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DIAGONAL MODEL STRUCTURES

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ABSTRACT. The category of bisimplicial presheaves carries a model structure for which the weak equivalences are defined by the diagonal functor and the cofibrations are monomorphisms. This model structure has the most cofibrations of a large family of model structures with weak equivalences defined by the diagonal. The diagonal structure for bisimplicial presheaves specializes to a diagonal model structure for bisimplicial sets, for which the fibrations are the Kan fibrations.

Introduction

The original purpose of this paper was to display a model structure for the category $s^2\mathbf{Set}$ of bisimplicial sets whose cofibrations are the monomorphisms and whose weak equivalences are the diagonal weak equivalences, and then show that this model structure is cofibrantly generated in a very precise way. The project grew to include analogous model structures on categories of bisimplicial presheaves. These are the diagonal model structures of the title.

The results of this paper have been collected here in anticipation of concrete applications. In particular, they are used in the analysis of homotopy types of diagrams and dynamical systems which appears in [9].

It is relatively painless to show that the diagonal model structures exist for all categories $s^2 \operatorname{Pre}(\mathcal{C})$ of bisimplicial presheaves — this result is Theorem 1.4. The proof is essentially a localization argument, since it involves a bounded cofibration statement which appears in Lemma 1.1.

Theorem 1.4 specializes immediately to the existence of a diagonal model structure for bisimplicial sets. The result for bisimplicial sets has already been displayed by other authors [2], [11].

It is also straightforward to show that the diagonal functor and its left adjoint d^* define a Quillen equivalence

$$d^* : s\operatorname{Pre}(\mathcal{C}) \leftrightarrows s^2 \operatorname{Pre}(\mathcal{C}) : d$$

between the injective model structure on simplicial presheaves and the diagonal structure for bisimplicial presheaves; this equivalence appears here as Proposition 1.5. A Quillen

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equivalence

$$d^* : s\mathbf{Set} \leftrightarrows s^2\mathbf{Set} : d$$

between the standard model structure on simplicial sets and the diagonal model structure on bisimplicial sets is an immediate consequence.

The Moerdijk model structure for bisimplicial sets [10], [4] is induced from the standard model structure for simplicial sets by the diagonal functor — this was the first published example of a model structure for bisimplicial sets whose weak equivalences are defined by the diagonal functor. We show that there is a plethora of such model structures intermediate between an analog of the Moerdijk structure for bisimplicial presheaves and the diagonal structure — the precise statement is Theorem 1.9. The proof of this result is a translation of the intermediate model structures story for simplicial presheaves of [8].

The Kan fibrations for bisimplicial sets are defined by a lifting property with respect to the bisimplicial analogues of inclusions of horns in simplices. A horn can be viewed as the part of boundary $\partial\Delta^{p,q}$ of a bisimplex $\Delta^{p,q}$ that results from removing a single cell of maximal total degree. The inclusions of the horns in their corresponding bisimplices are simple examples of anodyne extensions of bisimplicial sets.

The problem of showing that the fibrations of the diagonal model structure for bisimplicial sets are precisely the Kan fibrations is technically interesting, and is the subject of the second section of this paper, culminating in Theorem 2.14.

This theorem is the analogue of well known results for simplicial sets and cubical sets [1], [7]. It has been known in some form since 2003, at least to Cisinski and Joyal-Tierney, but was never published. The proof which is given here is direct, and does not use Cisinski's localization techniques, though some of his ideas are certainly involved.

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1. Bisimplicial presheaves

Recall that a *bisimplicial set* X is a functor

$$X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Set},$$

and a *morphism of bisimplicial sets* is a natural transformation of such functors. Write $X_{p,q} = X(\mathbf{p}, \mathbf{q})$ for ordinal numbers $\mathbf{p}$ and $\mathbf{q}$. Let $s^2\mathbf{Set}$ denote the category of bisimplicial sets.

The bisimplicial set $\hom(\ , (\mathbf{p}, \mathbf{q}))$ which is represented by the pair of ordinal numbers $(\mathbf{p}, \mathbf{q})$ is denoted by $\Delta^{p,q}$, and is called a *standard bisimplex*. The bisimplices are the cells for the category of bisimplicial sets.

As usual, the *diagonal* simplicial set $d(X)$ is defined by

$$d(X)_p = X_{p,p}.$$

This construction defines the diagonal functor

$$d : s^2\mathbf{Set} \to s\mathbf{Set}.$$

The diagonal functor has both a left adjoint d^* and a right adjoint d_*. The left adjoint d^* is defined by extending the assignment

$$d^*\Delta^n = \Delta^{n,n}$$

in a canonical way, while the right adjoint d_* is defined by

$$d_*(Y)_{p,q} = \hom(\Delta^p \times \Delta^q, Y),$$

All functorial constructions on bisimplicial sets extend to presheaves of bisimplicial sets. Let $\mathcal{C}$ be a small Grothendieck site, and let $s^2\,\mathrm{Pre}(\mathcal{C})$ denote the category of functors $X : \mathcal{C}^{op} \to s^2\mathbf{Set}$ and all natural transformations between them — this is the category of bisimplicial presheaves, or presheaves of bisimplicial sets on the site $\mathcal{C}$.

Say that a map $f : X \to Y$ of bisimplicial presheaves is a *diagonal weak equivalence* if the induced simplicial presheaf map $d(X) \to d(Y)$ is a local weak equivalence in the usual sense [5], [6]. A monomorphism of bisimplicial presheaves is a *cofibration*. An *injective fibration* of bisimplicial presheaves is a morphism which has the right lifting property with respect to trivial cofibrations.

