutator, such as in the strongly protomodular or in the action accessible cases ([Bou04, BJ07]). As a consequence (see Corollary 6.2), we find that in the case of categories of distributive Ω_2-groups, such as the categories of groups, rings, Lie algebras, and all the categories of groups with operations (see [Orz72, Por87]), the notion of internal crossed module can be given via Peiffer precrossed module conditions.

In the last part, we examine how a star-multiplication can induce a multiplication on a reflexive graph in the semiabelian context, and we exhibit a sufficient condition in order to get this case.

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2. Preliminaries

Let $\mathcal{C}$ be a semiabelian category. We recall the definition of two naturally equivalent pseudofunctors

$$\mathcal{C}^{op} \underset{Act}{\overset{Pt}{\rightrightarrows}} \mathbf{Cat},$$

where $\mathcal{C}$ is considered as a locally trivial 2-category.

The first is called *the pseudofunctor of points*: if B is an object of $\mathcal{C}$, $Pt(B)$ is the category of split epimorphisms over B, where objects are triples (A, α, β) with $\alpha : A \to B$, $\beta : B \to A$ and $\alpha\beta = 1_B$, and where a morphism $(A, \alpha, \beta) \to (A', \alpha', \beta')$ is an arrow $f : A \to A'$ such that $\alpha' f = \alpha$ and $f\beta = \beta'$. For an arrow $p : E \to B$, the functor $Pt(B) \to Pt(E)$ is given by pulling back along p.

The description of *Act* is slightly more complicated. For an object B of $\mathcal{C}$, the category $Act(B)$ of internal actions of B is the category of algebras $\mathcal{C}^{B\flat-}$ for the monad $\mathbf{B}\flat- = (B\flat-, \eta^B, \mu^B)$. This monad is canonically determined by the adjunction

$$\begin{tikzcd} Pt(\mathcal{C}) \arrow[r, shift right, "(i_B)^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "(i_B)_!"'] \end{tikzcd} . \tag{3}$$

(where $Pt(0) = \mathcal{C}$ since $\mathcal{C}$ is pointed) and right and left adjoints are respectively the pullback and the pushout functors induced by the initial arrow $i_B : 0 \to B$. Consequently the functor $B\flat- : \mathcal{C} \to \mathcal{C}$ is determined by the kernel diagram below:

$$B\flat A \xrightarrow{\kappa_{B,A}} B + A \xrightarrow{[1,0]} B \,.$$

For a morphism $p : E \to B$, the functor $Act(B) \to Act(E)$ is given by pre-composition with p. In fact, for an algebra $\xi^B : B\flat X \to X$ in $Act(B)$, the composition $\xi^B \circ B\flat p$ yields an algebra in $Act(E)$. Let us notice that pseudofunctor *Act* is strict, i.e. it is indeed a functor with values in the category underlying **Cat**.

This situation is classically dealt with, by turning pseudofunctors into fibrations, and the natural equivalence between them in a fiberwise categorical equivalence K:

$$\begin{tikzcd} Pt(B) \arrow[rr, "K(B)"] \arrow[d, hook] & & Act(B) \arrow[d, hook] \\ Pt \arrow[rr, "K"] \arrow[dr] & & Act \arrow[dl] \\ & \mathcal{C} & \end{tikzcd} \tag{4}$$

The forgetful functor $Pt \to \mathcal{C}$ extends the assignment $(A, B, \alpha, \beta) \mapsto B$, while $Act \to \mathcal{C}$ extends $(B, X; \xi^B) \mapsto B$.

In fact, the equivalence K is the comparison functor for the monad $\mathbf{T} = (T, \eta, \mu)$ determined by the adjunction

$$\begin{tikzcd} Pt \arrow[r, shift right, "i^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \times \mathcal{C} \arrow[l, shift right, "i_!"'] \end{tikzcd} , \tag{5}$$

where $i^*((A, B, \alpha, \beta)) = (B, Ker(\alpha))$ and $i_!((B, X)) = (B + X, B, [1_B, 0], \iota_0)$; one can compute $\eta_{(B,X)} = (1_B, \eta^B_X)$ and $\mu_{(B,X)} = (1_B, \mu^B_X)$.

In [Bou07] Bourn shows that when $\mathcal{C}$ is semiabelian, the right adjoint i^* is monadic. Henceforth *Act* can be defined as the category of algebras $(\mathcal{C} \times \mathcal{C})^{\mathbf{T}}$, its objects being of the form $\Xi = (1_B, \xi^B)$. Notice that the adjoint pair $(i_!, i^*)$ is a kind of *free/forgetful* pair, as i^* forgets the action on the kernel, while $i_!$ gives a free (conjugation) action of B over X.

Let us recall that the morphisms $\xi^B : B\flat X \to X$ are internal B-actions in the sense of [BJK05]. In fact these are the object-actions for the categorical action $-\flat- : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ of the monoidal category $(\mathcal{C}, +, 0)$ on $\mathcal{C}$.

An explicit description of the functor K will be useful in the sequel: for a 4-tuple (A, B, α, β) as above with (X, k) the kernel of α, the action is obtained by universality of kernels, i.e. $\Xi = (1_B, \xi^B)$, where ξ^B is the dotted arrow in the diagram below:

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, dotted, "\xi^B"'] & B+X \arrow[d, "{[\beta,k]}"] \arrow[r, "{[1,0]}"] & B \\ X \arrow[r, "k"'] & A \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{6}$$

The left adjoint of K defines internally semidirect products [BJ98]. In a category with finite coequalizers, this can be obtained by Beck's construction as the coequalizer

$$B + (B\flat X) \underset{1+\xi}{\overset{[\iota_0, \kappa_{B,X}]}{\rightrightarrows}} B + X \xrightarrow{\sigma_\xi} X \rtimes_\xi B, \tag{7}$$

where ι_0 denotes the first injection in the coproduct. Moreover, since coproduct injections are jointly (strongly) epimorphic, this is equivalent to taking the coequalizer

$$B\flat X \underset{\iota_0\xi}{\overset{\kappa_{B,X}}{\rightrightarrows}} B + X \xrightarrow{\sigma_\xi} X \rtimes_\xi B. \tag{8}$$

In the case of groups, diagram (8) describes in an easier way the semidirect product as a quotient of the free product of B and X. Explicitly this is done by identifying elements of the kind $(b, x, -b)$ of $B + X$ with $b \cdot x$, with $\cdot$ expressing the group action associated to ξ.

