

ON THE REPRESENTATIONS OF 2-GROUPS IN BAEZ-CRANS 2-VECTOR SPACES

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ABSTRACT. We study the theory of representations of a 2-group \mathbb{G} in Baez-Crans 2-vector spaces over a field k of arbitrary characteristic, and the corresponding 2-vector spaces of intertwiners. We also characterize the irreducible and indecomposable representations. Finally, it is shown that when the 2-group is finite and the base field k is of characteristic zero or coprime to the orders of the homotopy groups of \mathbb{G} , the theory essentially reduces to the theory of k -linear representations of the first homotopy group of \mathbb{G} , the remaining homotopy invariants of \mathbb{G} playing no role.

1. Introduction

In the last two decades there have been a few attempts to generalize the representation theory of groups to the higher dimensional setting of categories. See Baez *et al* [2], Bartlett [4], Crane and Yetter [6], Elgueta [8, 9], Ganter and Kapranov [12], and Ganter [11].

By analogy with the classical setting, it is natural to represent 2-groups in a suitable categorification of the category \mathcal{Vect}_k of (finite dimensional) vector spaces over a ground field k , often called the 2-category of *2-vector spaces* over k .

One of the first proposals of definition of 2-vector space is that of Baez and Crans [3]. According to these authors, a 2-vector space over k is an internal category in \mathcal{Vect}_k , and they proved that this is the same thing as a 2-term chain complex of vector spaces over k , i.e. a k -linear map $d : V_1 \rightarrow V_0$. To our knowledge, the unique existing work

The first author is supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações). Also supported by the Generalitat de Catalunya (Project: 2014 SGR 634) and from by DGI of Spain, Project MTM2011-22554. The first author would like to thank João Faria Martins for introducing him into this subject, and also the kind hospitality during his stay at the Departament de Matemàtiques, Universitat Politècnica de Catalunya.

The second author acknowledges the financial support from the Generalitat de Catalunya (Project: 2014 SGR 634), the Ministerio de Economía y Competitividad of Spain (Project: MTM2015-69135-9) and from the Centro de Matemática e Aplicações of the Universidade Nova de Lisboa (Project: UID/MAT/00297/2013), Portugal, and the kind hospitality during his stay at the Centro de Matemática e Aplicações.

Both authors would like to thank the anonymous referee, whose helpful comments greatly improved this paper.

Received by the editors 2016-07-25 and, in final form, 2016-10-04.

Transmitted by Ross Street. Published on 2016-10-06.

2010 Mathematics Subject Classification: 18D05, 18D10, 20L05.

Key words and phrases: 2-groups (categorical groups); 2-vector spaces; Representations; 2-categories.

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on the representation theory of 2-groups in these 2-vector spaces is the very preliminary presentation by Forrester-Barker [10].

The purpose of this paper is to further develop the representation theory of a 2-group \mathbb{G} in the 2-category of Baez-Crans 2-vector spaces over a field k (or more generally, in $\mathbf{Ch}_2(\mathcal{A})$ for any k -linear abelian category \mathcal{A} such that all short exact sequences split). As it should be expected, the theory strongly depends on the characteristic of k . However, in sharp contrast to what happens in the classical representation theory of finite groups, when the characteristic of k is zero or coprime to the orders of the homotopy groups of \mathbb{G} the resulting theory is not rich enough to make it possible to recover the 2-group from the corresponding 2-category of representations. In fact, we shall see that in this case the theory essentially reduces to the representation theory of the group of isomorphism classes of objects in \mathbb{G} , the remaining homotopy information about the 2-group being completely lost.

The paper is organized as follows. In Section 2, we briefly recall the definition of the 2-category $\mathbf{Ch}_2(\mathcal{A})$ of 2-term chain complexes of objects in any abelian category \mathcal{A} , and we discuss how this 2-category simplifies when \mathcal{A} is *split*, i.e. such that every short exact sequence splits. In Section 3, we give a detailed description of the 2-category of representations of a 2-group \mathbb{G} in $\mathbf{Ch}_2(\mathcal{A})$, and we study some features of this 2-category when \mathcal{A} is a split k -linear category. In particular, we describe the 2-vector spaces of intertwiners between any representations, we introduce two notions of monomorphisms and characterize them, and we identify the corresponding irreducible objects as well as the indecomposable representations. Finally, in Section 4 we prove that the theory "collapses" for finite 2-groups and in characteristic zero (or coprime to the orders of the homotopy groups of \mathbb{G}).

To avoid writing a too long paper, we will assume the reader is familiar with the notions of 2-group and 2-category, and with the corresponding notions of morphism, which are understood in the weak sense, including the notions of pseudonatural transformation and modification. We refer the reader to Leinster [13] or Borceux [5] for an introduction to 2-categories, and to Baez and Lauda [1] for an introduction to 2-groups.

Notation. We will use letters like $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ to denote categories, and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ to denote 2-categories. Vertical composition of 2-cells will be denoted by juxtaposition, and composition of 1-cells and horizontal composition of 2-cells by \circ .

Note. After finishing the first draft of this paper and uploading it to the arXiv, we were informed that N. Gurski and J. Copeland already proved a result similar to our Theorem 4.3 and presented it at the International Category Theory Conference 2008, but they never published it.

2. The 2-category of Baez-Crans 2-vector spaces

Let us start by describing the 2-category $\mathbf{Ch}_2(\mathcal{A})$ of 2-term chain complexes (i.e. chain complexes concentrated in degrees 1 and 0) in any abelian category \mathcal{A} . We will be mainly concerned with the case $\mathcal{A} = \mathcal{Vect}_k$, the category of finite dimensional vector spaces over a

field k . $\mathbf{Ch}_2(k)$ is short notation for $\mathbf{Ch}_2(\mathcal{Vect}_k)$. We will refer to $\mathbf{Ch}_2(k)$ as the 2-category of Baez-Crans 2-vector spaces over k .

2.1. Let \mathcal{A} be an arbitrary abelian category. An object of $\mathbf{Ch}_2(\mathcal{A})$ is a morphism of \mathcal{A} , that is $d_V = d : V_1 \rightarrow V_0$, denoted by V_\bullet . The morphism d is called the differential.

A 1-cell $f_\bullet = (f_1, f_0) : V_\bullet \rightarrow W_\bullet$ is a commutative square

$$\begin{array}{ccc} V_1 & \xrightarrow{d_V} & V_0 \\ f_1 \downarrow & & \downarrow f_0 \\ W_1 & \xrightarrow{d_W} & W_0, \end{array}$$

and a 2-cell $\sigma : f_\bullet \Rightarrow g_\bullet : V_\bullet \rightarrow W_\bullet$ is a morphism $\sigma : V_0 \rightarrow W_1$ in \mathcal{A} such that

$$\begin{aligned} d_W \circ \sigma &= g_0 - f_0 \\ \sigma \circ d_V &= g_1 - f_1. \end{aligned}$$

The composition of 1-cells is given by the composition in \mathcal{A} , that is, given $(f_1, f_0) : U_\bullet \rightarrow V_\bullet$ and $(g_1, g_0) : V_\bullet \rightarrow W_\bullet$ the composite is

$$(g_1, g_0) \circ (f_1, f_0) = (g_1 \circ f_1, g_0 \circ f_0) : U_\bullet \rightarrow W_\bullet,$$

and the identity morphisms are given by $1_{V_\bullet} = (1_{V_1}, 1_{V_0})$.

The vertical composite of $\sigma : f_\bullet \Rightarrow g_\bullet$ and $\tau : g_\bullet \Rightarrow h_\bullet$ is given by the addition in \mathcal{A} , that is

$$\tau \sigma = \tau + \sigma : f_\bullet \Rightarrow h_\bullet,$$

while horizontal composite of $\sigma : f_\bullet \Rightarrow g_\bullet : U_\bullet \rightarrow V_\bullet$ and $\sigma' : f'_\bullet \Rightarrow g'_\bullet : V_\bullet \rightarrow W_\bullet$ is given by the map

$$\begin{aligned} \sigma' \circ \sigma &= f'_1 \circ \sigma + \sigma' \circ g_0 \\ &= g'_1 \circ \sigma + \sigma' \circ f_0. \end{aligned}$$

Finally, identity 2-cells are given by $1_{f_\bullet} = 0 : V_0 \rightarrow W_1$ for any 1-cell $f_\bullet : V_\bullet \rightarrow W_\bullet$. In particular, whiskerings are given by

$$\sigma \circ 1_{f_\bullet} = \sigma \circ f_0, \quad 1_{f'_\bullet} \circ \sigma = f'_1 \circ \sigma.$$

It is straightforward to check that $\mathbf{Ch}_2(\mathcal{A})$ is a strict 2-category. In fact, it is a category enriched in groupoids. Each 2-cell τ is invertible with inverse $-\tau$.