Suppose that β is a cardinal number. A bisimplicial presheaf A is said to be *β-bounded* if $|A_{p,q}(U)| < \beta$ for all $p, q \geq 0$ and all objects U in $\mathcal{C}$.

Suppose that α is an infinite cardinal which is an upper bound for the site $\mathcal{C}$ in the sense that $\alpha > |\,\mathrm{Mor}(\mathcal{C}|$. We have the following "bounded cofibration lemma" for bisimplicial presheaves:

1.1. LEMMA. *Suppose that $i : X \to Y$ is a trivial cofibration of bisimplicial presheaves, and that A is an α-bounded subobject of Y. Then Y has an α-bounded subobject B such that $A \subset B$ and the cofibration $B \cap X \to B$ is a diagonal weak equivalence.*

PROOF. There is an induced diagram

$$\begin{tikzcd} & d(X) \arrow[d, "i_*"] \\ d(A) \arrow[r] & d(Y) \end{tikzcd}$$

where i_* is a trivial cofibration of simplicial presheaves and $d(A)$ is an α-bounded subobject of $d(Y)$. The bounded cofibration lemma for simplicial presheaves (this result first appeared as Lemma 12 of [6]) implies that there is an α-bounded subobject D_1 of $d(Y)$ such that $d(A) \subset D_1$ and $D_1 \cap d(X) \to D_1$ is a local weak equivalence. Since D_1 is α-bounded there is an α-bounded subobject A_1 of the bisimplicial presheaf Y such that $A \subset A_1$ and $D_1 \subset d(A_1)$. Repeat this construction inductively to find an ascending families of α-bounded subobjects

$$A \subset A_1 \subset A_2 \subset \cdots \subset Y$$

and

$$d(A) \subset D_1 \subset D_2 \subset \cdots \subset d(Y)$$

such that $D_i \subset d(A_{i+1})$ and the map $D_i \cap d(X) \to D_i$ is a local weak equivalence for all i. Set $B = \cup_i A_i$. Then the map $B \cap X \to B$ of bisimplicial presheaves is a diagonal weak equivalence. ∎

1.2. COROLLARY. *A map $p : X \to Y$ is an injective fibration of bisimplicial presheaves if and only if it has the right lifting property with respect to all α-bounded trivial cofibrations.*

The proof of this corollary is a standard Zorn's lemma argument.

Recall that every simplicial set K can be identified with a horizontally constant bisimplicial set having the same name in a standard way, with $K_{p,q} = K_q$.

I also use the same notation for a bisimplicial set B and its associated constant simplicial presheaf, so that $B(U) = B$ for all objects U of $\mathcal{C}$.

1.3. LEMMA. *A map $q : Z \to Y$ is an injective fibration and a diagonal weak equivalence if and only if it has the right lifting property with respect to all α-bounded cofibrations.*

PROOF. If q has the right lifting property with respect to all α-bounded cofibrations, then it has the right lifting property with respect to all cofibrations, by the usual Zorn's lemma argument. In this case, q has a section $\sigma : Y \to Z$, and the lifting exists in the diagram

$$\begin{tikzcd} Z \sqcup Z \arrow[rr, "(\sigma q, 1)"] \arrow[d] & & Z \arrow[d, "q"] \\ Z \times \Delta^1 \arrow[r, "pr"'] \arrow[urr, dotted] & Z \arrow[r, "q"'] & Y \end{tikzcd}$$

It follows that the induced map $d(q)$ is a simplicial homotopy equivalence, and hence a local weak equivalence.

Suppose that q is an injective fibration and a diagonal weak equivalence. Then q has a factorization

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "q"'] & X \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p has the right lifting property with respect to all α-bounded cofibrations and i is a cofibration. Then p is a diagonal weak equivalence, so the cofibration i is a diagonal weak equivalence, and the lift exists in the diagram

$$\begin{tikzcd} Z \arrow[r, "1"] \arrow[d, "i"'] & Z \arrow[d, "q"] \\ X \arrow[r, "p"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

The map q is therefore a retract of the map p, and has the right lifting property with respect to all α-bounded cofibrations. ∎

The function complex $\mathbf{hom}(X, Y)$ for bisimplicial sets X and Y is the simplicial set whose n-simplices are the bisimplicial set maps $X \times \Delta^n \to Y$.

1.4. Theorem. *Suppose that $\mathcal{C}$ is a small Grothendieck site. Then, with the definitions of cofibration, injective fibration and diagonal weak equivalence given above, the category $s^2 \operatorname{Pre}(\mathcal{C})$ of bisimplicial sets has the structure of a cofibrantly generated closed simplicial model category.*

Properness for the model structure of Theorem 1.4 is proved in Corollary 1.7 below.

Proof. The axioms **CM1**, **CM2** and **CM3** are easy to verify: in particular, **CM2** and **CM3** are straightforward consequences of the corresponding statements for the injective model structure on simplicial presheaves. Similarly, trivial cofibrations are closed under pushout, so that Corollary 1.2 and Lemma 1.3 imply the factorization axiom **CM5**. The lifting axiom **CM4** also follows from Lemma 1.3. The cofibrant generation follows from Corollary 1.2 and Lemma 1.3.

For the simplicial structure, we show that if $i : A \to B$ is a cofibration of bisimplicial presheaves and $j : K \to L$ is a cofibration of simplicial sets, then the cofibration

$$(B \times K) \cup (A \times L) \to B \times L$$

is trivial if either i or j is trivial, but this is a consequence of the corresponding statement for simplicial presheaves. ∎

The model structure of Theorem 1.4 is the *diagonal structure* on the category of bisimplicial presheaves. This result specializes to give diagonal model structures for all categories $s^2\mathbf{Set}^I$ of small diagrams of simplicial sets and to the category $s^2\mathbf{Set}$.

