INFINITESIMAL ASPECTS OF THE LAPLACE OPERATOR

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ABSTRACT. In the context of synthetic differential geometry, we study the Laplace operator an a Riemannian manifold. The main new aspect is a neighbourhood of the diagonal, smaller than the second neighbourhood usually required as support for second order differential operators. The new neighbourhood has the property that a function is affine on it if and only if it is harmonic.

Introduction

Recall [3], [4], [5], [7] that any manifold M, when seen in a model of Synthetic Differential Geometry (SDG), carries a reflexive symmetric relation $\sim_k$ ($k = 0, 1, 2, ...$), where $x \sim_k y$ reads "x and y are k-neighbours"; $x \sim_0 y$ means $x = y$; $x \sim_k y$ implies $x \sim_{k+1} y$. Also, $x \sim_k y, y \sim_l z$ implies $x \sim_{k+l} z$. The set of $(x, y) \in M \times M$ with $x \sim_k y$ is denoted $M_{(k)}$, the "k'th neighbourhood of the diagonal", and for fixed x, the set $\{y \in M \mid y \sim_k x\}$ is denoted $\mathcal{M}_k(x)$ ("the k-monad around x"). In R^n, $\mathcal{M}_k(0)$ is denoted $D_k(n)$. Its elements u are characterized by the condition that any homogeneous polynomial of degree $k + 1$ vanishes on u. In this context, a *Riemannian metric* on M can be given in terms of a map

$$g : M_{(2)} \to R$$

with $g(x, x) = 0$, and with $g(x, y) = g(y, x)$ to be thought of as the "square-distance between x and y", se [5], [6]. (Also g should be positive-definite, in a certain sense.)

Given a Riemannian metric g on M, in this sense, one can construct the Levi-Civita connection [5], volume form [6], and hence also a notion of *divergence* of a vector field. And to a function $f : M \to R$, one can construct its *gradient vector field,* and hence one can construct the Laplacian Δ by $\Delta(f) = div(grad(f))$. This is what we shall *not* do here, rather, we shall exploit the richness of synthetic language to give a more economic and more geometric construction of Δ. The construction is more economic in the sense that the definition of $\Delta f(x)$ only depends on knowing f on a certain subset $\mathcal{M}_L(x) \subseteq \mathcal{M}_2(x)$, where $\mathcal{M}_2(x)$ is what is required to make the usual *div grad* construction work, or for defining the individual terms in the formula $\Delta f(x) = \sum \partial^2 f / \partial x_i^2(x)$.

The description of $\mathcal{M}_L(x) \subseteq M$, or equivalently, the description of the L-neighbour relation $\sim_L$, is coordinate free, see Definition 2.1 below, and therefore, too, is the description of Δf and of the notion of harmonic function. We get a characterization of harmonic

functions, in terms of an average-value property, which is infinitesimal in character and does not involve integration, see Theorem 2.4 and Proposition 2.7.

In Section 3 we prove that diffeomorphisms which preserve the L-neighbour relation are precisely the conformal ones. Section 4 deals with the special case of the complex plane, and Section 5 explains the "support of the Laplacian" in systematic algebraic terms.

1. Preliminaries

Although the notions we use are introduced in a coordinate free way, we have no intention of avoiding use of coordinates as a tool of proof. This Section contains in fact mainly certain coordinate calculations, which we believe will be useful also in other contexts where Riemannian geometry is treated in the present synthetic manner.

Working in coordinates in M means that we are identifying (an open subset of) M with (an open subset of) R^n; for simplicity, we talk about these open subsets as if they were all of M and R^n, respectively; all our considerations are anyway only *local*. The Riemannian metric g on M then becomes identified with a Riemannian metric on R^n, likewise denoted g, and it may be written (for $x \sim_2 y$) in the form of a matrix product,

$$g(x, y) = (y - x)^T \cdot G(x) \cdot (y - x),$$

where $x - y \in R^n$ is viewed as a column matrix, and $G(x)$, for each x, is a symmetric positive definite $n \times n$ matrix.