2.2. LEMMA. A 2-term chain complex $d : V_1 \rightarrow V_0$ is equivalent in $\mathbf{Ch}_2(\mathcal{A})$ to the zero complex $0_\bullet = 0 \rightarrow 0$ if and only if the differential d is an isomorphism in \mathcal{A} .

PROOF. It readily follows from the definitions that $V_\bullet \simeq 0_\bullet$ if and only if there exists a 2-cell $1_{V_\bullet} \Rightarrow 0_{V_\bullet}$, and this happens if and only if d is an isomorphism. ■

2.3. LEMMA. *For any object W of \mathcal{A} and any object V_\bullet of $\mathbf{Ch}_2(\mathcal{A})$, the 2-term chain complexes $d : V_1 \rightarrow V_0$ and $d \oplus 1_W : V_1 \oplus W \rightarrow V_0 \oplus W$ are equivalent in $\mathbf{Ch}_2(\mathcal{A})$.*

PROOF. Let $\pi_i : V_i \oplus W \rightarrow V_i$, $\iota_i : V_i \rightarrow V_i \oplus W$ be the canonical projections and injections for $i = 0, 1$. Then the 1-cell $\iota_\bullet = (\iota_0, \iota_1) : V_\bullet \rightarrow V_\bullet \oplus W$ is an equivalence with $\pi_\bullet = (\pi_0, \pi_1) : V_\bullet \oplus W \rightarrow V_\bullet$ as a pseudoinverse. Indeed, $\pi_\bullet \circ \iota_\bullet = 1_{V_\bullet}$ while $\iota_\bullet \circ \pi_\bullet \cong 1_{V_\bullet \oplus W}$ via the 2-isomorphism $0 \oplus 1_W : V_0 \oplus W \rightarrow V_1 \oplus W$. ■

2.4. Let $\mathbf{Ch}'_2(\mathcal{A})$ be the full sub-2-category of $\mathbf{Ch}_2(\mathcal{A})$ with objects the zero morphisms $0 : V_1 \rightarrow V_0$ in \mathcal{A} . The significance of $\mathbf{Ch}'_2(\mathcal{A})$ comes from the fact that all objects in $\mathbf{Ch}_2(\mathcal{A})$ are equivalent to an object in $\mathbf{Ch}'_2(\mathcal{A})$ when \mathcal{A} is such that each short exact sequence splits, for instance when \mathcal{A} is \mathcal{Vect}_k . Such an \mathcal{A} will be called a *split abelian* category. More precisely, we have the following result, already implicit in [7, Proposition 305].

2.5. PROPOSITION. *Let \mathcal{A} be a split abelian category. Then $\mathbf{Ch}'_2(\mathcal{A})$ is biequivalent to $\mathbf{Ch}_2(\mathcal{A})$.*

PROOF. It is enough to see that each object of $\mathbf{Ch}_2(\mathcal{A})$ is equivalent to a zero morphism in \mathcal{A} . In fact, an object $d : U_1 \rightarrow U_0$ of $\mathbf{Ch}_2(\mathcal{A})$ is equivalent to the zero morphism $\ker d \xrightarrow{0} \operatorname{coker} d$. Indeed, we have the short exact sequences

$$\begin{aligned} 0 \rightarrow \ker d \rightarrow U_1 \rightarrow \operatorname{coker}(\ker d) \rightarrow 0 \\ 0 \rightarrow \ker(\operatorname{coker} d) \rightarrow U_0 \rightarrow \operatorname{coker} d \rightarrow 0, \end{aligned}$$

and $\operatorname{coker}(\ker d) \cong \ker(\operatorname{coker} d)$. As usual we identify both objects and denote them by $\operatorname{im} d$. It follows that we have a commutative square of the form

$$\begin{array}{ccc} U_1 & \xrightarrow{d} & U_0 \\ \cong \downarrow & & \downarrow \cong \\ \ker d \oplus \operatorname{im} d & \xrightarrow{0 \oplus 1} & \operatorname{coker} d \oplus \operatorname{im} d. \end{array}$$

In particular, the top and the bottom morphisms are equivalent as objects in $\mathbf{Ch}_2(\mathcal{A})$ (in fact, isomorphic). The result now follows from Lemma 2.3. ■

3. Representations in Baez-Crans 2-vector spaces

In this section we describe the 2-category of representations of a 2-group \mathbb{G} in the 2-category of Baez-Crans 2-vector spaces over a field k (or more generally, in $\mathbf{Ch}_2(\mathcal{A})$ for any split k -linear abelian category \mathcal{A}).

Without loss of generality, we assume that \mathbb{G} is the (non-strict) skeletal 2-group $\pi_1[1] \rtimes_z \pi_0[0]$ for some group π_0 (with unit element e), left π_0 -module π_1 , and normalized 3-cocycle $z : \pi_0^3 \rightarrow \pi_1$. This is the 2-group with the elements of π_0 as objects, the pairs

$(a, x) \in \pi_1 \times \pi_0$ as morphisms, with $(a, x) : x \rightarrow x$, composition given by the sum in π_1 , and the tensor product by the product in π_0 on objects and by

$$(a, x) \otimes (b, y) = (a + x \triangleright b, xy)$$

on morphisms (\triangleright stands for the left action of π_0 on π_1). The associator is given by $\alpha_{x_1x_2x_3} = (z(x_1, x_2, x_3), x_1x_2x_3)$ and the left and right unit isomorphisms are trivial. By Sinh’s theorem [14], any 2-group is of this type up to equivalence (see also Baez and Lauda [1]).

We start by describing the 2-category of representations of \mathbb{G} in $\mathbf{Ch}_2(\mathcal{A})$ for an arbitrary abelian category \mathcal{A} , and we next focus on the split k -linear case.

3.1. DESCRIPTION OF THE GENERIC 2-CATEGORY OF REPRESENTATIONS. Let $\mathbb{G}[1]$ be the one-object 2-groupoid with \mathbb{G} as 2-group of self-equivalences of the unique object. By definition, $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ is the (strict) 2-category of (normal) pseudofunctors from $\mathbb{G}[1]$ to $\mathbf{Ch}_2(\mathcal{A})$, pseudonatural transformations between them, and modifications between these.¹ When unpacked, this definition leads to the 2-category with the following cells in each dimension and composition laws for the 1- and 2-cells.

3.1.1. An object in $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ is given by the following data:

(O1) a 2-term chain complex $d : V_1 \rightarrow V_0$, also denoted by V_\bullet ;

(O2) a map $(\rho_1, \rho_0) : \pi_0 \rightarrow \mathcal{A}(V_1, V_1) \times \mathcal{A}(V_0, V_0)$ such that for each $x \in \pi_0$ the square in \mathcal{A}

$$\begin{array}{ccc} V_1 & \xrightarrow{d} & V_0 \\ \rho_1(x) \downarrow & & \downarrow \rho_0(x) \\ V_1 & \xrightarrow{d} & V_0 \end{array}$$

commutes;

(O3) a map $\tau : \pi_1 \times \pi_0 \rightarrow \mathcal{A}(V_0, V_1)$ such that

$$d \circ \tau(a, x) = \tau(a, x) \circ d = 0$$

for each $(a, x) \in \pi_1 \times \pi_0$;

(O4) a map $\sigma : \pi_0^2 \rightarrow \mathcal{A}(V_0, V_1)$ such that

$$\rho_0(xy) - \rho_0(x) \circ \rho_0(y) = d \circ \sigma(x, y), \tag{1}$$

$$\rho_1(xy) - \rho_1(x) \circ \rho_1(y) = \sigma(x, y) \circ d \tag{2}$$

for each $x, y \in \pi_0$;

¹For the sake of simplicity, we restrict to *normal* pseudofunctors, i.e. such that the identity 1-cells are strictly preserved. Any pseudofunctor is equivalent to a normal one.

Moreover, these data must satisfy the following axioms:

- (AO1) $\tau(a' + a, x) = \tau(a', x) + \tau(a, x)$ for every composable 1-cells $x \xrightarrow{(a,x)} x \xrightarrow{(a',x)} x$;
- (AO2) $\tau(0, x) = 0$ for each object $x \in \pi_0$;
- (AO3) $\rho_1(x) \circ \tau(b, y) + \tau(a, x) \circ \rho_0(y) = \tau(a + x \triangleright b, xy)$ for every 1-cells $(a, x) : x \rightarrow x$ and $(b, y) : y \rightarrow y$ in \mathbb{G} ;
- (AO4) $\rho_1(e) = 1_{V_1}$ and $\rho_0(e) = 1_{V_0}$;
- (AO5) $\tau(z(x_1, x_2, x_3), x_1x_2x_3) + \sigma(x_1, x_2x_3) + \rho_1(x_1) \circ \sigma(x_2, x_3) = \sigma(x_1x_2, x_3) + \sigma(x_1, x_2) \circ \rho_0(x_3)$ for every objects $x_1, x_2, x_3 \in \pi_0$;
- (AO6) $\sigma(x, e) = \sigma(e, x) = 0$ for each object $x \in \pi_0$.