Finally, if the base category is semiabelian, the functor K is an equivalence. In this case the left–hand square in diagram (6) is a pushout (and also a pullback, since the horizontal arrows are normal monomorphisms with isomorphic cokernels). In fact that diagram can be recasted as a semidirect product construction. This is displayed below:

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B+X \arrow[d, "\sigma_\xi"] \\ X \arrow[r, "\sigma_\xi\iota_0"'] & X \rtimes_\xi B. \end{tikzcd}$$

Following [BJ98], we will call the pointed category $\mathcal{C}$ a *pointed category with semidirect products* if the functor i^* above (exists and) is monadic.

Let us note that the functor K yields an equivalence of fibrations. This comes from the fact that the right adjoint i^* is itself fiberwise monadic, i.e. it is a morphism of fibrations over $\mathcal{C}$ whose restrictions to fibers $(i_B)^*$ are monadic, for every object B in $\mathcal{C}$.

In a semiabelian category, the jointly (strongly) epic pair of monomorphisms (k, β) of a split extension $x \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$ is characterized by the following (couniversal property of semidirect products):

2.1. Theorem. [Jan03] *Let $A, B, X, \alpha, \beta, k, \xi^B$ as above, and C an arbitrary object in $\mathcal{C}$ semiabelian. Then for every two morphisms $x : X \to C$ and $b : B \to C$ there exists exactly one morphism $a : A \to C$ such that $ak = x$ and $a\beta = b$ if and only if the square diagram on the right is commutative:*

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "x"'] & A \arrow[d, dotted, "a"] & B \arrow[l, "\beta"'] \arrow[dl, "b"] \\ & C & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi^B"'] \arrow[dr, phantom, "\equiv"] & B+X \arrow[d, "{[b,x]}"] \\ X \arrow[r, "x"'] & C, \end{tikzcd} \tag{9}$$

This theorem can be formulated more naturally in terms of equivariance. This is done in Lemma 2.3 below, but first we have to recall the internal version of the conjugation action.

For an object C in $\mathcal{C}$, the *conjugation in C* is the internal action corresponding to the split extension

$$C \xrightarrow{\langle 1,0\rangle} C \times C \underset{\Delta}{\overset{\pi_1}{\rightleftarrows}} C \ , \qquad \text{with } \Delta = \langle 1, 1\rangle,$$

and it can be computed explicitly as the composite

$$\chi_C : C\flat C \xrightarrow{\kappa_{C,C}} C + C \xrightarrow{[1,1]} C\ .$$

2.2. Remark. As it usually happens in the algebraic setting, the functor K assigns to a split epimorphism (A, B, α, β) the canonical action on the kernel of α given by conjugation *via* β. In fact, looking at the definition of ξ^B in diagram (6), one has $k\,\xi^B = [\beta, k]\kappa_{B,X} = [1, 1](\beta + k)\kappa_{B,X} = [1, 1]\kappa_{A,A}(\beta\flat k) = (\beta\flat k)\chi_A$.

Now we are ready to prove the

2.3. Lemma. [Translation Lemma].

(i) *Let $\mathcal{C}$ be a pointed category with finite limits and coproducts.*

The statement of Theorem 2.1 is equivalent to the following property (TL)*: given two split extensions*

$$X \xrightarrow{k} Y \underset{p}{\overset{s}{\leftrightarrows}} Z\ , \quad X' \xrightarrow{k'} Y' \underset{p'}{\overset{s'}{\leftrightarrows}} Z'$$

and two morphisms $x : X \to X'$ and $z : Z \to Z'$, the pair (x, z) extends univocally to a morphism $y : Y \to Y'$ with $ys = s'z$ and $yk = k'x$ if and only if (x, z) are equivariant w.r.t. the induced actions ξ and ξ':

$$
\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "x"'] & Y \arrow[r, shift right, "p"'] \arrow[d, dotted, "\exists! y"] & Z \arrow[l, shift right, "s"'] \arrow[d, "z"] \\
X' \arrow[r, "k'"'] & Y' \arrow[r, shift right, "p'"'] & Z \arrow[l, shift right, "s'"']
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
Z \flat X \arrow[r, "z \flat x"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\equiv"] & Z' \flat X' \arrow[d, "\xi'"] \\
X \arrow[r, "x"'] & X'.
\end{tikzcd}
$$

(ii) *If $\mathcal{C}$ has also semidirect products, the property* (TL) *holds and moreover the unique morphism $y : Y \to Y'$ gives rise to a morphism of split extensions (x, y, z). This is the main justification for denoting y by $x \rtimes z$.*

Proof. (i). First we suppose that the thesis of Theorem 2.1 holds. Let us consider the diagram below:

$$
\begin{tikzcd}
Z \flat X \arrow[r, "z \flat x"] \arrow[d, "\xi"'] \arrow[dr, phantom, "(j)"] & Z' \flat X' \arrow[r, "\kappa_{Z',X'}"] \arrow[d, "\xi'"] \arrow[dr, phantom, "(jj)"] & Z' + X' \arrow[d, "{[s',k']}"] \\
X \arrow[r, "x"'] & X' \arrow[r, "k'"'] & Y'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Z \flat X \arrow[r, "\kappa_{Z,X}"] \arrow[d, "\xi"'] \arrow[dr, phantom, "(jjj)"] & Z + X \arrow[d, "{[s'z,k'x]}"] \\
X \arrow[r, "k'x"'] & Y'
\end{tikzcd}
$$

Let us assume equivariance, i.e. that (j) commutes. Since diagram (jj) commutes by definition of ξ', we conclude that also the outer rectangle on the left commutes. Moreover $\kappa_{Z',X'} \circ z \flat x = (z + x) \circ \kappa_{Z,X}$, hence the equation of diagrams holds. By Theorem 2.1, there exists a unique y such that $yk = k'x$ and $ys = s'z$.

Conversely, given the arrow y such that $ys = s'z$ and $yk = k'x$, (jjj) commutes by Theorem 2.1 and (jj) commutes by definition. Since k' is a monomorphism, we deduce that (j) commutes too.