Using coordinates, we may form affine combinations of (the coordinate sets of) points of M, at least for sufficiently nearby points, and such combinations will in general have only little geometric significance, since they depend on the choice of the coordinate system. However, we have the following useful fact:

1.1. PROPOSITION. *Assume $y_1 \sim_1 x$ and $y_2 \sim_1 x$ (so $(x + y_2 - y_1) \sim_2 x$). Then*

$$g(x, x + y_2 - y_1) = g(y_1, y_2);$$

in particular, for $x = 0$,

$$g(0, y_2 - y_1) = g(y_1, y_2).$$

PROOF. We may assume $x = 0$. Then $g(0, y_2 - y_1)$ and $g(y_1, y_2)$ are given, respectively, by

$$(y_2 - y_1)^T \cdot G(0) \cdot (y_2 - y_1) = -2y_1^T \cdot G(0) \cdot y_2$$

and

$$(y_2 - y_1)^T \cdot G(y_1) \cdot (y_2 - y_1) = -2y_1^T \cdot G(y_1) \cdot y_2.$$

Now expand $G(y)$ as $G(0) + H(y)$ where H depends linearly on $y \sim_1 0$. The difference between our two expressions is then $-2y_1^T \cdot H(y_1) \cdot y_2$, which depends bilinearly on y_1 and therefore vanishes. ∎

We shall see below (Proposition 1.4) that the "coordinatewise" affine combination considered in Proposition 1.1 *does* have an invariant geometric meaning, provided the coordinate system is *geodesic*:

We say that the metric g on R^n (or equivalently, the coordinate system around $x_0 \in M$) is *geodesic* at $0 \in R^n$ (or at $x_0 \in M$, respectively), if the first partial derivatives of $G(x)$, as functions of $x \in R^n$, vanish at 0; equivalently, if $G(x) = G(0)$ for every $x \sim_1 0$. (This is in turn equivalent to the vanishing at x_0 of the Christoffel symbols of the metric, in the given coordinate system.) It is classical that for every point x_0, there exists a coordinate system which is geodesic at x_0. If $G(0)$ is the identity matrix, one talks about a *normal* coordinate system at x, and such also exist. Cf. e.g. [1] for such notions.

Recall from [5] formula (2) that any Riemannian metric $g : M_{(2)} \to R$ admits a unique symmetric extension $\overline{g} : M_{(3)} \to R$; in coordinates it is given by

$$\overline{g}(x, y) = (y - x)^T \cdot (G(x) + 1/2(D_{(y-x)}G)(x)) \cdot (y - x). \tag{1}$$

Recall also from [5] Theorem 3.6 that for $x \sim_2 z$ in a Riemannian manifold, and for $t \in R$, there exists a unique y_0 with $y_0 \sim_2 x$ and $y_0 \sim_2 z$ which is a critical point for the function of y given by

$$t\overline{g}(x, y) + (1 - t)\overline{g}(z, y); \tag{2}$$

We call this y_0 an (intrinsic) *affine combination* of x and z. We write it $tx + (1 - t)z$; this raises a compatibility problem in case we are working in coordinates, since we can then also form the "algebraic" affine combination of two coordinate n-tuples. However, in geodesic coordinates at x, there is no problem, according to the following Proposition, which extends Proposition 3.7 in [5]. Let us consider a coordinate system with x identified with 0.

1.2. PROPOSITION. *The critical point y_0 for the function in (2) is the algebraic affine combination $tx + (1 - t)z$, if either $x \sim_1 z$, or if the coordinate system is geodesic at x.*

PROOF. Since x is identified with 0 in the coordinate system, the affine combination in question is just $(1 - t)z$. To show that it is a critical value for (2) means that

$$t\overline{g}(0, (1 - t)z + v) + (1 - t)\overline{g}(z, (1 - t)z + v) \tag{3}$$

is independent of $v \sim_1 0$. We write g in terms of the symmetric matrices G, as above. Let us take a Taylor expansion of the function $G(y)$, writing

$$G(y) = G(0) + H(y),$$

where the entries of the matrix $H(y)$ are of degree ≥ 1 in y; and if the coordinate system is geodesic at $x = 0$, $H(y)$ is even of degree ≥ 2 in y. We then calculate. We get a "significant" part from each of the two terms in (3), and then some "error" terms, each of which will vanish for degree reasons, as we shall argue.