In particular, a cofibration for the diagonal structure on bisimplicial sets is a monomorphism, a weak equivalences is a bisimplicial set map $X \to Y$ such that the induced map $d(X) \to d(Y)$ is a weak equivalence of simplicial sets, and injective fibrations are defined by a right lifting property with respect to trivial cofibrations.

The left adjoint

$$d^* : s\mathbf{Set} \to s^2\mathbf{Set}$$

of the diagonal functor d preserves cofibrations and takes trivial cofibrations to diagonally trivial cofibrations [4, IV.3.12]. It follows that the functors d^* and d define a Quillen adjunction between the standard model structure on simplicial sets and the diagonal structure on bisimplicial sets.

The adjunction map $\eta : \Delta^n \to dd^*(\Delta^n)$ can be identified up to isomorphism with the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$, which map is a weak equivalence. The functors d and d^* both preserve colimits, cofibrations and trivial cofibrations, so an induction on skeleta shows that the adjunction map $\eta : X \to dd^*(X)$ is a weak equivalence for all simplicial sets X. A triangle identity argument then shows that the natural map $\epsilon : d^*d(Y) \to Y$ is a diagonal equivalence for all bisimplicial sets Y.

The corresponding simplicial presheaf map $\eta : X \to dd^*(X)$ is a sectionwise weak equivalence for all X, and it follows that the functor $d^* : s \operatorname{Pre}(\mathcal{C}) \to s^2 \operatorname{Pre}(\mathcal{C})$ takes local weak equivalences to diagonal weak equivalences for simplicial presheaves on a Grothendieck site $\mathcal{C}$. The functors d^* and d therefore determine a Quillen adjunction between the injective model structure for simplicial presheaves and the diagonal model structure for bisimplicial presheaves. The adjunction map $\epsilon : d^*d(Y) \to Y$ is also a sectionwise weak equivalence for all bisimplicial presheaves Y, and we have the following result:

1.5. Proposition. *Suppose that $\mathcal{C}$ is a small Grothendieck site. Then the adjoint functors*

$$d^* : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s^2\,\mathrm{Pre}(\mathcal{C}) : d$$

define a Quillen equivalence between the injective model structure on simplicial presheaves and the diagonal structure on bisimplicial presheaves on the site $\mathcal{C}$.

1.6. Corollary. *The adjoint functors*

$$d^* : s\mathbf{Set} \leftrightarrows s^2\mathbf{Set} : d$$

define a Quillen equivalence between the standard model structure on simplicial sets and the diagonal structure on bisimplicial sets.

1.7. Corollary. *The diagonal model structure on the category $s^2\,\mathrm{Pre}(\mathcal{C})$ is proper.*

Proof. All bisimplicial presheaves are cofibrant, so that pushouts of diagonal weak equivalences along cofibrations are diagonal weak equivalences [4, II.8.5].

The functor d preserves fibrations and pullbacks, and so right properness for the diagonal model structure on bisimplicial presheaves follows from right properness for the injective structure on simplicial presheaves. ■

1.8. Corollary. *The diagonal model structure on the category $s^2\mathbf{Set}$ of bisimplicial sets is proper.*

The Moerdijk model structure is another well known example of a model structure on the category $s^2\mathbf{Set}$ of bisimplicial sets for which the weak equivalences are the diagonal weak equivalences — see [10], and Section IV.3.3 of [4]. The Moerdijk structure is induced from the standard model structure on simplicial sets, in the sense that a bisimplicial set map $X \to Y$ is a fibration (respectively weak equivalence) if and only if the induced map $d(X) \to d(Y)$ is a Kan fibration (respectively weak equivalence) of simplicial sets. The Moerdijk structure is Quillen equivalent to the standard model structure on simplicial sets, via the diagonal functor d and its left adjoint d^*.

Suppose that S is a set of cofibrations of bisimplicial presheaves which contains the set S_0 of all maps $d^*A \to d^*B$ which are induced by α-bounded cofibrations $A \to B$ of simplicial presheaves. Suppose that S further satisfies the closure property that if the map $C \to D$ is in S, then so is the induced cofibration

$$(D \times \partial\Delta^n) \cup (C \times \Delta^n) \to D \times \Delta^n,$$

for all $n \geq 0$. (Here, $X \times K$, for a bisimplicial set X and a simplicial set K is the product of X with the horizontally constant bisimplicial set associated to K.) Let C_S be the saturation of the set S in the class of all cofibrations (monomorphisms) of the bisimplicial set category. I say that C_S is the class of S-cofibrations.

Say that a bisimplicial presheaf map $p : X \to Y$ is an S-fibration if it has the right lifting property with respect to all S-cofibrations which are diagonal weak equivalences.

The proof of the following result follows the outline established in [8]:

1.9. THEOREM. *The category $s^2\,\mathrm{Pre}(\mathcal{C})$ of bisimplicial presheaves, together with the S-cofibrations, diagonal weak equivalences and S-fibrations satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

PROOF. Every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where j is a member of C_S and q has the right lifting property with respect to all members of C_S. Then $q_* : d(Z) \to d(Y)$ is a trivial injective fibration of simplicial presheaves, so that q is a diagonal weak equivalence. The map q is an S-fibration.

The map $f : X \to Y$ also has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where i is a trivial cofibration and p is a fibration for the diagonal model structure of Theorem 1.4. The map p is an S-fibration. The cofibration i has a factorization $i = q \cdot j$ as above, where j is an S-cofibration and q is an S-fibration and a diagonal equivalence. The map j is a diagonal equivalence, so that f has a factorization $f = (p \cdot q) \cdot j$ such that $p\dot{q}$ is an S-fibration and j is an S-cofibration and a diagonal equivalence.