Data (O1)-(O3) give the action on 0-, 1- and 2-cells, respectively, of the pseudofunctor from $\mathbb{G}[1]$ to $\mathbf{Ch}_2(\mathcal{A})$, and (O4) gives the pseudofunctorial structure. Axioms (AO1)-(AO2) correspond to the functoriality of the assignments $(a, x) \mapsto \tau(a, x)$, axiom (AO3) to the naturality of $\sigma(x, y)$ in x, y , axiom (AO4) to the normal character of the pseudofunctor, and (AO5)-(AO6) to the coherence conditions. We will denote such an object by $(V_\bullet, \rho, \tau, \sigma)$ or just V_\bullet when the action of \mathbb{G} on V_\bullet is implicitly understood.

3.1.2. Given objects $(V_\bullet, \rho, \tau, \sigma)$ and $(V'_\bullet, \rho', \tau', \sigma')$, a 1-cell or 1-*intertwiner* from the first to the second consists of the following data:

- (I1) a pair $(r_1, r_0) \in \mathcal{A}(V_1, V'_1) \times \mathcal{A}(V_0, V'_0)$ which makes the square

$$\begin{array}{ccc} V_1 & \xrightarrow{d} & V_0 \\ r_1 \downarrow & & \downarrow r_0 \\ V'_1 & \xrightarrow{d'} & V'_0 \end{array}$$

commute;

- (I2) a map $\mu : \pi_0 \rightarrow \mathcal{A}(V_0, V'_1)$ such that

$$\rho'_1(x) \circ r_1 - r_1 \circ \rho_1(x) = \mu(x) \circ d, \quad (3)$$

$$\rho'_0(x) \circ r_0 - r_0 \circ \rho_0(x) = d' \circ \mu(x) \quad (4)$$

for each $x \in \pi_0$.

Moreover, these data must satisfy the following axioms:

- (AI1) $\tau'(a, x) \circ r_0 = r_1 \circ \tau(a, x)$ for each morphism $(a, x) \in \pi_1 \times \pi_0$;
- (AI2) $r_1 \circ \sigma(x, y) + \mu(xy) = \mu(x) \circ \rho_0(y) + \rho'_1(x) \circ \mu(y) + \sigma'(x, y) \circ r_0$ for every objects $x, y \in \pi_0$;

(AI3) $\mu(e) = 0$.

Data (I1)-(I2) give the (unique) 1-cell and the invertible 2-cells, respectively, of the corresponding pseudonatural transformation. Axiom (AI1) is the naturality of $\mu(x)$ in x , and axioms (AI2)-(AI3) are the coherence conditions.

3.1.3. Given 1-cells $(r_1, r_0, \mu), (s_1, s_0, \nu)$ between two representations V_\bullet and V'_\bullet , a 2-cell or *2-intertwiner* from (r_1, r_0, μ) to (s_1, s_0, ν) consists of a morphism $\omega : V_0 \rightarrow V'_1$ such that

$$\begin{aligned} s_1 - r_1 &= \omega \circ d, \\ s_0 - r_0 &= d' \circ \omega \end{aligned}$$

and satisfying the following naturality axiom:

(A2I) $\rho'_1(x) \circ \omega + \mu(x) = \nu(x) + \omega \circ \rho_0(x)$ for each object $x \in \pi_0$.

3.1.4. Composition of 1-cells corresponds to the vertical composition of pseudonatural transformations, and it is given by

$$(r'_1, r'_0, \mu') \circ (r_1, r_0, \mu) = (r'_1 \circ r_1, r'_0 \circ r_0, \mu' * \mu) \tag{5}$$

for 1-cells $(r_1, r_0, \mu) : V_\bullet \rightarrow V'_\bullet$ and $(r'_1, r'_0, \mu') : V'_\bullet \rightarrow V''_\bullet$, with $\mu' * \mu : \pi_0 \rightarrow \mathcal{A}(V_0, V''_1)$ defined by

$$(\mu' * \mu)(x) = r'_1 \circ \mu(x) + \mu'(x) \circ r_0, \quad x \in \pi_0. \tag{6}$$

Vertical and horizontal composition of 2-cells correspond to the appropriate compositions of modifications, and they are respectively given by the sum and composition of morphisms in \mathcal{A} . More precisely, for 2-cells $\omega : (r_1, r_0, \mu) \Rightarrow (s_1, s_0, \nu)$ and $\eta : (s_1, s_0, \nu) \Rightarrow (t_1, t_0, \xi)$, with $(r_1, r_0, \mu), (s_1, s_0, \nu), (t_1, t_0, \xi) : V_\bullet \rightarrow V'_\bullet$, their vertical composite is

$$\eta\omega = \eta + \omega, \tag{7}$$

and for ω as before and $\omega' : (r'_1, r'_0, \mu') \Rightarrow (s'_1, s'_0, \nu') : V'_\bullet \rightarrow V''_\bullet$ their horizontal composite is

$$\omega' \circ \omega = \omega' \circ s_0 + r'_1 \circ \omega = \omega' \circ r_0 + s'_1 \circ \omega. \tag{8}$$

Notice that $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ is locally a groupoid because $\mathbf{Ch}_2(\mathcal{A})$ is so.

3.2. CASE OF A SPLIT k -LINEAR ABELIAN CATEGORY. From now on, \mathcal{A} stands for a split k -linear abelian category. In this case, $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ is biequivalent to $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$ because of Proposition 2.5 and the general fact that for any biequivalent 2-categories \mathbf{C}, \mathbf{C}' the representation 2-categories $\mathbf{Rep}_{\mathbf{C}}(\mathbb{G})$ and $\mathbf{Rep}_{\mathbf{C}'}(\mathbb{G})$ are biequivalent. Therefore we may restrict to representations of \mathbb{G} in $\mathbf{Ch}'_2(\mathcal{A})$. The above general descriptions of the 0-, 1- and 2-cells reduce then to the following data and axioms.

3.2.1. A representation of \mathbb{G} in $\mathbf{Ch}'_2(\mathcal{A})$ consists of

- (O1') two representations of π_0 in \mathcal{A} , denoted by $\rho_i : \pi_0 \rightarrow \text{Aut}_{\mathcal{A}}(V_i)$, $i = 0, 1$;
- (O2') a morphism of (left) π_0 -modules $\beta : \pi_1 \rightarrow \mathcal{A}(V_0, V_1)_{\rho_1}^{\rho_0}$, where $\mathcal{A}(V_0, V_1)_{\rho_1}^{\rho_0}$ stands for the abelian group $\mathcal{A}(V_0, V_1)$ equipped with the (left) π_0 -action given by $x \triangleright f = \rho_1(x) \circ f \circ \rho_0(x^{-1})$, and
- (O3') a normalized 2-cochain $c : \pi_0^2 \rightarrow \mathcal{A}(V_0, V_1)_{\rho_1}^{\rho_0}$ such that $\partial c = \beta_*(z)$.

The representations ρ_1, ρ_0 are the maps in (O2) (equations (1)-(2) together with axiom (AO4) ensure that they are indeed representations of π_0). The morphism β is the restriction of τ to $\pi_1 \times \{e\}$. This restriction completely determines τ . Indeed, it follows from (AO3) that $\tau(a, x) = \beta(a) \circ \rho_0(x)$. The fact that β is a morphism of (left) π_0 -modules follows from axioms (AO1)-(AO3). Finally, c is given by

$$c(x, y) = \sigma(x, y) \circ \rho_0(xy)^{-1},$$

and the conditions on c follow from axioms (AO5)-(AO6). The representation so defined will be denoted by $(\rho_1, \rho_0, \beta, c)$, or (V_1, V_0, β, c) if the actions of π_0 are implicitly understood.