The two significant terms are the two terms in

$$t(((1-t)z+v)^T \cdot G(0) \cdot ((1-t)z+v)) + (1-t)((-tz+v)^T \cdot G(0) \cdot (-tz+v)).$$

Expanding out by bilinearity and symmetry, the terms involving v linearly cancel each other; and the terms involving v quadratically vanish because $v \sim_1 0$. So the significant terms, jointly, do not depend on $v \sim_1 0$.

The "error" terms are of two kinds: partly, arising from the replacement of $G(z)$ by $G(0)$; here, $H(z)$ enters; and partly there are correction terms when passing from g to $\overline{g}$ defined on pairs of third order neighbours. The error term of the first kind is a multiple of

$$(-tz+v)^T \cdot H(z) \cdot (-tz+v);$$

we expand this out by bilinearity, and use that $H(z)$ is of degree ≥ 1, and $v \sim_1 0$. We get four terms each of which vanish for degree reasons if *either* $z \sim_1 0$ *or* if $H(z)$ is of degree ≥ 2.

Finally, the correction terms for upgrading g to $\overline{g}$ do not occur if $z \sim_1 0$, since then g is only applied to pairs of second order neighbours. Thus the assertion of the Proposition about the case $z \sim_1 x$ is already proved. In general, the upgrading involves first partial derivatives of G, (see (1)), so in the case the coordinate system is geodesic at 0, no correction term is needed for $\overline{g}(0, (1-t)z)$, but only for $\overline{g}(z, (1-t)z+v)$. Using the formula (1), we see that the correction needed is a certain multiple of

$$(-tz+v)^T \cdot (D_{-tz+v}G)(z) \cdot (-tz+v),$$

hence a linear combination of terms

$$z \cdot D_z G(z) \cdot z, \ v \cdot D_z G(z) \cdot z, \ z \cdot D_v G(z) \cdot z,$$

and something that contains v in a bilinear way. All these terms vanish for degree reasons: for, since H vanishes in the first neighbourhood of 0, $D_v G(z)$ is of degree ≥ 1 in z, and $D_z G(z)$ is even of degree ≥ 2 in z. ∎

Essentially the same degree counting as in this proof gives the following result:

1.3. LEMMA. *Let $y \sim_1 x$ and $z \sim_2 x$; then using a geodesic coordinate system at $x = 0$, the quantity $\overline{g}(y,z)$ may be calculated as $(z-y)^T \cdot G(0) \cdot (z-y)$.*

Given a Riemannian manifold. If $x \sim_2 z$, the *mirror image z'* of z in x is by definition the affine combination $2x - z$, i.e. the y which is critical value for $2\overline{g}(x,y) - \overline{g}(z,y)$, [5] Theorem 3.6. Also, the parallelogram formation λ is described in [5]. Finally, if t is a tangent vector $D \to M$, its geodesic prolongation $\overline{t} : D_2 \to M$ is determined by the validity, for all $d_1, d_2 \in D$ of

$$\overline{t}(d_1 + d_2) = \lambda(t(0), t(d_1), t(d_2)).$$

(Recall that $D \subseteq R$ are the elements of square zero, D_2 the elements of cube zero.) Now Proposition 1.2 has the following Corollary:

1.4. PROPOSITION. *Let $x \sim_2 z$; then the mirror image z' of y w.r.to x may be calculated as follows: take a geodesic coordinate system at x with $x = 0$. Then $z' = -z$.*

Let $y \sim_1 x$, $z \sim_1 x$. Then $\lambda(x, y, z)$ may be calculated as follows: take a geodesic coordinate system at x with $x = 0$. Then $\lambda(x, y, z) = y + z$.