Now we assume that property (TL) holds. Then, given the data of Theorem 2.1, we can draw the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "x"'] & A \arrow[d, dotted, "y"] & B \arrow[l, "\beta"'] \arrow[d, "b"] \\
C \arrow[r, "{\langle 1,0 \rangle}"'] & C \times C & C \arrow[l, "\Delta"']
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
B \flat X \arrow[r, "z \flat x"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\equiv"] & C \flat C \arrow[d, "\chi_C"] \\
X \arrow[r, "x"'] & C.
\end{tikzcd}
$$

By the explicit definition of χ_C given above, the equivariance diagram on the right-hand side is equivalent to the commutativity of the square diagram in (9), and by property (TL), this is equivalent to saying that there exists a (unique) morphism y which makes the diagram on the left-hand side commute. So it remains to show that, in these circumstances, this is equivalent to the existence of a morphism $a : A \to C$ as in the left-hand diagram of (9), but this is clear if we define $a = \pi_0(y)$ and $y = (a, a)$ respectively.

(ii). If $\mathcal{C}$ has also semidirect products, since the equivalence K restricted to split exact sequences is still an equivalence, hence fully faithful, the pair (x, z) underlies a morphism of split extensions if and only if it gives a morphism of algebras for the monad $\mathbf{T}$. Actually, given any y such that $ys = s'z$ and $yk = k'x$, also $zp = p'y$, as one can easily check by precomposition with the pair (k, s), jointly strongly epic because of protomodularity. ∎

3. Internal precrossed modules

An internal reflexive graph is a split epic pair (A, B, α, β) endowed with a morphism $\gamma : A \to B$ such that $\gamma\beta = 1_B$. In semiabelian categories, this additional structure can be described in terms of internal actions equivariant w.r.t. (internal) conjugation. This is categorically meaningful, since, as recalled in the previous section, equivariance of actions is nothing but a morphism of algebras for a specific monad.

3.1. Definition. *An internal precrossed module in $\mathcal{C}$ is a 4-tuple (B, X, ξ, ∂) where (B, X, ξ) is an object in Act, and $\partial : X \to B$ a morphism in $\mathcal{C}$ making the diagram below commute:*

$$\begin{tikzcd} B\flat X \arrow[r, "\xi"] \arrow[d, "B\flat\partial"'] & X \arrow[d, "\partial"] \\ B\flat B \arrow[r, "\chi_B"'] & B \end{tikzcd} \tag{10}$$

Our definition is equivalent to the one given in [Jan03], where, instead of diagram (10), precrossed modules are defined by the diagram below:

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B + X \arrow[d, "{[1,\partial]}"] \\ X \arrow[r, "\partial"'] & B \end{tikzcd} \tag{11}$$

In fact $[1_B, \partial]\kappa_{B,X} = [1_B, 1_B](1_B + \partial)\kappa_{B,X} = [1_B, 1_B]\kappa_{B,B}(B\flat\partial) = \chi_B(B\flat\partial)$.

Internal precrossed modules in $\mathcal{C}$ naturally organize in a category denoted by $PCM(\mathcal{C})$, whence equivalence K of diagram (4) determines an equivalence of categories $K : RG(\mathcal{C}) \to PCM(\mathcal{C})$. This is easily seen if we apply the Translation Lemma 2.3 to diagram (10), thus getting

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\partial"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, dotted, "\partial\rtimes 1_B"] & B \arrow[l, shift left, "\beta"] \arrow[d, "1_B"] \\ B \arrow[r, "{\langle 1,0\rangle}"'] & B \times B \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

If we denote by $\vee B$ the point $B \times B \underset{\Delta}{\overset{\pi_1}{\rightleftarrows}} B$, the above condition says that $\partial \rtimes 1_B$ is an object of the comma category $\mathrm{Pt}_B(\mathcal{C})/\vee B$. This is equivalent to saying that $A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$ can be given the structure of a reflexive graph
$$\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$$
, with $\gamma = \pi_0 \left(\partial \rtimes 1_B\right)$. So we easily get Janelidze's equivalence

$$PCM(\mathcal{C}) \quad \cong \quad RG(\mathcal{C})$$

4. Internal crossed modules

In the semiabelian context, the notion of internal category gets simplified: an internal category in a Mal'cev category is just a multiplicative graph (see [CPP92]).

4.1. Definition. *A multiplicative graph is a reflexive graph*
$$\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$$
, together with a binary multiplication, i.e. a morphism $A \times_B A \xrightarrow{m} A$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\langle 1,\beta\alpha\rangle"] \arrow[dr, "1"'] & A \times_B A \arrow[d, "m"] & A \arrow[l, "\langle \beta\gamma,1\rangle"'] \arrow[dl, "1"] \\ & A & \end{tikzcd} \tag{12}$$

Using Theorem 2.1 again, one can translate this in terms of the precrossed module corresponding to the reflexive graph (α, β, γ), that is, in a semiabelian category one can characterize those precrossed modules underlying a multiplicative graph. Indeed, this is not straightforward at all, and protomodularity of the base category $\mathcal{C}$ plays an important role (see [Jan03] for details), as also the fact that the above definition of a multiplicative graph is partially redundant. Anyway, the final result of such a study is the following

4.2. Definition. *([Jan03]) An internal crossed module in $\mathcal{C}$ is an internal precrossed module (B, X, ξ, ∂) in $\mathcal{C}$ such that the following diagram commutes*

$$\begin{tikzcd} (B+X)\flat X \arrow[r, "{[1,\partial]\flat 1}"] \arrow[d, "{[1,\iota_0]^\sharp}"'] & B\flat X \arrow[d, "\xi"] \\ B\flat X \arrow[r, "\xi"'] & X, \end{tikzcd} \tag{13}$$

where $[1, \iota_0]^\sharp$ is the restriction to kernels of $[1, \iota_0] : (B + X) + X \to B + X$, modulo the morphism $[1, 0] : B + X \to B$.

It is natural to wonder if it is possible to simplify the definition of internal crossed module only in terms of equivariance of actions (as we have done for precrossed module), by requiring the following:

4.3. Peiffer Condition. *For a precrossed module (B, X, ξ, ∂), the diagram*

$$\begin{tikzcd} X\flat X \arrow[r, "\chi_X"] \arrow[d, "\partial\flat 1"'] & X \arrow[d, dash, "1"] \\ B\flat X \arrow[r, "\xi"'] & X, \end{tikzcd} \tag{14}$$

commutes.