We have verified the model category axiom **CM5**. If $p : X \to Y$ is an S-fibration and a diagonal equivalence, then it is a retract of a map which has the right lifting property with respect to all S-cofibrations, giving **CM4**. The rest of the model category axioms are easily verified.

The simplicial model axiom **SM7** is a consequence of the construction of the class C_S and the instance of this axiom for the injective model structure on simplicial presheaves. The left properness of this structure is an easy consequence of left properness for the diagonal structure on $s^2\,\mathrm{Pre}(\mathcal{C})$, while right properness follows from right properness for the injective structure on $s\,\mathrm{Pre}(\mathcal{C})$.

The cofibrant generation is proved with what is now a familiar trick. Every α-bounded trivial cofibration $\beta : A \to B$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j_\beta"] \arrow[dr, "\beta"'] & Z_\beta \arrow[d, "q_\beta"] \\ & B \end{tikzcd}$$

as in the first paragraph, where j_β is an S-cofibration and q_β has the right lifting property with respect to all S-cofibrations. Then both j_β and q_β are diagonal equivalences. One shows that if $i : C \to D$ is an α-bounded S-cofibration and there is a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ D \arrow[r] & Y \end{tikzcd}$$

where f is a diagonal equivalence, then the diagram has a factorization

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & A \arrow[r] \arrow[d, "j_\beta"] & X \arrow[d, "f"] \\ D \arrow[r] & Z_\beta \arrow[r] & Y \end{tikzcd}$$

for some β.

Finally, if $j : E \to F$ is an S-cofibration and a diagonal equivalence, then j has a factorization

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[dr, "j"'] & V \arrow[d, "p"] \\ & F \end{tikzcd}$$

where p has the right lifting property with respect to all j_β and i is in the saturation of the set of all maps j_β. But then j and p are diagonal equivalences, and the construction of the last paragraph shows that p has the right lifting property with respect to all members of C_S, so that i is a retract of j. This means that the set of all maps j_β generates the class of trivial cofibrations in the model structure defined by the set of cofibrations S. ∎

Say that the model structure of Theorem 1.9 is the *S-model structure* on the category of bisimplicial presheaves.

The S_0-model structure on bisimplicial sets (for whatever infinite cardinal α) is the Moerdijk structure, and the S_0-model structure for bisimplicial presheaves is a locally defined analogue of the Moerdijk structure. An obvious comparison with the various intermediate model structures for simplicial presheaves [8] says that the S_0-model structure for bisimplicial presheaves is a "projective" model structure, while the diagonal model structure of Theorem 1.4 is an "injective" model structure, and all S-model structures have classes of cofibrations lying between these two extremes.

2. Bisimplicial sets

Suppose that K and L are simplicial sets, and let $K\tilde{\times}L$ be the bisimplicial set which is defined by

$$(K\tilde{\times}L)_{p,q} = K_p \times L_q.$$

The bisimplicial set $K\tilde{\times}L$ is the *external product* of K and L.

Examples: 1) The standard bisimplex $\Delta^{p,q}$ has the form

$$\Delta^{p,q} = \Delta^p\tilde{\times}\Delta^q.$$

2) Set

$$\partial\Delta^{p,q} = (\partial\Delta^p\tilde{\times}\Delta^q) \cup (\Delta^p\tilde{\times}\partial\Delta^q) \subset \Delta^p\tilde{\times}\Delta^q = \Delta^{p,q}.$$

Then the *boundary* $\partial\Delta^{p,q}$ of the bisimplex $\Delta^{p,q}$ is generated as a subcomplex by the images of the maps $(d^i, 1) : \Delta^{p-1,q} \to \Delta^{p,q}$ and $(1, d^j) : \Delta^{p,q-1} \to \Delta^{p,q}$.

The following statement about simplicial sets is well known — it is sometimes called the Eilenberg-Zilber Lemma (see [3, (8.3)]) and is used, however silently [4, I.2.3], in all discussions of the standard skeletal decomposition of a simplicial set. The proof is usually left as an exercise.

2.1. Lemma. *Suppose that x, y are non-degenerate simplices of a simplicial set X, and suppose that s, t are ordinal number epimorphisms such that $s^*(x) = t^*(y)$. Then $x = y$ and $s = t$.*

Suppose that X is a bisimplicial set and that $x \in X_{p,q}$. The number $p + q$ is the *total degree* of x.

Suppose that A is a subcomplex of a bisimplicial set X and that $x \in X_{p,q}$ is a bisimplex of $X - A$ of minimal total degree. Write $x : \Delta^{p,q} \to X$ for the classifying map of the bisimplex x. The bisimplices $(d_i, 1)(x)$ and $(1, d_j)(x)$ have smaller total degree than x and are therefore in A, and it follows that there is a pullback diagram

$$\begin{tikzcd} \partial\Delta^{p,q} \arrow[r, "\alpha"] \arrow[d] & A \arrow[d, "i"] \\ \Delta^{p,q} \arrow[r, "x"'] & X \end{tikzcd}$$

of bisimplicial set maps.

2.2. Lemma. *Suppose that A is a subcomplex of a bisimplicial set X and that $x \in X_{p,q}$ is a bisimplex of $X - A$ of minimal total degree. Form the pushout*

$$\begin{tikzcd} \partial\Delta^{p,q} \arrow[r, "\alpha"] \arrow[d] & A \arrow[d, "i"] \\ \Delta^{p,q} \arrow[r, "x"'] & B \end{tikzcd}$$

Then the induced bisimplicial set map $B \to X$ is a monomorphism.