Let t be a tangent vector $D \to M$ at $x \in M$. Then the geodesic prolongation $\overline{t} : D_2 \to M$ of t may be calculated as follows: take a geodesic coordinate system at x with $x = 0$. Let u be the unique vector in R^n so that $t(d) = d \cdot u$ for all $d \in D$. Then for $\delta \in D_2$, $\overline{t}(\delta) = \delta \cdot u$

(The vector $u \in R^n$ appearing in the last clause is usually called the *principal part* of t, relative to the coordinate system.)

If t and s are tangent vectors at the same point x of a Riemannian manifold M, g, we define their inner product $\langle t, s \rangle$ by the validity, for all $d_1, d_2 \in D$, of

$$d_1 d_2 \langle t, s \rangle = -\frac{1}{2} g(t(d_1), s(d_2)). \tag{4}$$

In this way, the tangent vector space $T_x M$ is made into an inner product space, (and this is the contact point with the classical formulation of Riemannian metric).

If u and v are the principal parts of tangent vectors t and s at $x \in M$, in some coordinate system at $x = 0$ (not necessarily geodesic), one has $\langle t, s \rangle = u^T \cdot G(0) \cdot v$; this follows easily from Proposition 1.1.

Combining Lemma 1.3 and Proposition 1.4, one gets

1.5. LEMMA. *Let t be a tangent vector. Then for $d \in D$, $\delta \in D_2$, we have*

$$\overline{g}(t(d), \overline{t}(\delta)) = (\delta^2 - 2d\delta) \cdot \langle t, t \rangle.$$

We are going to define the *orthogonal projection* of z ($z \sim_2 x$) onto a proper tangent t at x. We first define the *scalar component* of z along t; this is the unique number $\alpha(z, t)$ so that

$$d \cdot \alpha(z, t) = \frac{1}{2} \frac{(g(x, z) - \overline{g}(t(d), z))}{\langle t, t \rangle} \tag{5}$$

for all $d \in D$. Note that if $z = x$, $\alpha(z, t) = 0$, and from this follows that for any $z \sim_2 x$, $\alpha(z, t) \sim_2 0$, in other words $\alpha(z, t) \in D_2$. From Lemma 1.5, applied twice (once with $d = 0$, once with a general $d \in D$), it is immediate to deduce that if z is of the form $\overline{t}(\delta)$ for a $\delta \in D_2$, then $\alpha(z, t) = \delta$.

We define the orthogonal projection $proj_t(z)$ by

$$proj_t(z) = \overline{t}(\alpha(z, t)).$$

Note that it is a second-order neighbour of x. It follows from the above that if z is of the form $\overline{t}(\delta)$, then $proj_t(z) = z$.

2. Laplacian neighbours

Here is the crucial definition:

2.1. DEFINITION. *Let $z \sim_2 x$. We say that z is a* Laplacian neighbour *of x (written $z \sim_L x$) if for every proper tangent t at x, we have*

$$g(x, z) = n \cdot g(x, proj_t(z)), \tag{6}$$

where n is the dimension of the manifold.

Maybe one of the names "isotropic, harmonic, or conformal, neighbour" would be more appropriate.

Clearly $z \sim_1 x$ implies $z \sim_L x$; for if z is a first-order neighbour of x, then so is its orthogonal projection, and hence both the g-quantities to be compared in (6) are zero. If the dimension n is 1, $\sim_L$ is the same as $\sim_2$ but in general, the set $\mathcal{M}_L(x)$ of L-neighbours of x is much smaller than the set $\mathcal{M}_2(x)$ of second-order neighbours; in fact, the ring of functions on $\mathcal{M}_L(x)$ is a finite dimensional vector space which is just one dimension bigger than the ring of functions on $\mathcal{M}_1(x)$, as we shall see in the proof of Proposition 2.3 below.