3.2.2. Given two representations $(\rho_1, \rho_0, \beta, c)$, $(\rho'_1, \rho'_0, \beta', c')$ as in § 3.2.1, on objects (V_1, V_0) and (V'_1, V'_0) of $\mathbf{Ch}'_2(\mathcal{A})$, respectively, a 1-cell between them is given by

- (I1') two intertwiners $r_i : V_i \rightarrow V'_i$, $i = 0, 1$, which make the diagram

$$\begin{array}{ccc} \pi_1 & \xrightarrow{\beta} & \mathcal{A}(V_0, V_1) \\ \beta' \downarrow & & \downarrow r_{1*} \\ \mathcal{A}(V'_0, V'_1) & \xrightarrow{r_0^*} & \mathcal{A}(V_0, V_1) \end{array}$$

commute, and

- (I2') a normalized 1-cochain $u : \pi_0 \rightarrow \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0}$ such that the diagram

$$\begin{array}{ccc} \pi_0^2 & \xrightarrow{c} & \mathcal{A}(V_0, V_1) \\ c' \downarrow & & \downarrow r_{1*} \\ \mathcal{A}(V'_0, V'_1) & \xrightarrow{r_0^*} & \mathcal{A}(V_0, V_1) \end{array}$$

commutes up to the coboundary of u .

The intertwiners r_0, r_1 are the maps in (I1) (equations (3)-(4) ensure that they are indeed intertwiners). As for the 1-cochain u , it is given by

$$u(x) = \mu(x) \circ \rho_0(x)^{-1}. \tag{9}$$

The condition (I1') on r_0, r_1 follows from (AI1) and condition (I2') on u from (9) and (AI2). The 1-cell so defined will be denoted by (r_1, r_0, u) .

In particular, for any representations $\rho_1, \rho_0, \rho'_1, \rho'_0$ of π_0 , a 1-cell from $(\rho_1, \rho_0, 0, 0)$ to $(\rho'_1, \rho'_0, 0, 0)$ simply amounts to two arbitrary intertwiners $r_i : V_i \rightarrow V'_i, i \in \{0, 1\}$, together with an arbitrary normalized 1-cocycle $u : \pi_0 \rightarrow \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0}$.

It easily follows from (9) that the composition of 1-cells is given by the same formulas as in § 3.1.4 with u, u' instead of μ, μ' .

3.2.3. Given two 1-cells $(r_1, r_0, u), (s_1, s_0, v) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ as before, a 2-cell between them exists only when $r_i = s_i$, for $i = 0, 1$, and if so it is given by a 0-cochain $\omega : 1 \rightarrow \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0}$ as in § 3.1.3 such that $\partial\omega = v - u$. This condition follows readily from (A2I). Vertical and horizontal compositions of 2-cells are given by the same formulas as in § 3.1.4.

3.3. PROPOSITION. *A 1-intertwiner $(r_1, r_0, u) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ is an equivalence if and only if r_1 and r_0 are isomorphisms.*

PROOF. Suppose (r_1, r_0, u) is an equivalence with pseudo-inverse $(\bar{r}_1, \bar{r}_0, \bar{u})$. Then we have 2-cells $(r_1 \circ \bar{r}_1, r_0 \circ \bar{r}_0, u * \bar{u}) \Rightarrow (1_{V_1}, 1_{V_0}, 0)$ and $(\bar{r}_1 \circ r_1, \bar{r}_0 \circ r_0, \bar{u} * u) \Rightarrow (1_{V'_1}, 1_{V'_0}, 0)$. It follows from § 3.2.3 that $r_i \circ \bar{r}_i = 1_{V_i}$ and $\bar{r}_i \circ r_i = 1_{V'_i}$ for $i = 0, 1$ and hence, r_1, r_0 are isomorphisms.

Conversely, let us assume that r_1 and r_0 are isomorphisms. Then it is easy to check that $(r_1^{-1}, r_0^{-1}, \bar{u})$, with \bar{u} defined by

$$\bar{u} : \pi_0 \xrightarrow{-u} \mathcal{A}(V_0, V'_1) \xrightarrow{(r_1^{-1})_* \circ (r_0^{-1})^*} \mathcal{A}(V'_0, V_1),$$

is a 1-cell from $(\rho'_1, \rho'_0, \beta', c')$ to $(\rho_1, \rho_0, \beta, c)$ and a pseudo-inverse of (r_1, r_0, u) . ■

It follows that a necessary condition for $(\rho_1, \rho_0, \beta, c)$ and $(\rho'_1, \rho'_0, \beta', c')$ to be equivalent representations is that $V_1 \cong V'_1$ and $V_0 \cong V'_0$ as representations of π_0 . However, this condition is far from being sufficient, as it is made clear in the next concrete cases.

3.4. COROLLARY. *Let $\rho_1, \rho_0, \beta, c, c'$ be as before. If c, c' differ by a coboundary the representations $(\rho_1, \rho_0, \beta, c)$ and $(\rho_1, \rho_0, \beta, c')$ are equivalent.*

PROOF. If $c' - c = \partial u$, then $(1_{V_1}, 1_{V_0}, u)$ is an equivalence between $(\rho_1, \rho_0, \beta, c)$ and $(\rho_1, \rho_0, \beta, c')$. ■

3.5. COROLLARY. A representation $(\rho_1, \rho_0, \beta, c)$ is equivalent to a representation of the form $(\rho'_1, \rho'_0, 0, 0)$ if and only if $\beta = 0$ and $c = \partial u$ for some 1-cochain u .

PROOF. Let $(\rho_1, \rho_0, \beta, c)$ be equivalent to $(\rho'_1, \rho'_0, 0, 0)$. Then there exists isomorphisms of representations $r_i : V_i \rightarrow V'_i, i = 0, 1$, and a normalized 1-cochain $\tilde{u} : \pi_0 \rightarrow \mathcal{A}(V_0, V'_1)$ such that (1) $r_1 \circ \beta(a) = 0$ for each $a \in \pi_1$, and (2) $c(x, y) = r_1^{-1} \circ (\partial \tilde{u})(x, y)$ for each $x, y \in \pi_0$. Condition (1) clearly implies $\beta = 0$, and (2) implies that $c = \partial u$ with u the normalized 1-cochain defined by $u(x) = r_1^{-1} \circ \tilde{u}(x)$. The converse is a consequence of Corollary 3.4. ■

3.6. THE BAEZ-CRANS 2-VECTOR SPACES OF INTERTWINERS. Let us fix two representations $(\rho_1, \rho_0, \beta, c)$ and $(\rho'_1, \rho'_0, \beta', c')$. Then it follows from § 3.2.2 that the set of 1-cells between them has a natural structure of k -vector space induced by the k -linear enrichment of \mathcal{A} . Let us call this space $H_1 \left(\begin{matrix} \rho_1 & \rho_0 & \beta & c \\ \rho'_1 & \rho'_0 & \beta' & c' \end{matrix} \right)$, or just H_1 . Similarly, we can consider the set of all 2-cells between these 1-cells. To make explicit the involved 1-cells, we shall denote such a 2-cell by

$$(r_1, r_0, u, \omega) : (r_1, r_0, u) \Rightarrow (r_1, r_0, u + \partial\omega).$$

Then the set of these 2-cells is also a k -vector space $H_2 \left(\begin{matrix} \rho_1 & \rho_0 & \beta & c \\ \rho'_1 & \rho'_0 & \beta' & c' \end{matrix} \right)$, or just H_2 , with k -linear structure induced again by the k -linear enrichment of \mathcal{A} . Moreover, the source and target maps $s, t : H_2 \rightarrow H_1$ are k -linear, and the same is true of the identity-assigning map $i : H_1 \rightarrow H_2$, given by $(r_1, r_0, u) \mapsto (r_1, r_0, u, 0)$, and the composition map $\circ : H_2 \times_{H_1} H_2 \rightarrow H_2$, given by

$$(r_1, r_0, u + \partial\omega, \eta) \circ (r_1, r_0, u, \omega) = (r_1, r_0, u, \omega + \eta)$$

(cf. equation (7)). Therefore the hom-category between any pair of fixed representations $(\rho_1, \rho_0, \beta, c)$ and $(\rho'_1, \rho'_0, \beta', c')$ is in fact an internal category in \mathcal{Vect}_k and hence, equivalent to the Baez-Crans 2-vector space

$$t|_{\ker(s)} : \ker(s) \rightarrow H_1 \tag{10}$$

(see [3]). Now, since $s(r_1, r_0, u, \omega) = (r_1, r_0, u)$ we have

$$\ker(s) \cong \mathcal{A}(V_0, V'_1),$$

and (10) is equivalent to the linear map $d : \mathcal{A}(V_0, V'_1) \rightarrow H_1$ given by

$$d(\omega) = (0, 0, \partial\omega).$$

In order to identify the equivalent object in $\mathbf{Ch}'_2(k)$, notice that

$$\ker(d) = H^0(\pi_0, \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0}) = \text{Hom}_{\pi_0}(V_0, V'_1),$$

where $\text{Hom}_{\pi_0}(V_0, V'_1)$ denotes the k -vector space of intertwiners from ρ_0 to ρ'_1 . Moreover, the cokernel of d is the k -vector space of all triples

$$(r_1, r_0, [u]) \in \text{Hom}_{\pi_0}(V_1, V'_1) \times \text{Hom}_{\pi_0}(V_0, V'_0) \times \tilde{H}^1(\pi_0, \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0})$$

such that $r_1 \circ \beta = \beta' \circ r_0$ and $\partial u = r_1 \circ c - c' \circ r_0$, where $\tilde{H}^1(\pi_0, \mathcal{A}(V_0, V'_1))$ is the space of 1-cochains modulo coboundaries. We shall denote this space by $H(\rho_1, \rho_0, \rho'_1, \rho'_0)$. Notice that it depends on β, β', c, c' although this is not made explicit. When β, β', c and c' are zero, it reduces to the space $\text{Hom}_{\pi_0}(V_1, V'_1) \times \text{Hom}_{\pi_0}(V_0, V'_0) \times H^1(\pi_0, \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0})$. Therefore, we have proved the following.