PROOF. If $x = s(y)$ for some degeneracy s (vertical or horizontal), then y has smaller total degree, and so $y \in A$ and $x \in A$. It follows that x is vertically and horizontally non-degenerate.

There is a decomposition

$$B_{r,s} = A_{r,s} \sqcup \{u \times v : \mathbf{r} \times \mathbf{s} \to \mathbf{p} \times \mathbf{q},\ u, v \text{ epi}\}.$$

in all bidegrees.

If $a \in A_{r,s}$ and $u \times v$ have the same image in X, then $a = (u \times v)^*(x)$ is in A so that $x \in A$ by applying a suitable section of $u \times v$, which can't happen. The restriction of $B_{r,s} \to X_{r,s}$ to $A_{r,s}$ is the monomorphism $i : A_{r,s} \to X_{r,s}$. Finally, if the epis $u \times v, u' \times v' : \mathbf{r} \times \mathbf{s} \to \mathbf{p} \times \mathbf{q}$ have the same image in X, then $(u \times v)^*(x) = (u' \times v')^*(x)$ in X.

The bisimplex $(1 \times v)^*(x)$ is horizontally non-degenerate. Otherwise,

$$(1 \times v)^*(x) = (s \times 1)^*(y)$$

for some y and non-trivial ordinal number epi s, and if d is a section of v then

$$x = (1 \times d)^*(1 \times v)^*(x) = (1 \times d)^*(s \times 1)^*(y) = (s \times 1)^*(1 \times d)^*(y)$$

so that x is horizontally degenerate. Similarly, $(1 \times v')^*(x)$ is horizontally non-degenerate, and so Lemma 2.1 and the relations

$$(u \times 1)^*(1 \times v)^*(x) = (u' \times 1)^*(1 \times v')^*(x)$$

together imply that $u = u'$ and $(1 \times v)^*(x) = (1 \times v')^*(x)$, so that $v = v'$ ∎

2.3. COROLLARY. *The set of inclusions $\partial\Delta^{p,q} \subset \Delta^{p,q}$ generates the class of cofibrations of $s^2\mathbf{Set}$.*

The class $\mathcal{A}$ of *anodyne extensions* of $s^2\mathbf{Set}$ is the saturation of the set of bisimplicial set maps S, which consists of all morphisms

$$(\Lambda^r_k \tilde{\times} \Delta^s) \cup (\Delta^r \tilde{\times} \partial\Delta^s) \subset \Delta^r \tilde{\times} \Delta^s = \Delta^{r,s}$$

as well as all morphisms

$$(\partial\Delta^r \tilde{\times} \Delta^s) \cup (\Delta^r \tilde{\times} \Lambda^s_j) \subset \Delta^r \tilde{\times} \Delta^s = \Delta^{r,s}$$

The class $\mathcal{A}$ contains the set of all cofibrations

$$(A \tilde{\times} D) \cup (B \tilde{\times} C) \subset B \tilde{\times} D$$

induced by cofibrations $A \to B$ and $C \to D$, where one of the two maps is a trivial cofibration of simplicial sets. The diagonal of such a map is the trivial cofibration

$$(A \times D) \cup (B \times C) \subset B \times D.$$

in simplicial sets.

In particular, we have the following:

2.4. Lemma. *Every anodyne extension of bisimplicial sets is a diagonal weak equivalence.*

Say that a map $p : X \to Y$ of bisimplicial sets is a *Kan fibration* if it has the right lifting property with respect to all anodyne extensions.

Every injective fibration is a Kan fibration. The purpose of the remainder of this section is to prove the converse assertion, so that the injective fibrations of bisimplicial sets are precisely the Kan fibrations. This statement appears as Theorem 2.14 below.

Suppose that X is a bisimplicial set and that K is a simplicial set. The bisimplicial set $X \times K$ has bisimplices defined by the assignment

$$(X \times K)_{p,q} = X_{p,q} \times K_q.$$

There is a natural isomorphism

$$d(X \times K) \cong d(X) \times K.$$

The construction $(X, K) \mapsto X \times K$ preserves diagonal weak equivalences in bisimplicial sets X and weak equivalences in simplicial sets K.

2.5. Lemma. *Suppose that $i : A \to B$ is a cofibration of bisimplicial sets and that $j : K \to L$ is a cofibration of simplicial sets. Then the induced map*

$$(i, j)_* : (B \times K) \cup (A \times L) \to B \times L$$

is a cofibration which is an anodyne extension if either i or j is a trivial cofibration of simplicial sets.

Proof. The map

$$(\Delta^{r,s} \times K) \cup (\partial\Delta^{r,s} \times L) \to \Delta^{r,s} \times L$$

can be identified with the map

$$(\partial\Delta^r \tilde{\times} (\Delta^s \times L)) \cup (\Delta^r \tilde{\times}((\partial\Delta^s \times L) \cup (\Delta^s \times K))) \to \Delta^r \tilde{\times} (\Delta^s \times L),$$

which is a cofibration.

The simplicial set map

$$(\partial\Delta^s \times L) \cup (\Delta^s \times K) \to \Delta^s \times L$$

is a trivial cofibration if j is a trivial cofibration, so that the bisimplicial set map $(i, j)_*$ is an anodyne extension in general if j is a trivial cofibration.