We conjecture that the relation $\sim_L$ is symmetric, but we haven't been able to do the necessary calculations, except in the case of R^n, where the symmetry is easy to prove, using Proposition 2.2 below.

Note the following curious phenomenon in dimension $n \geq 2$: if $z \sim_L x$, then z does not connect to x by *any* geodesic $D_2 \to M$ (given by a proper tangent vector t), except perhaps in the trivial case when $g(x, z) = 0$. In other words, the L-neighbours of x are genuinely *isotropic*, in the sense that they are in *no* preferred direction t (hence the alternative name "isotropic neighbour" suggested). Nevertheless, there are sufficiently many L-neighbours of x to define the Laplacian differential operator Δ, see Theorem 2.4 below.

Let us assume the manifold in question has dimension n. Then we have

2.2. PROPOSITION. *In any geodesic normal coordinate system at $x = 0$, $z = (z_1, \ldots, z_n)$ is $\sim_L 0$ if and only if*

$$z_i^2 = z_j^2 \text{ for all } i, j, \text{ and } z_i z_j = 0 \text{ for } i \neq j$$

(and $z_i^3 = 0$ for all i; this latter condition follows from the other two if $n \geq 2$).

PROOF. First, if t and s are tangent vectors at $x = 0$ with principal parts u and v, respectively (meaning $t(d) = du, s(d) = dv$), then $\langle t, s \rangle = u \bullet v$, where $\bullet$ denotes the usual dot product of vectors in R^n. Also, if t is a tangent at $x = 0$ with principal part u, then $\alpha(z, t) = (z \bullet u)/(u \bullet u)$; for, calculating the enumerator in (5) gives (using Proposition 1.1)

$$z \bullet z - \overline{g}(du, z) = z \bullet z - (z - du) \bullet (z - du) = 2d \, z \bullet u.$$

From the third clause in Proposition 1.4, we then get the familiar looking

$$proj_t(z) = \frac{z \bullet u}{u \bullet u} u. \qquad (7)$$

In particular, if t is the (proper) tangent vector with principal part $e_i \in R^n$ (= $(0, \ldots, 1, \ldots 0)$ (with 1 in the i'th position, 0's elsewhere), then $proj_t(z_1, \ldots, z_n) = z_i e_i$. In particular $g(0, proj_t(z)) = z_i^2$. If, on the other hand, t is the tangent vector with principal part $e_{i,j}$ (the vector with 1's in the i'th and in the j'th position, $i \neq j$, 0's elsewhere), then $proj_t(z)$ has $(z_i + z_j)/2$ in the i'th and in the j'th position, and 0's elsewhere. In particular,

$$g(0, proj_t(z)) = \frac{1}{2}(z_i^2 + z_j^2) + z_i z_j.$$

If z therefore is an L-neighbour of 0, we conclude that $z_i^2 = z_j^2$ for all i, j, and that $z_i z_j = 0$ if $i \neq j$.

Conversely, assume that in some geodesic normal coordinate system at $x = 0$, the coordinates $(z_1, \ldots, z_n)$ satisfy the equations $z_i^2 = z_j^2$, $z_i z_j = 0$ for $i \neq j$, and let t be a proper tangent vector at x with principal part $u = (u_1, \ldots, u_n)$. Then

$$proj_t(z) = \frac{z \bullet u}{u \bullet u} u,$$

and therefore

$$g(x, proj_t(z)) = (\frac{z \bullet u}{u \bullet u} u) \bullet (\frac{z \bullet u}{u \bullet u} u),$$

which we calculate by arithmetic to be

$$\frac{(\sum_i u_i z_i)(\sum_j u_j z_j)}{u \bullet u} = \frac{\sum_{ij} u_i u_j z_i z_j}{u \bullet u},$$

but since $z_i z_j = 0$ for $i \neq j$, only the "diagonal" terms survive, and we are left with

$$\frac{\sum_i u_i u_i z_i z_i}{u \bullet u}.$$

But $z_i z_i = z_1 z_1$ for all i, so this factor can go outside the sum sign in the enumerator, and we get $z_1^2(\sum_i u_i u_i)/u \bullet u = z_1^2$, which is $1/n$ times $\sum z_i^2$ since all the z_i^2 are equal. This proves the Proposition. ∎