3.7. THEOREM. *For any fixed representations $(\rho_1, \rho_0, \beta, c), (\rho'_1, \rho'_0, \beta', c')$ the hom-category of intertwiners between them is equivalent to the Baez-Crans 2-vector space*

$$0 : \text{Hom}_{\pi_0}(V_0, V'_1) \rightarrow H(\rho_1, \rho_0, \rho'_1, \rho'_0).$$

In particular, when β, β', c and c' are all zero, this 2-vector space is

$$0 : \text{Hom}_{\pi_0}(V_0, V'_1) \rightarrow \text{Hom}_{\pi_0}(V_1, V'_1) \times \text{Hom}_{\pi_0}(V_0, V'_0) \times H^1(\pi_0, \mathcal{A}(V_0, V'_1)_{\rho'_1}^{\rho_0}).$$

3.8. MONOMORPHISMS OF REPRESENTATIONS. There are various possible notions of monomorphism in a 2-category \mathbf{C} . The most standard ones are perhaps the (representably) faithful or fully faithful morphisms. However, all 2-categories we work with have a zero object and hence, monomorphisms can also be defined in terms of the 2-kernel of the morphism in question. In this subsection we introduce two such notions of monomorphism, and we characterize which 1-cells (r_1, r_0, u) in $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ are monomorphisms in each sense. We start by recalling the general notion of 2-kernel in a 2-category with a zero object.

Let \mathbf{C} be a 2-category with a zero object 0 , which for the sake of simplicity we will assume it is strict, i.e. such that for any other object X in \mathbf{C} the categories $\mathbf{C}(0, X)$ and $\mathbf{C}(X, 0)$ are *isomorphic* to the terminal category. For instance, for any zero object 0 of \mathcal{A} the zero complex $0_\bullet = 0 \rightarrow 0$ is a strict zero object of $\mathbf{Ch}_2(\mathcal{A})$. For any pair of objects X, Y of \mathbf{C} the (unique) 1-cell $X \rightarrow 0 \rightarrow Y$ is denoted by $0 : X \rightarrow Y$.

Then the *2-kernel* of a 1-cell $f : X \rightarrow Y$ in \mathbf{C} is the 2-limit of the diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows_0 \end{array} Y.$$

This means that it is a triple (K, k, κ) , with K an object of \mathbf{C} , $k : K \rightarrow X$ a 1-cell, and $\kappa : f \circ k \Rightarrow 0 : K \rightarrow Y$ an invertible 2-cell satisfying the (bi)universal property:

(K1) Any other triple (L, a, α) , with $a : L \rightarrow X$ and $\alpha : f \circ a \Rightarrow 0$ an invertible 2-cell of \mathbf{C} , factors through (K, k, κ) , i.e. there exists a pair (a', α') , with $a' : L \rightarrow K$ and

$\alpha' : a \Rightarrow k \circ a'$, such that

$$\begin{array}{ccc}
 & K & \\
 a' \nearrow & & \searrow 0 \\
 L & \xrightarrow{0} & Y \\
 a \searrow & \underline{\cong} & \nearrow f \\
 & X &
 \end{array}
 =
 \begin{array}{ccc}
 & K & \\
 a' \nearrow & & \searrow 0 \\
 L & \xrightarrow{\underline{\cong} g'} & Y \\
 a \searrow & k & \nearrow \underline{\cong} \kappa \\
 & X &
 \end{array}$$

(K2) The factorization is unique up to a unique invertible 2-cell, i.e. if (a'_1, α'_1) and (a'_2, α'_2) are two factorizations of (L, a, α) through (K, k, κ) , there exists a unique invertible 2-cell $\gamma : a'_1 \Rightarrow a'_2$ such that

$$\alpha'_2 = (1_k \circ \gamma)\alpha'_1.$$

As any (bi)universal object, the 2-kernel of a 1-cell is unique up to equivalence.

3.9. DEFINITION. Let $f : X \rightarrow Y$ be a 1-cell in \mathbf{C} .

- f is a monomorphism if its 2-kernel is zero, i.e. $(0, 0, 1_0)$.
- f is a weak monomorphism if its 2-kernel (K, k, κ) is such that $k \cong 0$ (the object K need not be zero).

Clearly, every monomorphism is a weak monomorphism but the converse is false as it will become clear below. In fact, it can be shown that the 2-kernel (K, k, κ) of a 1-cell f is zero if and only if $k \cong 0$ and there exists a 2-isomorphism $\kappa' : k \Rightarrow 0$ such that $\kappa = f * \kappa'$ (see [7, Proposition 90]). It is this last additional condition that should be left out to go from monomorphisms to the more general notion of weak monomorphism.

3.10. PROPOSITION. Let $(r_1, r_0, u) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ be an arbitrary 1-cell in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$. Then the following are equivalent:

- (1) (r_1, r_0, u) is a monomorphism.
- (2) the 2-kernel of the 1-cell (r_1, r_0) in $\mathbf{Ch}'_2(\mathcal{A})$ is zero.
- (3) r_1 is an isomorphism and r_0 a monomorphism in \mathcal{A} .

PROOF. The equivalence of (1) and (2) follows from the fact that 2-limits in 2-categories of pseudofunctors, and in particular 2-kernels, are computed componentwise, together with the fact that a representation is a zero object if and only if its underlying Baez-Crans 2-vector space is a zero object.

To prove the equivalence of (2) and (3), we make use of the fact that the 2-kernel of a 1-cell $f_\bullet : V_\bullet \rightarrow W_\bullet$ in $\mathbf{Ch}_2(\mathcal{A})$ is given by the commutative square

$$\begin{array}{ccc}
 V_1 & \xrightarrow{d \times f_1} & V_0 \times_{W_0} W_1 \\
 1 \downarrow & & \downarrow p_{V_0} \\
 V_1 & \xrightarrow{d} & V_0
 \end{array}$$

together with the projection $p_{W_1} : V_0 \times_{W_0} W_1 \rightarrow W_1$ as 2-cell $(f_1, f_0 \circ p_{V_0}) \Rightarrow 0$ (cf. [7, 6.3.3]). When $d_V = 0$ and $d_W = 0$ we have

$$V_0 \times_{W_0} W_1 \cong \ker f_0 \times W_1,$$

and the 2-term chain complex $V_1 \xrightarrow{0 \times f_1} \ker f_1 \times W_1$ is equivalent to the zero morphism $0 : \ker f_1 \rightarrow \ker f_0 \times \text{coker } f_1$ (cf. proof of Proposition 2.5). Hence the corresponding 1-cell of the 2-kernel in $\mathbf{Ch}'_2(\mathcal{A})$ is

$$\begin{array}{ccc} \ker f_1 & \xrightarrow{0} & \ker f_0 \times \text{coker } f_1 \\ \iota_1 \downarrow & & \downarrow \iota_0 \times 0 \\ V_1 & \xrightarrow{0} & V_0 \end{array}$$

where ι_0, ι_1 denote the canonical inclusions. Hence the 2-kernel is 0 if and only if $\ker f_1, \ker f_0$ and $\text{coker } f_1$ are all zero, i.e. if and only if f_1 is an isomorphism and f_0 a monomorphism. ■

3.11. **REMARK.** The monomorphisms in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$ as defined above are in fact the (representably) fully faithful morphisms. This is because in any 2-category the fully faithful morphisms can also be characterized as the 1-cells $f : X \rightarrow Y$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ 1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is a 2-pullback. Since 2-limits in 2-categories of pseudofunctors are computed pointwise, it follows that a 1-cell in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$ is fully faithful if and only if the underlying morphism of 2-term chain complexes is fully faithful, and it is easy to check that this happens if and only if its 2-kernel is zero. More generally, Dupond proved that the fully faithful morphisms coincide with the morphisms whose 2-kernel is zero in any 2-category he calls "2-Puppe exact" (cf. Propositions 90, 180 and 292 in [7]). Both $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$ and $\mathbf{Ch}'_2(\mathcal{A})$ will be examples of such 2-categories.