The answer in general is *no* (see Example 7.5), although the crossed module condition implies Peiffer Condition, according to:

4.4. Proposition. *Let (B, X, ξ, ∂) be a crossed module in a semiabelian category $\mathcal{C}$. Then it satisfies Peiffer Condition.*

Proof. It suffices to precompose diagram (13) with the morphism

$$\iota_1\flat 1 : X\flat X \longrightarrow (B + X)\flat X$$

On one side, by functoriality of $(-)\flat X$ we get

$$\xi \circ [1_B, \partial]\flat 1_X \circ \iota_1\flat 1_X = \xi \circ ([1_B, \partial] \circ \iota_1)\flat 1_X = \xi \circ \partial\,\flat 1_X.$$

Turning to the other side, let us consider the following diagram

$$\begin{tikzcd} X\flat X \arrow[r, "\kappa_{X,X}"] \arrow[d, "\iota_1\flat 1"'] & X + X \arrow[d, "\iota_1+1"'] \arrow[dr, "{[1,1]}"] & \\ (B + X)\flat X \arrow[r, "\kappa_{B+X,X}"] \arrow[d, "{[1,\iota_1]^\sharp}"'] & (B + X) + X \arrow[d, "{[1,\iota_1]}"'] & X \arrow[dl, "\iota_1"] \\ B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B + X \arrow[d, "\sigma_\xi"] & \\ X \arrow[r, "\sigma_\xi\iota_1"'] & X \rtimes_\xi B & \end{tikzcd}$$

The three rectangles on the left–hand side commute by definition, while on the right–hand side one can easily compute

$$[1, \iota_1] \circ (\iota_1 + 1) = [\iota_1, \iota_1] = \iota_1 \circ [1, 1].$$

The outer perimeter yields the equation

$$\sigma_\xi \circ \iota_1 \circ \xi \circ [1, \iota_1]^\sharp \circ (\iota_1\flat 1) = \sigma_\xi \circ \iota_1 \circ [1, 1] \circ \kappa_{X,X} = \sigma_\xi \circ \iota_1 \circ \chi_X,$$

cancelling the (normal) monomorphism $\sigma_\xi \circ \iota_1$ from both sides we get

$$\xi \circ [1, \iota_1]^\sharp \circ (\iota_1\flat 1) = \chi_X$$

and then the result. ∎

This proof does not use the equivalence K of diagram (4), so that the property above holds in a context more general than the semiabelian one. In the sequel we will get a simpler proof of its reformulation via the Translation Lemma (i.e. by using K).

5. Peiffer precrossed modules, Peiffer reflexive graphs and star-multiplication

Peiffer Condition is another form of the condition given in Theorem 4.6 of [Jan03]. In a semiabelian category, this condition characterizes those precrossed modules corresponding (under the equivalence K) to star-multiplicative graphs:

5.1. Definition. *In a pointed category with finite limits, a star-multiplicative graph is a reflexive graph* $\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$*, together with a chosen kernel of* α $(X, k : X \to A)$ *and a partial composition law* $A \times_B X \xrightarrow{m_*} A$ *that makes diagram below commute:*

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] \arrow[dr, "1"'] & A \times_B X \arrow[d, "m_*"] & X \arrow[l, "\langle \beta\gamma k,1\rangle"'] \arrow[dl, "1"] \\ & X & \end{tikzcd} \quad . \tag{15}$$

Semantically:

$$\begin{tikzcd} y & & 0 \arrow[ll, bend right=10, "g\circ f"'] \arrow[dl, "f"] \\ & x \arrow[ul, "g"] & \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} y & & 0 \arrow[ll, bend right=10, "g"'] \arrow[dl, "1_0"] \\ & 0 \arrow[ul, "g"] & \end{tikzcd} \qquad \begin{tikzcd} x & & 0 \arrow[ll, bend right=10, "f"'] \arrow[dl, "f"] \\ & x \arrow[ul, "1_x"] & \end{tikzcd}$$

With our formulation of the same condition in terms of equivariance of actions, using again the Translation Lemma, we get the notion of a Peiffer reflexive graph.

5.2. Definition. *In a pointed category with finite limits, a* Peiffer (reflexive) graph *is a reflexive graph* $\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$*, together with a chosen kernel* $(X, k : X \to A)$ *of* α *and a (unique)* $d : X \times X \to A$ *such that* $(1_X, d, \partial = \gamma k)$ *is a morphism of split extensions:*

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] \arrow[d, "1_X"'] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[d, "d"] & X \arrow[l, shift left, "\Delta"] \arrow[d, "\partial"] \\ X \arrow[r, "k"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

5.3. THEOREM. *Let $\mathcal{C}$ be an homological category, that is a pointed protomodular regular category (see [BB04]). Given a reflexive graph* $\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$*, a kernel $(X, k : X \to A)$ of α and $\partial = \gamma k$, the following are equivalent:*

1. it is a Peiffer graph
2. there exists a (unique) $d : X \times X \to A$ such that (d, ∂) is a discrete cofibration

$$\begin{tikzcd} X \times X \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_0"'] \arrow[d, "d"'] & X \arrow[l, "\Delta" description] \arrow[d, "\partial"] \\ A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$$

3. it is a star-multiplicative graph
4. it is a star-divisible graph, i.e. there exists a (unique) partial division law $d : X \times X \to A$ that makes diagram below commute:

$$\begin{tikzcd} X \arrow[r, "\langle 1,0 \rangle"] \arrow[dr, "k"'] & X \times X \arrow[d, "d"] & X \arrow[l, "\Delta"'] \arrow[dl, "\beta\partial"] \\ & A & \end{tikzcd} .$$

Semantically:

$$\begin{tikzcd} y & & x \arrow[ll, bend right, "g\circ f^{-1}"'] \\ & 0 \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} y & & 0 \arrow[ll, bend right, "g"'] \\ & 0 \arrow[ul, "g"] \arrow[ur, "1_0"'] & \end{tikzcd} \quad \begin{tikzcd} x & & x \arrow[ll, bend right, "1_x"'] \\ & 0 \arrow[ul, "f"] \arrow[ur, "f"'] & \end{tikzcd}$$

PROOF. 1. $\Rightarrow$ 2.

Let $\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta" description] \end{tikzcd}$ be a Peiffer graph, so that there exists a (unique) $d : X \times X \to A$ such that $(d, \partial = \gamma k)$ is a morphism of points. Firstly, (d, ∂) is actually a morphism of reflexive graphs, as one can see by following the chain of obvious equalities below:

$$\partial\pi_0 = \gamma k \pi_0 = \gamma d\langle 1, 0\rangle \pi_0 = \gamma g.$$

Since π_1 is a regular epimorphism and its kernel is isomorphic to the kernel of α, the so called pointed Barr-Kock condition for protomodular categories (see e.g. [BB04]) implies that the square

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "d"'] & X \arrow[d, "\partial"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

is a pullback square, making (d, ∂) be a discrete cofibration.

2. $\Rightarrow$ 3.