From Propositions 1.2 and 2.2, one immediately deduces that if $z \sim_L x$, then also $z' \sim_L x$ for any affine combination $z' = tx + (1 - t)z$ ($t \in R$).

2.3. PROPOSITION. *If two functions f_1 and f_2: $\mathcal{M}_L(x) \to R$ agree on $\mathcal{M}_1(x)$ there is a unique number $c \in R$ such that for all $z \sim_L x$*

$$f_1(z) - f_2(z) = c \cdot g(x, z).$$

PROOF. Using a geodesic normal coordinate system at $x = 0$, it is a matter of analyzing the ring of functions $\mathcal{M}_L(0) \to R$ for the case where $M = R^n$ with standard inner-product metric. The Proposition gives that $\mathcal{M}_L(0)$ may be described as $D_L(n) \subseteq R^n$, defined by

$$D_L(n) := \{(d_1, \ldots, d_n) \in R^n \mid d_i^2 = d_j^2, \text{ and } d_i d_j = 0 \text{ for } i \neq j\}, \tag{8}$$

(for $n \geq 2$; for $n = 1$, $D_L(1) = D_2 = \{\delta \in R \mid \delta^3 = 0\}$). This is (for $n \geq 2$) the object represented by the Weil algebra $\mathcal{O}(D_L(n)) := k[Z_1, \ldots, Z_n]/I$, where I is the ideal generated by the $Z_i^2 - Z_j^2$, and by $Z_i Z_j$ for $i \neq j$. It is immediate to calculate that, as a vector space, this ring is $(n + 2)$-dimensional, with linear generators

$$1, Z_1, \ldots, Z_n, Z_1^2 + \ldots + Z_n^2.$$

By the general (Kock-Lawvere) axiom scheme for SDG [4], [8], this means that any function $f : D_L(n) \to R$ is of the form

$$f(z_1, \ldots, z_n) = a + \sum_i b_i z_i + c(\sum_i z_i^2),$$

for unique $a, b_1, \ldots, b_n, c \in R$, or equivalently

$$f(z_1, \ldots, z_n) = a + \sum_i b_i z_i + cg(0, z).$$

Since the restriction of f to $D(n)$ is given by the data $a, b_1, \ldots, b_n$, the unique existence of c follows. ∎

The following Theorem deals with an arbitrary Riemannian manifold M, g of dimension n, and gives a coordinate free characterization of the Laplacian operator Δ.

2.4. THEOREM. *For any $f : \mathcal{M}_L(x) \to R$, there is a unique number L with the property that for any $z \sim_L x$*

$$f(z) + f(z') - 2f(x) = L \cdot g(x, z),$$

where z' denotes the mirror image of z in x. We write $\Delta f(x) := nL$.

Put differently,

$$f(z) + f(z') - 2f(x) = \frac{\Delta f(x)}{n} g(x, z).$$

If the function f is *harmonic* at x, meaning that $\Delta f(x) = 0$, it follows that it has a strong *average value property*: the value at x equals the average value of f over any pair of points z and z' (L-neighbours of x) which are symmetrically located around x.

PROOF. Again, we pick a normal geodesic coordinate system at $x = 0$, so identify $\mathcal{M}_L(x)$ with $D_L(n)$; then z' gets identified with $-z$, by Proposition 1.4. The left hand side of the expression in the Theorem then has restriction 0 to $D(n)$, being (with notation as above) $(a + \sum b_i z_i) + (a + \sum b_i(-z_i)) - 2a$. Hence the unique existence of L follows from Proposition 2.3. ∎

The following Proposition serves to as partial justification of the use of the name "Laplacian" for the Δ considered in the Theorem. We consider the standard Riemannian metric on R^n, $g(x,z) = ||z - x||^2$, for $z \sim_2 x$.