Computing explicitly the 2-kernel of an arbitrary 1-cell in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$ looks difficult. However, in order to identify which 1-cells are weak monomorphisms it is enough to compute the 2-kernel of a 1-cell $(r_1, r_0, u) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ such that r_1 and r_0 are monomorphisms.

Let be given such a 1-cell (r_1, r_0, u) . We can decompose the object V'_1 of \mathcal{A} as the direct sum $V'_1 \cong V_1 \oplus \text{coker}(r_1)$ (we are assuming that \mathcal{A} is split). Moreover, the decomposition is such that the inclusion $r_1 : V_1 \rightarrow V'_1$ and the projection $p_c : V'_1 \rightarrow \text{coker}(r_1)$ are both π_0 -morphisms. We shall denote by $i_c : \text{coker}(r_1) \rightarrow V'_1$ and $p_m : V'_1 \rightarrow V_1$ the remaining inclusion and projection, respectively. Then let $\hat{u} : \pi_0 \rightarrow \mathcal{A}(\text{coker}(r_1), V_1)$ be the map defined by

$$\hat{u}(x)(k) = -(p_m \circ \rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) \tag{11}$$

for all $x \in \pi_0$ and $k \in \text{coker}(r_1)$, where $\bar{\rho}'_1$ denotes the canonical action of π_0 on $\text{coker}(r_1)$ induced by the action ρ'_1 on V'_1 .

3.12. PROPOSITION. *The triple $((0, \bar{\rho}'_1, 0, 0), (0, 0, \hat{u}), i_c)$ is a 2-kernel of (r_1, r_0, u) , where $\bar{\rho}'_1$ is the representation on $\text{coker}(r_1)$ induced by ρ'_1 .*

In order to prove this result, we shall make use of the following lemma.

3.13. LEMMA. *Let \hat{u} be the 1-cochain defined in Eq. (11). Then*

$$(r_1)_* \circ \hat{u} = -\partial i_c.$$

PROOF. Let $x \in \pi_0$ and $k \in \text{coker}(r_1)$. We have

$$\begin{aligned} ((r_1)_* \circ \hat{u}(x))(k) &= -(r_1 \circ p_m \circ \rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) \\ &= -(\rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) + (i_c \circ p_c \circ \rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) \\ &= -(\rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) + i_c(k) \\ &= -(\partial i_c)(x)(k), \end{aligned}$$

where we have used that $r_1 \circ p_m + i_c \circ p_c = 1$ and that p_c is a π_0 -morphism. ■

PROOF OF PROPOSITION 3.12. First, we need to check that

$$(0, 0, \hat{u}) : (0, \text{coker}(r_1), 0, 0) \rightarrow (V_1, V_0, \beta, c)$$

is a 1-intertwiner, which in this case reduces to checking that \hat{u} is a 1-cocycle with values in $\mathcal{A}(\text{coker}(r_1), V_1)_{\bar{\rho}'_1}$. Now, Lemma 3.13 and the fact that r_1 is a π_0 -morphisms imply that

$$(r_1)_* \circ (\partial \hat{u}) = \partial((r_1)_* \circ \hat{u}) = -\partial^2 i_c = 0,$$

and hence $\partial \hat{u} = 0$ because r_1 is a monomorphism.

Secondly, i_c is a 2-cell $(r_1, r_0, u) \circ (0, 0, \hat{u}) \Rightarrow (0, 0, 0)$ because by Lemma 3.13 again we have

$$u * \hat{u} = (r_1)_* \circ \hat{u} = -\partial i_c.$$

Now we only need to check that it verifies the universal property.

Suppose that we have a triple $((W_1, W_0, \tilde{\beta}, \tilde{c}), (t_1, t_0, \tilde{u}), \chi)$ as in (K1). The fact that $\chi : (r_1, r_0, u) \circ (t_1, t_0, \tilde{u}) \Rightarrow (0, 0, 0)$ implies that $r_1 \circ t_1 = 0$ and $r_0 \circ t_0 = 0$ and hence $t_1 = 0$ and $t_0 = 0$ because r_1 and r_0 are both monomorphisms. Then let $s_0 : W_0 \rightarrow \text{coker}(r_1)$ be the map given by $s_0 = p_c \circ \chi$. It is a π_0 -morphism because

$$\begin{aligned} \partial s_0 &= \partial(p_c \circ \chi) \\ &= (p_c)_* \circ \partial \chi \\ &= -(p_c)_* \circ (u * \tilde{u}) \\ &= -(p_c)_* \circ ((t_0)^* \circ u + (r_1)_* \circ \tilde{u}) \\ &= -(p_c \circ r_1)_* \circ \tilde{u} \\ &= 0. \end{aligned}$$

Therefore we have a 1-intertwiner $(0, s_0, 0) : (W_1, W_0, \tilde{\beta}, \tilde{c}) \rightarrow (0, \text{coker}(r_1), 0, 0)$. Moreover, if $\hat{\chi} : W_0 \rightarrow V_1$ is the map given by $\hat{\chi} = -p_m \circ \chi$ we have

$$\begin{aligned} (r_1)_* \circ \partial \hat{\chi} &= -\partial(r_1 \circ p_m \circ \chi) \\ &= -\partial \chi + \partial(i_c \circ p_c \circ \chi) \\ &= (r_1)_* \circ \tilde{u} + \partial(i_c \circ s_0) \\ &= (r_1)_* \circ \tilde{u} + (s_0)^* \circ \partial i_c \\ &= (r_1)_* \circ \tilde{u} - (r_1)_* \circ (s_0)^* \circ \hat{u}, \end{aligned}$$

where we have used that s_0 is a π_0 -morphism and Lemma 3.13. Since r_1 is a monomorphism this implies that $\partial \hat{\chi} = \tilde{u} - (s_0)^* \circ \hat{u}$ and hence, $\hat{\chi} : (0, s_0, 0) \circ (0, 0, \hat{u}) \Rightarrow (t_1, t_0, \tilde{u})$ is indeed a 2-intertwiner. Furthermore it satisfies the condition in (K1) because

$$\chi + (r_1)_*(\hat{\chi}) = \chi - r_1 \circ p_m \circ \chi = i_c \circ p_c \circ \chi = (s_0)^*(i_c).$$

Finally, let $((0, \bar{s}, 0), \bar{\chi})$ be another factorization of $((W_1, W_0, \tilde{\beta}, \tilde{c}), (t_1, t_0, \tilde{u}), \chi)$. This means that

$$(r_1)_*(\bar{\chi} - \hat{\chi}) = (s_0)^*(i_c) - (\bar{s})^*(i_c) = i_c \circ (s_0 - \bar{s}),$$

and applying p_c to this equality we get $s_0 = \bar{s}$, and hence also $\bar{\chi} = \hat{\chi}$ because r_1 is a monomorphism. This proves (K2) with the identity as the unique 2-intertwiner $(0, s_0, 0) \Rightarrow (0, \bar{s}, 0)$. ■

3.14. PROPOSITION. *Let $(r_1, r_0, u) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ be an arbitrary 1-cell in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$. Then the following are equivalent:*

1. (r_1, r_0, u) is a weak monomorphism.
2. Both r_1 and r_0 are monomorphisms in $\mathcal{R}ep(\pi_0)$, with r_1 a split one.

PROOF. Let us suppose that r_1, r_0 are monomorphisms in $\mathcal{R}ep(\pi_0)$, with r_1 split. It follows that $V'_1 \cong V_1 \oplus \text{coker}(r_1)$ in $\mathcal{R}ep(\pi_0)$. In particular, the projection $p_m : V'_1 \rightarrow V_1$ is a π_0 -morphism. Then using the definition of \hat{u} in Eq. (11), we have

$$\hat{u}(x)(k) = -(p_m \circ \rho'_1(x) \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) = -(\rho_1(x) \circ p_m \circ i_c \circ \bar{\rho}'_1(x)^{-1})(k) = 0.$$

Hence the 2-kernel morphism is 0.