By the condition of cofibration, it follows that there is an isomorphism $\theta : A \times_B X \to X \times X$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \times_B X \arrow[dr, "\theta"] \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_0"'] & & \\
& X \times X \arrow[r, "\pi_1"] \arrow[d, "d"'] & X \arrow[d, "\partial"] \\
& A \arrow[r, "\alpha"'] & B.
\end{tikzcd}
$$

It easy to see that $m_* = \pi_0\theta$ makes $A \mathrel{\substack{\xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow[\gamma]{}}} B$ a star-multiplicative graph.

3. $\Rightarrow$ 4.

If we have a star-multiplicative graph, since $< k, 0 >: X \to A \times_B X$ is a kernel of the regular epimorphism π_1, the pointed Barr-Kock condition tells us immediately that the square

$$
\begin{tikzcd}
A \times_B X \arrow[r, "\pi_1"] \arrow[d, "m_*"'] & X \arrow[d, "0"] \\
X \arrow[r, "0"'] & 0
\end{tikzcd}
$$

is a pullback square, hence, as before, we get an isomorphism $\tau : X \times X \to A \times_B X$ such that the following diagram commutes:

$$
\begin{tikzcd}
X \times X \arrow[dr, "\tau"] \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_0"'] & & \\
& A \times_B X \arrow[r, "\pi_1"] \arrow[d, "m_*"'] & X \arrow[d] \\
& X \arrow[r] & 0
\end{tikzcd}
$$

and $d = \pi_0\tau$ makes $A \mathrel{\substack{\xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow[\gamma]{}}} B$ a star-divisible graph.

4. $\Rightarrow$ 1.

If $A \mathrel{\substack{\xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow[\gamma]{}}} B$ is a star-divisible graph, we get the commutativity of the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "k"'] \arrow[ddr, "0"'] & X \times X \arrow[d, "d"] & X \arrow[l, "\Delta"'] \arrow[dl, "\beta\partial"] \arrow[ddl, "\partial"] \\
& A \arrow[d, "\alpha"] & \\
& B &
\end{tikzcd} .$$

so that αd makes the outer triangles commute. But $\partial\pi_1$ does the same, and, as before, this implies that $\alpha d = \partial\pi_1$. We can conclude then that $A \underset{\gamma}{\overset{\alpha}{\underset{\leftarrow \beta -}{\rightrightarrows}}} B$ is a Peiffer graph. ■

5.4. Remark. Under the hypothesis of the above theorem, any groupoid (=multiplicative graph, see [CPP92]) is always star-multiplicative and Peiffer. In general, the converse is not true (see Example 7.5 in the category of digroups, that is protomodular, but not strongly protomodular). In the sequel we will give some sufficient conditions in order to get an equivalence between the two notions in a restricted protomodular context.

6. Peiffer graphs vs groupoids

Our approach to Peiffer Condition reveals a connection with categorical commutator theory, introduced by M. C. Pedicchio in 1995 ([Ped95]) and deeply developed by D. Bourn and M. Gran. In fact, the partial division morphism d is a cooperator [Bou02] between k and the inversion morphism $\sigma = d\langle 0,1\rangle$:

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "k"'] & X \times X \arrow[d, "d"] & X \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "\sigma"] \\
& A &
\end{tikzcd} .$$

In the language of commutators for the regular case, the existence of such d is equivalent to $[X, \sigma(X)] = 0$.

The morphism σ can be factorized also through the kernel Y of γ:

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\sigma'"'] & & A \\
& Y \arrow[ur, tail, "h"'] &
\end{tikzcd} ,$$

so that $\sigma(X)$ is a subobject of Y. The fruitful case is when the two coincide, so that $[X, Y] = 0$, and this can give, under suitable conditions, a characterization of internal groupoids among internal reflexive graphs.

In a finitely complete Mal'cev category, a reflexive graph is a groupoid iff $[R[\alpha], R[\gamma]] = \Delta$ (see e.g. [BB04]). We are interested in categories where the following condition holds:

Condition (C) A reflexive graph $A \mathrel{\substack{\xrightarrow{\alpha}\\ \xleftarrow{\beta}\\ \xrightarrow[\gamma]{}}} B$ is a groupoid iff $[ker[\alpha], ker[\gamma]] = 0$.

This is not always the case, even if the base category $\mathcal{C}$ is semiabelian (e.g. the category of digroups of Example 7.5). Nevertheless this condition is satisfied in many remarkable context, e.g. when $\mathcal{C}$ is strongly protomodular ([Bou04]) or when it is action representative, or even action accessible ([BB07], [BJ07]).

6.1. Theorem. *Let $\mathcal{C}$ be a homological category satisfying* Condition (C)*. Given a Peiffer reflexive graph* $A \mathrel{\substack{\xrightarrow{\alpha}\\ \xleftarrow{\beta}\\ \xrightarrow[\gamma]{}}} B$ *, the following are equivalent:*

1. *it is a groupoid*
2. *(d, ∂) is a discrete fibration*
3. *the image of the kernel of α through the inversion morphism is the kernel of γ, i.e. $\sigma(X) = Y$*

Proof. 1. $\Rightarrow$ 2. Trivial, since a morphism of reflexive graphs between two groupoids is a discrete fibration if and only if is a discrete cofibration

2. $\Rightarrow$ 3.

If (d, ∂) is a discrete fibration, in the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\langle 0,1 \rangle"] \arrow[d, "\sigma'"'] \arrow[dr, "\sigma"] & X \times X \arrow[r, "\pi_0"] \arrow[d, "d"'] & X \arrow[d, "\partial"] \\
Y \arrow[r, "h"'] & A \arrow[r, "\gamma"] & B
\end{tikzcd}$$

the right square is a pullback and then σ' is an isomorphism, which means $\sigma(X) = Y$.

3. $\Rightarrow$ 1.