2.5. PROPOSITION. *Let $f : R^n \to R$ and let $x \in R^n$. Then*

$$\Delta f(x) = \sum_i \frac{\partial^2 f}{\partial x_i^2}(x).$$

PROOF. For simplicity, let $x = 0$, so that $z' = -z$. We Taylor expand $f(z)$ and $f(-z)$ from 0, and consider $f(z) + f(-z) - 2f(0)$; then terms of degree ≤ 1 cancel, and we get

$$\sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} z_i z_j + \text{ higher terms;}$$

now the calculation proceeds much like the one in the proof of Proposition 2.2 above: if $z \sim_L 0$, only the diagonal terms in the sum survive, all the z_i^2 are equal to z_1^2, which we move outside the parenthesis, and get z_1^2 times the classical Laplacian $\sum \partial^2 f/\partial x_i^2(0)$. But $z_1^2 = 1/n \cdot g(0, z)$. ∎

Similarly, one proves by Taylor expansion

2.6. PROPOSITION. *If $z \sim_L x$ in R^n, then for any $f : R^n \to R$,*

$$f(z) = f(x) + df_x(z - x) + \frac{1}{2n} \Delta f(x) ||z - x||^2.$$

Recall that any function $M \to R$ looks affine on any 1-monad $\mathcal{M}_1(x)$; functions that look affine on the larger L-monads $\mathcal{M}_L(x)$ are precisely the harmonic ones:

2.7. PROPOSITION. *Assume $f : M \to R$ is harmonic at x. Then for an $z \sim_L x$, f preserves affine combinations of x and z. Conversely, if for given x, f preserves affine combinations of x and z for every $z \sim_L x$, then f is harmonic at x. ("Harmonic" at x here in the sense: $\Delta f(x) = 0$.)*

(Recall that the affine combination $tx + (1 - t)z$ is defined as the critical point y_0 in (2).)

PROOF. Assume $z \sim_L x$, and pick a geodesic normal coordinate system at $x = 0$. Without loss of generality, we may assume $f(0) = 0$. Then to say that f preserves affine combinations of x and z is to say that for all $s \in R$, $f(sz) = sf(z)$. For $z \sim_L 0$, we have by Proposition 2.6 that $f(z) = \sum a_i z_i + c \sum z_i^2$ for unique a_i and c ($c = \Delta f(0)/2n$); $f(sz)$ and $sf(z)$ have the same terms of first order in z; their second order terms are respectively $cs^2 \sum z_i^2$ and $sc \sum z_i^2$, and if these two expressions are to be equal for all s and all z, we must have $c = 0$. This means that f is harmonic at x. Conversely, if f preserves affine combinations of the kind mentioned, it preserves the affine combination $2x - z$, or equivalently, the left hand side of the expression in Theorem 2.4 is 0, hence it follows that L and hence $\Delta f(x)$ is 0. ∎

REMARK. With some hesitation, I propose to call a map $f : M \to N$ between Riemannian manifolds *harmonic* if it preserves affine combinations of L-neighbours in M and if it preserves the property of being L-neighbours. I have not been able to compare the proposed definition with a certain classical concept of harmonic map between Riemannian manifolds. But at least, when the codomain is R, the definition is the basic classical one of harmonic function, by Proposition 2.7. For, preservation of the $\sim_L$ relation is automatic when the codomain is R, since in R, $\sim_L$ is the same as $\sim_2$.