Conversely, let us suppose that the 2-kernel morphism of (r_1, r_0, u) is isomorphic to zero. In particular, the 2-kernel morphism of the 1-cell (r_1, r_0) in $\mathbf{Ch}'(\mathcal{A})$ must also be isomorphic to zero. Now, this morphism is determined by the inclusions $\iota_1 : \ker(r_1) \rightarrow V_1$ and $\iota_0 : \ker(r_0) \rightarrow V_0$, so that it is isomorphic to zero if and only if both r_1 and r_0 are monomorphisms. Hence the 2-kernel of (r_1, r_0, u) is as described in Proposition 3.12. Let $\phi : (0, 0, \hat{u}) \Rightarrow (0, 0, 0)$ be the isomorphism given by hypothesis, so that $\phi : \text{coker}(r_1) \rightarrow V_1$, and let be

$$\bar{p} = p_m + \phi \circ p_c : V'_1 \rightarrow V_1.$$

It is a π_0 -morphism since

$$\begin{aligned} (r_1)_* \circ \partial \bar{p} &= (r_1)_* \circ \partial(p_m + \phi \circ p_c) \\ &= \partial(r_1 \circ p_m) + (p_c)^* \circ (r_1)_* \circ \partial \phi \\ &= \partial 1_{V'_1} - \partial(i_c \circ p_c) - (p_c)^* \circ (r_1)_* \circ \hat{u} = 0 \end{aligned}$$

and hence $\partial \bar{p} = 0$ because r_1 is a monomorphism. Finally we have

$$\bar{p} \circ r_1 = p_m \circ r_1 + \phi \circ p_c \circ r_1 = 1_{V_1},$$

and hence, r_1 is split. ■

In the rest of this section we study the irreducible and indecomposable objects in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$. In fact, we introduce two different notions of irreducible object, both of them respectively derived in the standard way from the above two notions of monomorphism.

3.15. DEFINITION. *A representation $(\rho_1, \rho_0, \beta, c)$ is called a (weak) subrepresentation of another one $(\rho'_1, \rho'_0, \beta', c')$ if it is the zero representation $(0, 0, 0, 0)$ or if there exists a (weak) monomorphism $(r_1, r_0, \mu) : (\rho_1, \rho_0, \beta, c) \rightarrow (\rho'_1, \rho'_0, \beta', c')$ in $\mathbf{Rep}_{\mathbf{Ch}'_2(\mathcal{A})}(\mathbb{G})$. The representation $(\rho_1, \rho_0, \beta, c)$ is called (weakly) irreducible if it has no non-trivial (weak) subrepresentations.*

Let us emphasize that, in spite of the terminology, every weakly irreducible representation is irreducible because every monomorphism is a weak monomorphism.

3.16. PROPOSITION. *The only irreducible representations of \mathbb{G} are those of the form $(\rho, 0, 0, 0)$ for any representation of π_0 , and $(0, \rho, 0, 0)$ for any irreducible representation of π_0 .*

The only weakly irreducible representations of \mathbb{G} are those of the form $(\rho, 0, 0, 0)$ for any indecomposable representation of π_0 , and $(0, \rho, 0, 0)$ for any irreducible representation of π_0 .

PROOF. It follows from Proposition 3.10 that a representation of any of the forms in the first statement is irreducible. Conversely, let $(\rho_1, \rho_0, \beta, c)$ be an irreducible representation with $V_0 \neq 0$. Then the map $(1_{V_1}, 0, 0) : (\rho_1, 0, 0, 0) \rightarrow (\rho_1, \rho_0, \beta, c)$ is a monomorphism and hence, we necessarily have $V_1 = 0$. This implies $\beta = 0$ and $c = 0$. This proves the first statement. The second one is shown in a similar way. ■

There is also a natural notion of (direct) sum of representations induced by the direct sum we have in $\mathbf{Ch}_2(\mathcal{A})$, and given as follows.

3.17. DEFINITION. *Given $(\rho_1, \rho_0, \beta, c)$ and $(\rho'_1, \rho'_0, \beta', c')$ two representations their sum is the representation*

$$(\rho_1 \oplus \rho'_1, \rho_0 \oplus \rho'_0, \beta \oplus \beta', c \oplus c'),$$

where $\beta \oplus \beta'$ is the composite

$$\pi_1 \xrightarrow{\beta \times \beta'} \mathcal{A}(V_0, V_1) \times \mathcal{A}(V'_0, V'_1) \xrightarrow{\oplus} \mathcal{A}(V_0 \oplus V'_0, V_1 \oplus V'_1)$$

and $c \oplus c'$ is the composite

$$\pi_0^2 \xrightarrow{c \times c'} \mathcal{A}(V_0, V_1) \times \mathcal{A}(V'_0, V'_1) \xrightarrow{\oplus} \mathcal{A}(V_0 \oplus V'_0, V_1 \oplus V'_1).$$

3.18. DEFINITION. *A representation is indecomposable if it is not equivalent to a sum of two non-zero representations.*

Starting with decompositions of ρ_1, ρ_0 in indecomposable representations of π_0 , it is easy to identify the indecomposable representations of \mathbb{G} . Thus suppose we are given decompositions

$$\begin{aligned} \rho_1 &= \bigoplus_{i \in I} \rho_i^1, \\ \rho_0 &= \bigoplus_{j \in J} \rho_j^0, \end{aligned}$$

both as a sum of indecomposable representations of π_0 . If $\rho_1 = 0$ (resp. $\rho_0 = 0$) it is understood that $I = \emptyset$ (resp. $J = \emptyset$). Let $\iota_i^e : \rho_i^e \rightarrow \rho_e$ be the corresponding inclusions, and $\phi_i^e : \rho_e \rightarrow \rho_i^e$ the corresponding projections, with $e \in \{0, 1\}$. Then we have two families of morphisms $\{\beta_{ij}\}_{i \in I, j \in J}$ and $\{c_{ij}\}_{i \in I, j \in J}$ defined by

$$\begin{aligned} \beta_{ij} : \pi_1 &\xrightarrow{\beta} \mathcal{A}(V_0, V_1) \xrightarrow{(\iota_j^0)^*(\phi_i^1)^*} \mathcal{A}(V_j^0, V_i^1), \\ c_{ij} : \pi_0^2 &\xrightarrow{c} \mathcal{A}(V_0, V_1) \xrightarrow{(\iota_j^0)^*(\phi_i^1)^*} \mathcal{A}(V_j^0, V_i^1), \end{aligned}$$

Let us consider a bipartite graph $\mathcal{G}(I, J)$ whose set of vertices is $I \amalg J$, and with an edge $\{i, j\}$ if and only if $\beta_{ij} \neq 0$ or $c_{ij} \neq \partial u$ for some 1-cochain $u : \pi_0 \rightarrow \mathcal{A}(V_j^0, V_i^1)$. Then we have the following.

3.19. PROPOSITION. *A representation $(\rho_1, \rho_0, \beta, c)$ is indecomposable if and only if for each pair of decompositions of ρ_1 and ρ_0 as before, the corresponding graph $\mathcal{G}(I, J)$ is connected.*

PROOF. Let us suppose that there exists a decomposition such that $\mathcal{G}(I, J)$ is disconnected. This means that there exist $\emptyset \subsetneq I' \subsetneq I$ and $\emptyset \subsetneq J' \subsetneq J$ such that $\beta_{i', j} = 0$ and $c_{i', j}$ is a coboundary for all $i' \in I'$ and $j \in J \setminus J'$, and $\beta_{i, j'} = 0$ and $c_{i, j'}$ is a coboundary for all $i \in I \setminus I'$ and $j' \in J'$.

Put $\rho'_1 = \bigoplus_{i' \in I'} \rho_{i'}^1$, $\rho''_1 = \bigoplus_{i \in I \setminus I'} \rho_i^1$, $\rho'_0 = \bigoplus_{j' \in J'} \rho_{j'}^0$ and $\rho''_0 = \bigoplus_{j \in J \setminus J'} \rho_j^0$. Then we have $\rho_1 = \rho'_1 \oplus \rho''_1$ and $\rho_0 = \rho'_0 \oplus \rho''_0$, and by composing with the appropriate projections and inclusions we can define β', β'', c', c'' as

$$\begin{aligned} \beta' : \pi_1 &\xrightarrow{\beta} \mathcal{A}(V_0, V_1) \longrightarrow \mathcal{A}(V'_0, V'_1), \\ \beta'' : \pi_1 &\xrightarrow{\beta} \mathcal{A}(V_0, V_1) \longrightarrow \mathcal{A}(V''_0, V''_1), \\ c' : \pi_0^2 &\xrightarrow{c} \mathcal{A}(V_0, V_1) \longrightarrow \mathcal{A}(V'_0, V'_1), \\ c'' : \pi_0^2 &\xrightarrow{c} \mathcal{A}(V_0, V_1) \longrightarrow \mathcal{A}(V''_0, V''_1). \end{aligned}$$

It is straightforward to check that $\beta = \beta' \oplus \beta''$ and that $c - c' \oplus c''$ is a coboundary. Therefore

$$(\rho_1, \rho_0, \beta, c) \simeq (\rho'_1, \rho'_0, \beta', c') \oplus (\rho''_1, \rho''_0, \beta'', c'').$$

Conversely, let us now suppose that

$$(\rho_1, \rho_0, \beta, c) \simeq (\rho'_1, \rho'_0, \beta', c') \oplus (\rho''_1, \rho''_0, \beta'', c''),$$

and let $\rho'_1 = \bigoplus_{i' \in I'} \rho_{i'}^1, \rho''_1 = \bigoplus_{i'' \in I''} \rho_{i''}^1$ and $\rho'_0 = \bigoplus_{j' \in J'} \rho_{j'}^0, \rho''_0 = \bigoplus_{j'' \in J''} \rho_{j''}^0$. It is clear that these give decompositions of ρ_1 and ρ_0 such that the corresponding graph $\mathcal{G}(I' \amalg I'', J' \amalg J'')$ is disconnected. ■

Notice that, according to this result, a representation $(\rho_1, \rho_0, \beta, c)$ can be indecomposable even when ρ_1 and ρ_0 are decomposable.