Since we have a Peiffer graph, $[X, \sigma(X) = Y] = 0$, and by Condition (C) this implies that $A \mathrel{\substack{\xrightarrow{\alpha}\\ \xleftarrow{\beta}\\ \xrightarrow[\gamma]{}}} B$ is a groupoid. ∎

6.2. Corollary. *Let $\mathcal{V}$ be a pointed strongly protomodular or action accessible variety. If the associated Mal'cev operation p has the following additional property:*

$$p(0, p(x, y, 0), x) = y$$

then $\quad$ *Peiffer* $PCM(\mathcal{V}) \;\cong\; \mathbf{Gpd}(\mathcal{V}) \;\cong\; CM(\mathcal{V})$

Proof. In a Mal'cev variety, the inversion morphism of a Peiffer graph is necessary given by $\sigma(f) = p(0, f, \beta\gamma(f))$. So, given h in Y and taking $f = p(\beta\alpha(h), h, 0)$, it follows that $\gamma(f) = p(\gamma\beta\alpha(h), \gamma(h), 0) = p(\alpha(h), 0, 0) = \alpha(h)$,

and by the additional property,

$$h = p(0, p(\beta\alpha(h), h, 0), \beta\alpha(h)) = p(0, f, \beta\gamma(f)) = \sigma(f).$$

This shows that $\sigma(X) = Y$. ∎

In order to introduce next Example, we recall the notion of *category of* Ω*-groups* (according to [Hig56]).

A category $\mathcal{V}$ of Ω-groups is a variety of groups (in the sense of the universal algebra) such that:

- the group identity is the only operation of arity 0, i.e. the variety is pointed;
- all other operations different from group operation (here written additively), inverse and identity, have arity n, with $n \geq 1$. We shall denote the sets of these operations respectively with Ω_n.

If these data satisfy also the following axioms, for all $\omega \in \Omega_n, \quad n \geq 1$:

$$\omega(x_1, .., x_{i-1}, x + y, x_{i+1}, .., x_n) = \omega(x_1, .., x_{i-1}, x, x_{i+1}, .., x_n) + \omega(x_1, .., x_{i-1}, y, x_{i+1}, .., x_n),$$

$\mathcal{V}$ is said to be a category of *distributive* Ω-groups.

We consider now categories of *distributive* Ω_2-groups, i.e. Ω-groups with only unary and binary operations, as it happens for the *categories of groups with operations* in the sense of G. Orzech (see [Orz72, Por87, Pao]). Examples of such categories are the categories of groups, rings, Lie algebras and many others.

Being varieties of groups, categories of Ω-groups are clearly semiabelian. For distributive Ω_2-groups, they are also strongly semiabelian, i.e. semiabelian and strongly protomodular.

In order to check strong protomodularity, one can apply the useful sufficient condition detailed in [BB04], 6.2.1. Namely, one may show that given any diagram of split exact sequences:

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "\tau"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "\sigma"] & B \arrow[l, shift left, "\beta"] \arrow[d, equal] \\ X' \arrow[r, "\xi'"'] & A' \arrow[r, shift left, "\alpha'"] & B \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

if τ is a normal monomorphism, then the composition $\xi'\tau$ has to be a normal monomorphism as well. Let us recall from [Hig56] that the notion of normal monomorphism, i.e. of a kernel, for a category of distributive Ω-groups can be casted as closure w.r.t. conjugation for the group operation (as for the definition of normal subgroups of a group), together with closure w.r.t. left and right multiplication for the other binary operations (as for the definition of ideals of a ring). With this in mind, applying the criterion above is a matter of pasting the well known proofs in the case of groups and in the case of rings.

6.3. Example. For $\mathcal{V} = \mathbf{Gr}$ or any variety of distributive Ω_2-groups , the condition of the previous corollary is satisfied, since in these cases

$$p(0, p(x, y, 0), x) = -(x - y) + x = y.$$

This gives an explanation why the notion of internal crossed module in the categories of groups, rings, Lie algebras, can be given via Peiffer precrossed module conditions.

7. Peiffer precrossed modules vs crossed modules

In this section we approach the problem of directly extending a star-multiplication to a multiplication and we present some sufficient conditions on semiabelian categories under which the statement of Proposition 4.4 can be inverted.

In a finitely complete pointed category $\mathcal{C}$, let us consider a reflexive graph

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta"'] \end{tikzcd}$$

with a (X,k) kernel of α. It is straightforward to show that the pullback $A\times_B X$ of the pair $(\alpha,\gamma k)$ is (isomorphic to) the kernel of $\alpha\pi_1$. Hence we can consider the split extension

$$\begin{tikzcd} A\times_B X \arrow[r, "h"] & A\times_B A \arrow[r, shift right, "\alpha\pi_1"'] & B \arrow[l, shift right, "\langle\beta,\beta\rangle"'] \end{tikzcd}$$

and, if the $\mathcal{C}$ is semiabelian, the induced action $\bar\xi$ is defined by the diagram:

$$\begin{tikzcd} B\flat(A\times_B X) \arrow[r, "\kappa_{B,A\times_B X}"] \arrow[d, dotted, "\bar\xi"'] & B+(A\times_B X) \arrow[d, "{[\langle\beta,\beta\rangle,h]}"] \\ A\times_B X \arrow[r, "h"'] & A\times_B A. \end{tikzcd}$$

This can be conveniently described by means of the two projections of the codomain, i.e. $\bar\xi$ is univocally determined by the following diagram:

$$\begin{tikzcd} B\flat A \arrow[r, "\beta\flat A"] & A\flat A \arrow[r, "\chi_A"] & A \\ B\flat(A\times_B X) \arrow[u, "B\flat\pi_0"] \arrow[d, "B\flat\pi_1"'] \arrow[rr, dotted, "\bar\xi"] & & A\times_B X \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\ B\flat X \arrow[rr, "\xi"'] & & X \end{tikzcd} \tag{16}$$

The following proposition clarifies the relation between star-multiplicative and multiplicative graphs.

7.1. Proposition. *In a semiabelian category $\mathcal{C}$, let us consider a star-multiplicative graph*

$$\Big(\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & B \arrow[l, "\beta"'] \end{tikzcd},\ \begin{tikzcd} A\times_B X \arrow[r, "m_*"] & X \end{tikzcd}\Big).$$

Then the following statements are equivalent:

1. *The graph is multiplicative;*

2. *there exists an arrow $m : A \times_B A \to A$ such that diagram below is a morphism of split extensions*

$$\begin{tikzcd} A \times_B X \arrow[r, "h"] \arrow[d, "m_*"'] & A \times_B A \arrow[r, shift right, "\alpha\pi_0"'] \arrow[d, "m"] & B \arrow[l, shift right, "\langle\beta,\beta\rangle"'] \arrow[d, "1"] \\ X \arrow[r, "k"'] & A \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "\beta"'] \end{tikzcd} \tag{17}$$

3. *$(m_*, 1_B)$ is an equivariant pair, i.e. the following is a commutative diagram:*

$$\begin{tikzcd} B\flat(A \times_B X) \arrow[r, "B\flat m_*"] \arrow[d, "\bar{\xi}"'] & B\flat X \arrow[d, "\xi"] \\ A \times_B X \arrow[r, "m_*"'] & X \end{tikzcd} \tag{18}$$

Proof. 1. $\Rightarrow$ 2. We must show that m induces a morphism of split extensions. To this end, just compose the right–hand side of diagram (12) with the split epimorphic pair (α, β), thus obtaining the morphism of points

$$m : (\alpha\pi_1, \langle\beta, \beta\rangle) \to (\alpha, \beta).$$

Clearly m_* is the restriction of m to kernels, and this yields the desired morphism of split extensions.