The remaining assertion, that $(i, j)_*$ is an anodyne extension if i is an anodyne extension, has a similar proof. ∎

Suppose that X and Y are bisimplicial sets. The collection of bisimplicial set maps

$$X \times \Delta^n \to Y$$

is the set of n-simplices of the simplicial set $\mathbf{hom}(X,Y)$. If $p : X \to Y$ is a Kan fibration and A is a bisimplicial set, then the induced map $p_* : \mathbf{hom}(A,X) \to \mathbf{hom}(A,Y)$ is a fibration of simplicial sets since all maps $A \times \Lambda^n_k \to A \times \Delta^n$ are anodyne extensions by Lemma 2.5.

If $f : A \to Y$ is a map of bisimplicial sets, then f is a vertex of the simplicial set $\mathbf{hom}(A,Y)$, and we can form the pullback diagram

$$\begin{tikzcd} \mathbf{hom}_f(A,X) \arrow[r] \arrow[d] & \mathbf{hom}(A,X) \arrow[d, "p_*"] \\ * \arrow[r, "f"'] & \mathbf{hom}(A,Y) \end{tikzcd}$$

The simplicial set $\mathbf{hom}_f(A,X)$ is the *space of liftings of the map* f. It is a Kan complex since the bisimplicial set map p is a Kan fibration.

The n-simplices of $\mathbf{hom}_f(A,X)$ are commutative diagrams of the form

$$\begin{tikzcd} A \times \Delta^n \arrow[r] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ A \arrow[r, "f"'] & Y \end{tikzcd}$$

The functor $s^2\mathbf{Set}/Y \to s\mathbf{Set}$ which takes an object $f : A \to Y$ to the simplicial set $\mathbf{hom}_f(A,X)$ has a left adjoint which takes a simplicial set K to the object

$$A \times K \xrightarrow{pr} A \xrightarrow{f} Y.$$

A map

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

of bisimplicial sets over Y is said to be an *anodyne equivalence* over Y if the simplicial set maps

$$\mathbf{hom}_g(B,X) \xrightarrow{\alpha^*} \mathbf{hom}_f(A,X)$$

are weak equivalences for all Kan fibrations $p : X \to Y$.

2.6. Lemma. *Suppose that the map $A \xrightarrow{\alpha} B \xrightarrow{g} Y$ of bisimplicial sets over Y is defined by a cofibration α, and let $f = g \cdot \alpha$. Suppose that $p : X \to Y$ is a Kan fibration. Then the induced map*

$$\alpha^* : \mathbf{hom}_g(B,X) \to \mathbf{hom}_f(A,X)$$

is a Kan fibration. If α is an anodyne extension, then α^* is a trivial Kan fibration.

PROOF. Use Lemma 2.5 to see that the lifting exists in all diagrams

$$\begin{tikzcd}
(B \times \Lambda^n_k) \cup (A \times \Delta^n) \arrow[rr] \arrow[d] & & X \arrow[d, "p"] \\
B \times \Delta^n \arrow[r, "pr"'] \arrow[urr, dotted] & B \arrow[r, "g"'] & Y
\end{tikzcd}$$

Similarly, if $\alpha : A \to B$ is an anodyne extension, then the lifting exists in all diagrams

$$\begin{tikzcd}
(B \times \partial\Delta^n) \cup (A \times \Delta^n) \arrow[rr] \arrow[d] & & X \arrow[d, "p"] \\
B \times \Delta^n \arrow[r, "pr"'] \arrow[urr, dotted] & B \arrow[r, "g"'] & Y
\end{tikzcd}$$

so that α^* is a trivial fibration. ∎

2.7. COROLLARY. *Suppose that $\alpha : A \to B$ is an anodyne extension. Then any map $A \xrightarrow{\alpha} B \to Y$ is an anodyne equivalence of bisimplicial sets over Y.*

2.8. LEMMA. *If $\alpha : K \to K'$ and $\beta : L \to L'$ are weak equivalences of simplicial sets, then any map*

$$\alpha \tilde{\times} \beta : K \tilde{\times} L \to K' \tilde{\times} L' \to Y$$

is an anodyne equivalence of bisimplicial sets over Y.

PROOF. We show that the map

$$\alpha \times 1 : K \tilde{\times} L \to K' \tilde{\times} L \to Y$$

is an anodyne weak equivalence.

This is true if α is a trivial cofibration by Corollary 2.7, and is therefore true in general since all simplicial sets are cofibrant. ∎

If X is a bisimplicial set, then the simplicial set maps

$$\Delta^n \times X_{n,m} \to X_{*,m}$$

induce bisimplicial set maps

$$\gamma_n : \Delta^n \tilde{\times} X_n \to X.$$

The bisimplicial set X has a filtration $\mathrm{sk}_n X$ by (horizontal) skeleta, and there are natural pushout diagrams

$$\begin{tikzcd}
s_{[r]} X_{n-1} \arrow[r, "s_{r+1}"] \arrow[d] & s_{[r]} X_n \arrow[d] \\
X_n \arrow[r, "s_{r+1}"'] & s_{[r+1]} X_n
\end{tikzcd} \tag{1}$$

of simplicial sets and pushout diagrams

$$\begin{tikzcd} (\Delta^{n+1} \tilde{\times} s_{[n]} X_n) \cup (\partial\Delta^{n+1} \tilde{\times} X_{n+1}) \arrow[r] \arrow[d] & \mathrm{sk}_n X \arrow[d] \\ \Delta^{n+1} \tilde{\times} X_{n+1} \arrow[r] & \mathrm{sk}_{n+1} X \end{tikzcd} \tag{2}$$

of bisimplicial sets, in which the vertical maps are cofibrations. The subobject

$$s_{[r]} X_n := \cup_{i \leq r}\ s_i(X_{n-1})$$

is a union of images of horizontal degeneracies. See also [4, IV.1.7].