4. Case π_0, π_1 finite and $\text{char}(k) = 0$ or coprime to the orders of π_0, π_1

As mentioned in the introduction, in this case the theory “collapses”. The main reason is the following.

4.1. PROPOSITION. *Let us assume that \mathbb{G} is finite (i.e. π_0 and π_1 are finite), and that \mathcal{A} is k -linear, with k a field whose characteristic is zero or coprime to the orders of π_1 and π_0 . Then for any representation $(\rho_1, \rho_0, \beta, c)$ we have:*

(i) $\beta = 0$, and

(ii) $(\rho_1, \rho_0, 0, c)$ is equivalent to $(\rho_1, \rho_0, 0, 0)$.

In particular, up to equivalence a representation of \mathbb{G} in $\mathbf{Ch}'_2(\mathcal{A})$ is completely given by two representations of π_0 in \mathcal{A} .

PROOF. If \mathcal{A} is k -linear, $\mathcal{A}(V_0, V_1)$ is a k -vector space. Since the characteristic of k is relatively prime with the order of π_1 the only morphism of abelian groups $\beta : \pi_1 \rightarrow \mathcal{A}(V_0, V_1)$ is $\beta = 0$. In this case, the 2-cochain is a 2-cocycle, and the second statement follows then from the next lemma (applied to G -bimodules with trivial right action of G) together with Corollary 3.4. ■

4.2. LEMMA. *Let G be a finite group, and V a k -vector space equipped with a structure of G -bimodule. If the characteristic of k is zero or coprime to the order of G , then $H^n(G, V) = 0$ for all $n \geq 1$.*

PROOF. Let $z : G^n \rightarrow V$ be an n -cocycle, with $n \geq 1$. Then an n -cochain with boundary z is given by the map $c : G^{n-1} \rightarrow V$ defined by

$$c(g_1, \dots, g_{n-1}) = \frac{(-1)^n}{|G|} \sum_{k \in G} z(g_1, \dots, g_{n-1}, k) \cdot k^{-1}.$$

The reader may easily check that the cocycle condition $\partial z = 0$ indeed implies $\partial c = z$. ■

If as usual by the homotopy category of a 2-category \mathbf{K} we mean the category with the same objects as \mathbf{K} and with the 2-isomorphism classes of 1-cells in \mathbf{K} as morphisms, the main theorem in this case may be stated as follows.

4.3. THEOREM. *Let \mathcal{A} be a split k -linear abelian category. Then for any finite 2-group \mathbb{G} such that the characteristic of k is zero or coprime to the orders of π_0 and π_1 , the homotopy category of $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ is equivalent to the product category $\mathcal{R}ep_{\mathcal{A}}(\pi_0) \times \mathcal{R}ep_{\mathcal{A}}(\pi_1)$. Moreover, if $(r_1, r_0), (r'_1, r'_0) : (\rho_1, \rho_0) \rightarrow (\rho'_1, \rho'_0)$, there is a 2-cell between them in $\mathbf{Rep}_{\mathbf{Ch}_2(\mathcal{A})}(\mathbb{G})$ only if $r_i = r'_i, i = 0, 1$, and in this case a 2-cell is just an intertwiner from ρ_0 to ρ'_0 .*

PROOF. By Proposition 4.1, we may restrict to representations $(\rho_1, \rho_0, \beta, c)$ with β and c equal to zero. From Theorem 3.7 together with Lemma 4.2 we can see that the 2-vector space of intertwiners between two such representations $(\rho_1, \rho_0, 0, 0), (\rho'_1, \rho'_0, 0, 0)$ is

$$0 : \text{Hom}_{\pi_0}(V_0, V'_0) \rightarrow \text{Hom}_{\pi_0}(V_1, V'_1) \times \text{Hom}_{\pi_1}(V_0, V'_0).$$

The result follows easily from this. ■

To finish, let us point out that, as it occurs in the classical setting of groups, the irreducible and indecomposable representations of \mathbb{G} in this case coincide.

4.4. PROPOSITION. *If the characteristic of the field k does not divide the orders of π_1 and π_0 , then a representation is indecomposable if and only if it is weakly irreducible.*

PROOF. By Proposition 4.1 any representation of \mathbb{G} is equivalent to one of the form $(\rho_1, \rho_0, 0, 0)$, and so it is indecomposable if and only if one of the ρ_i is indecomposable and the other is 0, which happens if and only if the representation is weakly irreducible by Proposition 3.16. ■

As the next examples shows, this is not true in general.

4.5. EXAMPLE. Let $k = \mathbb{F}_3$ be the field with 3 elements, C_2 and C_3 the cyclic groups of orders 2 and 3, respectively, and

$$\mathbb{G} = C_3[1] \rtimes C_2[0]$$

where we are considering $C_3 = \{1, s, s^2\}$ as a C_2 -module with trivial action. Consider \mathbb{F}_3 as the trivial representation of C_2 and let $\beta : C_3 \rightarrow \mathcal{V}ect_{\mathbb{F}_3}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3$ be the map given by

$$1 \mapsto 0, \quad s \mapsto 1, \quad s^2 \mapsto 2.$$

Then it easily follows from Proposition 3.19 that the representation $(\mathbb{F}_3, \mathbb{F}_3, \beta, 0)$ is indecomposable. However, it is not weakly irreducible by Proposition 3.16.

5. Final comments

We have shown that the representation theory of a finite 2-group \mathbb{G} in Baez-Crans 2-vector spaces over a field k is a bad one when the characteristic of k is zero because a lot of information on \mathbb{G} gets lost. In fact, a similar result is expected to hold for *compact topological 2-groups* and their *topological representations* in a suitable 2-category of *topological Baez-Crans 2-vector spaces*. Thus a topological Baez-Crans 2-vector space should reasonably be defined as an object in $\mathbf{Ch}_2(\mathcal{A})$ for some split abelian category \mathcal{A} of topological vector spaces over a suitable *topological* field. Although the category of all topological vector spaces over an arbitrary topological field is non abelian (images and coimages do not necessarily coincide), it will be so if one restricts to finite dimensional vector spaces over the field of real or complex numbers with the usual topology. Thus Proposition 2.5 will also hold in this setting. Moreover, if \mathbb{G} is a compact topological 2-group, the groups π_0 and π_1 will be compact topological groups, and in a topological representation the homomorphism β is expected to be continuous. However, there are no compact subgroups in the underlying topological abelian group of a finite dimensional real or complex vector space. Also, the proof of Lemma 4.2 is expected to work for compact topological groups if one replaces the sum over the elements of G by the corresponding Haar integral, so that both Lemma 4.2 and Proposition 4.1 are expected to be also true in this topological setting. However, the question would deserve a more careful study. In fact, the theory of topological 2-groups may look different to that of 2-groups. For instance, it is even unclear if any topological 2-group is equivalent to a skeletal topological 2-group because there is no axiom of choice in the category of topological spaces.

However, things seem to be different in positive characteristic. In this case, the roles of the second homotopy group of \mathbb{G} and its Postnikov invariant are no longer passive, and we conjecture that the theory is in this case rich enough to be able to prove a reconstruction theorem of the Tannaka-Krein type. To prove this, it is necessary to study the representability of the forgetful 2-functor $\mathbf{U} : \mathbf{Rep}_{\mathbf{Ch}'_2(k)}(\mathbb{G}) \rightarrow \mathbf{Ch}'_2(k)$. This makes sense because we already know that the hom-categories in $\mathbf{Rep}_{\mathbf{Ch}'_2(k)}$ are indeed objects in $\mathbf{Ch}'_2(k)$ (cf. Theorem 3.7). Then the 2-category Yoneda lemma should allow us to prove that \mathbb{G} can indeed be recovered as the 2-group of monoidal automorphism of \mathbf{U} . This is still work in progress.

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