2. $\Rightarrow$ 1. Conversely, let us be given diagram (17) above. Then m makes diagrams (12) commute. Since $\mathcal{C}$ is protomodular, (k, β) is a jointly (strongly) epic pair, then we can prove the commutativity right–hand side triangle of (12) by precomposing with k and β. In the first case we get

$$m \circ \langle\beta\gamma, 1_A\rangle \circ k = m \circ \langle\beta\gamma k, k\rangle = m \circ h \circ \langle\beta\gamma k, 1_X\rangle = k \circ m_* \circ \langle\beta\gamma k, 1_X\rangle = k \circ 1_X = k,$$

where the first equality is a restriction to kernels, the second by left–hand square in (17) and the third by the star-multiplication axiom (15). In the second case we obtain exactly the right–hand side of (17). Similarly we prove the commutativity of the right–hand side triangle of (12).

2. $\Leftrightarrow$ 3. The equivalence of (2.) and (3.) follows by *Translation Lemma 2.3*. ∎

Let us consider the diagram

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] & A \times_B X \arrow[r, shift left, "\pi_1"] & X. \arrow[l, shift left, "\langle\beta\gamma k,1\rangle"] \end{tikzcd} \tag{19}$$

Clearly $\pi_1 \circ \langle\beta\gamma k\rangle = 1_X$. Moreover one can show that $\langle k, 0\rangle$ is the kernel of $\langle\beta\gamma k, 1\rangle$. Then the pair $(\langle k, 0\rangle, \langle\beta\gamma k\rangle)$ is jointly strongly epic.

It is a natural question to ask whether the pair

$$B\flat X \xrightarrow{B\flat\langle k,0\rangle} B\flat(A\times_B X) \xleftarrow{B\flat\langle \beta\gamma k,1\rangle} B\flat X$$

is still (strongly) jointly epic. The answer in general is *no*, as the Counterexample 7.5 clarifies. If this was the case, next proposition would imply that the notions of star-multiplicative and multiplicative graph would coincide in any semiabelian category. Nevertheless there are important situations where this does happen, as witnessed by the example in the category of groups detailed below.

7.2. PROPOSITION. *Let $\mathcal{C}$ be a semiabelian category, and let (α,β,γ) be a star-multiplicative graph (notation as above). If the pair $(B\flat\langle k,0\rangle, B\flat\langle\beta\gamma k\rangle)$ is jointly epic, then the star-multiplication can be extended to a multiplication, i.e. (α,β,γ) is an internal category in $\mathcal{C}$.*

PROOF. Let us consider the precompositions below

$$\begin{tikzcd}
B\flat X \arrow[dr, shift left, "B\flat\langle\beta\gamma k,1\rangle"] \arrow[dr, shift right, "B\flat\langle k,0\rangle"'] & & \\
& B\flat(A\times_B X) \arrow[r, "B\flat m_*"] \arrow[d, "\bar\xi"'] & B\flat X \arrow[d, "\xi"] \\
& A\times_B X \arrow[r, "m_*"'] & X.
\end{tikzcd}$$

By universality of pullbacks we can describe the composite $\bar\xi\circ B\flat\langle k,0\rangle$ by computing its projections onto A and X:

$$\begin{aligned}
&\pi_0\circ\bar\xi\circ B\flat\langle k,0\rangle \overset{(i)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat A\circ B\flat\pi_0\circ B\flat\langle k,0\rangle\\
&\overset{(ii)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat A\circ B\flat k \overset{(iii)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat k =\\
&\overset{(iv)}{=} [1,1]\circ\beta+k\circ\kappa_{B,X} \overset{(v)}{=} [\beta,k]\circ\kappa_{B,X} \overset{(vi)}{=} k\circ\xi
\end{aligned}$$

where (i) holds by the explicit description of $\bar\xi$ of diagram (16), (ii) and (iii) by functoriality of $B\flat-$, (iv) by its own definition, (v) by universal property of sums, (vi) is the definition of ξ. On the other side,

$$\pi_1\circ\bar\xi\circ B\flat\langle k,0\rangle \overset{(i)}{=} \xi\circ B\flat\pi_1\circ B\flat\langle k,0\rangle \overset{(ii)}{=} \xi\circ B\flat 0 \overset{(iii)}{=} \xi\circ 0 = 0,$$

where (i) holds as above, (ii) by functoriality of $-B\flat$, (iii) by its definition.

Finally we get the equality $\bar\xi\circ B\flat\langle k,0\rangle = \langle k\xi,0\rangle$, and we can compute

$$\begin{aligned}
&m_*\circ\bar\xi\circ B\flat\langle k,0\rangle = m_*\circ\langle k\xi,0\rangle \overset{(i)}{=} m_*\circ\langle k,0\rangle\circ\xi \overset{(ii)}{=} \xi =\\
&\overset{(iii)}{=} \xi\circ B\flat(m_*\circ\langle k,0\rangle) \overset{(iv)}{=} \xi\circ B\flat m_*\circ B\flat\langle k,0\rangle,
\end{aligned}$$

where (i) holds by universality of pullbacks, (ii) by left–hand side of diagram (15), (iii) and (iv) by functoriality.