2.9. Lemma. *Suppose that $A \xrightarrow{\alpha} B \xrightarrow{g} Y$ is a map of bisimplicial sets over Y such that the map $\alpha : A_n \to B_n$ is a weak equivalence of simplicial sets in each horizontal degree n. Then α is an anodyne equivalence over Y.*

Proof. Write $f = g \cdot \alpha$. Suppose that $p : X \to Y$ is a Kan fibration of bisimplicial sets.

The functor which takes $f : A \to Y$ to $\mathbf{hom}_f(A, X)$ takes cofibrations to Kan fibrations by Lemma 2.6. It follows that anodyne weak equivalences satisfy a patching property for pushouts along cofibrations. One can then show inductively that the induced maps $s_{[r]} A \to s_{[r]} B \to Y$ and $\mathrm{sk}_n A \to \mathrm{sk}_n B \to Y$ are anodyne equivalences over Y.

The vertical maps in the diagram

$$\begin{tikzcd} \mathrm{sk}_0 A \arrow[r] \arrow[d] & \mathrm{sk}_1 A \arrow[r] \arrow[d] & \mathrm{sk}_2 A \arrow[r] \arrow[d] & \dots \\ \mathrm{sk}_0 B \arrow[r] & \mathrm{sk}_1 B \arrow[r] & \mathrm{sk}_2 B \arrow[r] & \dots \end{tikzcd}$$

are anodyne weak equivalences, and the horizontal maps are cofibrations. It follows that the induced map

$$\mathbf{hom}_g(B, X) \cong \varprojlim_n \mathbf{hom}_g(\mathrm{sk}_n B, X) \to \varprojlim_n \mathbf{hom}_f(\mathrm{sk}_n A, X) = \mathbf{hom}_f(A, X)$$

is a weak equivalence. ∎

2.10. Lemma. *Suppose that the map $Z \xrightarrow{\pi} W \xrightarrow{g} Y$ of bisimplicial sets over Y is defined by a map π which is a fibration and a diagonal weak equivalence. Then the map π is an anodyne equivalence of bisimplicial sets over Y.*

Proof. Write $f = g \cdot \pi$.

The composite

$$Z \times \Delta^1 \xrightarrow{pr} Z \xrightarrow{f} X$$

is a cylinder for f in $s^2\mathbf{Set}/Y$.

The map π is a trivial fibration of $s^2\mathbf{Set}/Y$, and all objects of this category are cofibrant. It follows that the map $\pi : f \to g$ is a fibre homotopy equivalence, for the choice of cylinder above. If the maps $\alpha, \beta : Z \to W \to X$ are fibre homotopic, then they induce the same maps

$$\alpha^*, \beta^* : \mathbf{hom}_g(W, X) \to \mathbf{hom}_f(Z, X)$$

in the homotopy category for simplicial sets, for all Kan fibrations $p : X \to Y$. It follows that the map

$$\pi^* : \mathbf{hom}_g(W, X) \to \mathbf{hom}_f(Z, X)$$

induces an isomorphism in the homotopy category, and is therefore a weak equivalence of simplicial sets for all Kan fibrations $p : X \to Y$. ■

2.11. Lemma. *Suppose that every diagonal weak equivalence $\alpha : f \to g$ over a bisimplicial set Y is an anodyne equivalence over Y. Then every Kan fibration $p : X \to Y$ is a diagonal fibration.*

Proof. Every Kan fibration $p : X \to Y$ has the right lifting property with respect to all cofibrations $j : A \to B$ which define anodyne equivalences $A \xrightarrow{j} B \xrightarrow{\beta} Y$. In effect, the corresponding simplicial set maps $\mathbf{hom}_\beta(B, X) \to \mathbf{hom}_{\beta \cdot j}(A, X)$ are trivial fibrations and are therefore surjective in degree 0.

Thus, if every diagonal weak equivalence over Y is an anodyne equivalence over Y, then every Kan fibration $p : X \to Y$ has the right lifting property with respect to all cofibrations which are diagonal equivalences. ■

2.12. Lemma. *Suppose that in the diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & \Delta^{p,q} & \end{tikzcd}$$

the map f is a diagonal weak equivalence of bisimplicial sets. Then the map f defines an anodyne equivalence over $\Delta^{p,q}$.

Proof. We can suppose that the maps π and π' are Kan fibrations.

If the map

$$\pi : X \to \Delta^{p,q} = \Delta^p \tilde{\times} \Delta^q$$

is a Kan fibration, then all maps

$$X_n \to \bigsqcup_{\mathbf{n} \to \mathbf{p}} \Delta^q$$

are fibrations of simplicial sets, and all diagrams

$$\begin{tikzcd} X_n \arrow[r, "\theta^*"] \arrow[d] & X_m \arrow[d] \\ \bigsqcup_{\mathbf{n} \to \mathbf{p}} \Delta^q \arrow[r, "\theta^*"'] & \bigsqcup_{\mathbf{m} \to \mathbf{p}} \Delta^q \end{tikzcd} \tag{3}$$

are homotopy cartesian.

The claim that the diagram (3) is homotopy cartesian is proved by forming the pullback diagram

$$\begin{tikzcd} v^*X \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^{p,0} \arrow[r, "1\tilde{\times}v"'] & \Delta^{p,q} \end{tikzcd}$$

corresponding to a vertex $v : \Delta^0 \to \Delta^q$. Then the diagram

$$\begin{tikzcd} v^*X_n \arrow[r, "\theta^*"] \arrow[d] & v^*X_m \arrow[d] \\ \bigsqcup_{\mathbf{n}\to\mathbf{p}} \Delta^0 \arrow[r, "\theta^*"'] & \bigsqcup_{\mathbf{m}\to\mathbf{p}} \Delta^0 \end{tikzcd} \tag{4}$$

is weakly equivalent to the diagram (3), and so one can assume that $q = 0$.