Finally, we are interested in the projections of $B\flat\langle\beta\gamma k,1\rangle$ onto X and A:

$$\begin{aligned}
&\pi_0\circ\bar{\xi}\circ B\flat\langle\beta\gamma k,1\rangle \overset{(i)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat A\circ B\flat\pi_0\circ B\flat\langle\beta\gamma k,1\rangle =\\
&\overset{(ii)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat A\circ B\flat(\beta\gamma k) \overset{(iii)}{=} [1,1]\circ\kappa_{A,A}\circ\beta\flat(\beta\gamma k) =\\
&\overset{(iv)}{=} [1,1]\circ(\beta+\beta\gamma k)\circ\kappa_{B,X} \overset{(v)}{=} [\beta,\beta\gamma k]\circ\kappa_{B,X}
\end{aligned}$$

where (i) holds by the explicit description of $\bar{\xi}$ of diagram (16), (ii) and (iii) by functoriality of $(-)\flat(-)$, (iv) by its definition and (v) by universal property of sums. On the other side,

$$\pi_1\circ\bar{\xi}\circ B\flat\langle\beta\gamma k,1\rangle \overset{(i)}{=} \xi\circ B\flat\pi_1\circ B\flat\langle\beta\gamma k,1\rangle \overset{(ii)}{=} \xi$$

where (i) as above, (ii) holds by functoriality of $B\flat-$. Finally we get the second equality $\bar{\xi}\circ B\flat\langle\beta\gamma k,1\rangle = \langle[\beta,\beta\gamma k]\circ\kappa_{B,X},\xi\rangle$, and we can compute

$$\begin{aligned}
&m_*\circ\bar{\xi}\circ B\flat\langle\beta\gamma k,1\rangle = m_*\circ\langle[\beta,\beta\gamma k]\circ\kappa_{B,X},\xi\rangle \overset{(i)}{=} m_*\circ\langle[\beta\gamma\beta,\beta\gamma k]\circ\kappa_{B,X},\xi\rangle =\\
&\overset{(ii)}{=} m_*\circ\langle\beta\gamma\circ[\beta,k]\circ\kappa_{B,X},\xi\rangle \overset{(iii)}{=} m_*\circ\langle\beta\gamma k\circ\xi,\xi\rangle \overset{(iv)}{=} m_*\circ\langle\beta\gamma k,1\rangle\circ\xi =\\
&\overset{(v)}{=} \xi \overset{(vi)}{=} \xi\circ B\flat(m_*\circ\langle\beta\gamma k,1\rangle) \overset{(vii)}{=} \xi\circ B\flat m_*\circ B\flat\langle\beta\gamma k,1\rangle
\end{aligned}$$

where (i) introduces the identity $\gamma\beta=1_B$, (ii) holds by universal property of sums, (iii) by the definition of ξ, (iv) by universal property of pullbacks, (v) and (vi) by right–hand side of diagram (15), (vii) by functoriality of $B\flat-$. This concludes the proof. ■

7.3. EXAMPLE. Let $\mathcal{C}$ be a semiabelian category, and let $X\xrightarrow{k}Y\underset{p}{\overset{s}{\rightleftarrows}}Z$ be a split exact sequence. By protomodularity one knows that the pair (k,s) is jointly strongly epic. If the pair $(B\flat k,B\flat s)$ is still jointly epic for any split exact sequence (k,p,s), then one can apply Proposition 7.2. In other terms, in categories with this property, all star-multiplicative graphs are multiplicative. This happens in *good* semiabelian categories, as groups and rings.

For instance, let a split exact sequence of groups (k,p,s) be given. We want to show that the group homomorphism $[B\flat k,B\flat s]:B\flat X+B\flat Z\to B\flat Y$ is surjective. In fact, let us consider any element υ of $B\flat Y$. Since $B\flat Y$ is generated by triples as $(b,y,-b)$, we can suppose

$$\upsilon=(b_1,y_1,-b_1)\cdots(b_n,y_n,-b_n)$$

Furthermore, since Y is a semidirect product of groups, any element can be written in the form $y_i=x_i+s(z_i)$, with $x_i\in X$ and $z_i\in Z$. Hence one has

$$\begin{aligned}
\upsilon &= (b_1,x_1+s(z_1),-b_1)\cdots(b_n,x_n+s(z_n),-b_n)\\
&= (b_1,x_1,-b_1)(b_1,s(z_1),-b_1)\cdots(b_n,x_n,-b_n)(b_n,s(z_n),-b_n)\\
&= [B\flat k,B\flat s]((b_1,x_1,-b_1)(b_1,z_1,-b_1)\cdots(b_n,x_n,-b_n)(b_n,z_n,-b_n))
\end{aligned}$$

7.4. Remark. It might be interesting to know if the sufficient condition of Proposition 7.2 is somehow related with the conditions given in Theorem 6.1. Even for strongly protomodular varieties, we do not know the answer to this question. In our opinion, this last condition should hold in any category of interest in the sense of G. Orzech (see [Orz72]), but not in any distributive Ω_2-group. It remains open the problem of finding a semiabelian variety which is not strongly protomodular fulfilling the sufficient condition of Proposition 7.2.

7.5. Counterexample. We consider the semiabelian category of digroups, whose object are sets with two group structures sharing the same unit and morphisms preserving these structures (see [Bou00]). We endow the set $\mathbb{Z}^2$ of ordered pairs of integers with the following two group operations: the ordinary componentwise sum $+$ and an additional operation $\oplus$ defined by:

$$(n,m) \oplus (p,q) = \varphi^{-1}\left(\varphi((n,m)) + \varphi((p,q))\right),$$

where φ is the bijection swapping only $(1,1)$ with $(1,2)$. The pair $(0,0)$ acts as a unit also for $\oplus$, so this way we get the digroup $(\mathbb{Z}^2,+,\oplus)$.

Let us consider the (totally disconnected) reflexive graph given by $\begin{tikzcd} \mathbb{Z}^2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, shift right=2, "\pi_0"'] & \mathbb{Z} \arrow[l, "i_o"'] \end{tikzcd}$, where π_0 is the projection on the first component.

This is a star-multiplicative graph, with $m_*\left((0,n),(0,m)\right) = (0,n+m)$ morphism of digroups.

It is not a multiplicative graph.

Indeed, since $\mathcal{C}$ is unital, if there exists a multiplication $m : \mathbb{Z}^2 \times_{\mathbb{Z}} \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$, it is unique and it extends m_*. As the domain is equal to the codomain, for any p,q,r, $m[(p,q),(p,r)] = (p, f_p(q,r))$ for an appropriate f_p, with $f_0 = m_*$. Since m must preserve the $+$-structure, it must be $f_p(q,r) = q+r$. But this is not compatible with the $\oplus$-structure. In fact

$$m[(0,1),(0,0)] \oplus m[(1,1),(1,1)] = (0,1) \oplus (1,2) = (1,1),$$

while

$$m[(0,1) \oplus (1,1),(0,0) \oplus (1,1)] = m[(1,3),(1,1)] = (1,4).$$

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Dipartimento di Matematica,
Università degli Studi di Milano,
Via C. Saldini, 50,
20133 Milano, Italia
Email: `sandra.mantovani@unimi.it`
`giuseppe.metere@unimi.it`

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