In horizontal degree n, $X_n = \sqcup_{\sigma:\mathbf{n}\to\mathbf{p}} X_\sigma$, where X_σ is the fibre over σ for the map $X_n \to \Delta^p_n$. It is enough to show that every ordinal number monomorphism $d : \mathbf{m} \to \mathbf{n}$ induces trivial fibrations $X_\sigma \to X_{d^*\sigma}$. The solution of the lifting problem

$$\begin{tikzcd} \partial\Delta^r \arrow[r] \arrow[d] & X_\sigma \arrow[d] \\ \Delta^r \arrow[r] \arrow[ur, dotted] & X_{d^*\sigma} \end{tikzcd}$$

is equivalent to a solution of the corresponding lifting problem

$$\begin{tikzcd} (\Delta^n\tilde{\times}\partial\Delta^r) \cup (\Delta^m\tilde{\times}\Delta^r) \arrow[r] \arrow[d] & X \\ \Delta^n\tilde{\times}\Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

in bisimplicial sets. The dotted arrow extension exists in the diagram

$$\begin{tikzcd} (\Delta^n\tilde{\times}\partial\Delta^r) \cup (\Delta^m\tilde{\times}\Delta^r) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n\tilde{\times}\Delta^r \arrow[r, dotted] & \Delta^p\tilde{\times}\Delta^0 \end{tikzcd}$$

and so the desired lifting problem is solved since the map

$$X \to \Delta^{p,0} = \Delta^p\tilde{\times}\Delta^0$$

is a Kan fibration.

In particular, given a Kan fibration $X \to \Delta^{p,q}$, the bisimplicial set X is determined by simplicial sets X_σ, one for each $\sigma : \mathbf{n} \to \mathbf{p}$, and weak equivalences $X_\sigma \to X_{\theta^*\sigma}$ which are functorial in maps between simplices of Δ^p.

Let $1 : \mathbf{p} \to \mathbf{p}$ be the generating simplex for Δ^p. The weak equivalences $X_1 \to X_\sigma$ define a map

$$\begin{tikzcd} \Delta^p \tilde{\times} X_1 \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \Delta^{p,q} & \end{tikzcd}$$

which is a levelwise equivalence, hence an anodyne equivalence over $\Delta^{p,q}$.

The induced map

$$1 \tilde{\times} f_* : \Delta^p \tilde{\times} X_1 \to \Delta^p \tilde{\times} Y_1$$

is a diagonal equivalence, and it follows that the map $f_* : X_1 \to Y_1$ is a weak equivalence of simplicial sets. The map $1 \tilde{\times} f_*$ is therefore a levelwise equivalence, and hence an anodyne equivalence over $\Delta^{p,q}$. The original map f is therefore an anodyne equivalence. ∎

We then have the following consequence of Lemma 2.11 and Lemma 2.12:

2.13. Corollary. *Every Kan fibration $p : X \to \Delta^{p,q}$ is a diagonal fibration.*

We close with the main result of this section.

2.14. Theorem. *The map $p : X \to Y$ is a diagonal fibration if and only if it is a Kan fibration.*

Proof. We show that every Kan fibration which is a diagonal weak equivalence has the right lifting property with respect to all cofibrations.

Suppose that this is so, and let $i : A \to B$ be a cofibration which is a diagonal weak equivalence. Find a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that j is anodyne and p is a Kan fibration. Then, subject to the claim of the first paragraph, the map p is a diagonal weak equivalence and the lifting exists in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ B \arrow[r, "1"'] \arrow[ur, dotted] & B \end{tikzcd}$$

Then the map i is a retract of j, and is therefore an anodyne extension. Thus, the classes of diagonal trivial cofibrations and anodyne extensions coincide, so the classes of diagonal fibrations and Kan fibrations coincide.

Suppose that $p : X \to Y$ is a Kan fibration and a diagonal equivalence. Form the pullback diagrams

$$\begin{tikzcd} p^{-1}(\sigma) \arrow[r] \arrow[d, "p_*"'] & X \arrow[d, "p"] \\ \Delta^{p,q} \arrow[r, "\sigma"'] & Y \end{tikzcd} \tag{5}$$

for all bisimplices σ. If

$$\begin{tikzcd} \Delta^{r,s} \arrow[rr] \arrow[dr, "\tau"'] & & \Delta^{p,q} \arrow[dl, "\sigma"] \\ & Y & \end{tikzcd}$$

is a map of simplices, then the maps p_* in the pullback diagram

$$\begin{tikzcd} p^{-1}(\tau) \arrow[r] \arrow[d, "p_*"'] & p^{-1}(\sigma) \arrow[d, "p_*"] \\ \Delta^{r,s} \arrow[r] & \Delta^{p,q} \end{tikzcd}$$

are diagonal fibrations by Corollary 2.13, so that the map $p^{-1}(\tau) \to p^{-1}(\sigma)$ is a diagonal equivalence since the diagonal model structure is proper. It follows from Quillen's Theorem B [4, IV.5.7] that all diagrams (3) are homotopy cartesian for the diagonal model structure.

In particular, the maps p_* are diagonal equivalences, so that the lifts exist in all diagrams

$$\begin{tikzcd} \partial\Delta^{p,q} \arrow[r] \arrow[d] & p^{-1}(\sigma) \arrow[d, "p_*"] \\ \Delta^{p,q} \arrow[r, "1"'] \arrow[ur, dotted] & \Delta^{p,q} \end{tikzcd}$$

The map $p : X \to Y$ is therefore a trivial diagonal fibration. ∎

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