3. Conformal maps

We consider a diffeomorphism $f : M \to N$ between Riemannian manifolds $(M, g), (N, h)$. To say that f is an *isometry* at $x \in M$ is to say that for all $z \sim_2 x$, $g(x, z) = h(f(x), f(z))$. To say that f is *conformal* at $x \in M$ with constant $k = k(x) > 0$ is to say that for all $z \sim_2 x$, $h(f(x), f(z)) = k(x)g(x, z)$ (so if $k(x) = 1$, f is an isometry at x). The terminology agrees with classical usage, as we shall see below. We first prove

3.1. PROPOSITION. *Assume $f : M \to N$ is conformal at $x \in M$ with*
$h(f(x), f(z)) = kg(x, z)$. *Then for all $y_1 \sim_1 x$, $y_2 \sim_1 x$,*

$$h(f(y_1), f(y_2)) = kg(y_1, y_2),$$

and conversely.

PROOF. We choose coordinates, and assume $x = 0$ and $f(x) = 0$; the metrics in M and N are then given by functions g and h, respectively, and they are in turn given by symmetric matrices $G(y)$, and $H(z)$ for all $y \in M$ and $z \in N$. We now calculate $kg(y_1, y_2)$. We have, by Proposition 1.1 that

$$kg(y_1, y_2) = kg(0, y_2 - y_1) = h(0, f(y_2 - y_1)).$$

Now there is a bilinear $B(-, -)$ such that for all pairs of 1-neighbours y_1, y_2 of 0, we have $f(y_2 - y_1) = f(y_2) - f(y_1) + B(y_1, y_2)$. So the calculation continues

$$= h(0, f(y_1) - f(y_2) + B(y_1, y_2))$$

$$= (f(y_1) - f(y_2) + B(y_1, y_2))^T \cdot H(0) \cdot (f(y_1) - f(y_2) + B(y_1, y_2)).$$

Since f depends in a linear way of $y_1 \sim_1 0$ and $y_2 \sim_1 0$, this whole expression multiplies out by linearity, and for degree reasons all terms involving B, as well as some others, vanish, and we are left with $-2f(y_1)^T \cdot H(0) \cdot f(y_2)$. On the other hand, $h(f(y_1), h(f(y_2)) = h(0, f(y_2) - f(y_1))$, by Proposition 1.1, and writing this in terms of $H(0)$ gives the same expression.

The converse is proved in the same way for $z \sim_2 0$ of the form $y_2 - y_1$ with $y_1 \sim_1 0$ and $y_2 \sim_1 0$, but this suffices to get the result for all $z \sim_2 0$, by general principles of SDG ("R, and hence any manifold, perceives the addition map $D(n) \times D(n) \to D_2(n)$ to be epic".) ∎

Call a linear map $F : U \to V$ between inner product vector spaces *conformal* with constant $k > 0$ if for all $u_1, u_2 \in U$

$$\langle F(u_1), F(u_2)\rangle = k\langle qu_1, u_2\rangle.$$

It follows immediately from Proposition 3.1, and from the construction of inner product in the vector space of tangents at x, and at $f(x)$, that if f is conformal at x with constant k, then $df_x : T_xM \to T_{f(x)}N$ is a conformal linear map with the same constant k. The converse also holds; for if df_x is conformal with constant k, we deduce that for all pairs of tangents t and s at x

$$g(f(t(d_1)), f(s(d_2))) = k \cdot g(t(d_1), s(d_2)),$$

and hence

$$g(f(y_1)), f(y_2)) = k \cdot g(y_1, y_2) \tag{9}$$

for all y_i's of the form $t(d)$ for a tangent vector t and a $d \in D$. Again by general principles, any manifold N "perceives all 1-neighbours of x to be of this form". From Proposition 3.1 we therefore deduce that f is conformal at x with constant k.

3.2. THEOREM. *A diffeomorphism f is conformal at $x \in M$ if and only if f maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(f(x))$.*

PROOF. Assume f maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(f(x))$. We may pick normal coordinates at x as well as at $f(x)$. The neighbourhoods $\mathcal{M}_L(x)$ and $\mathcal{M}_L(f(x))$ then both get identified with $D_L(n)$, and $x = 0$, $f(x) = 0$. The restriction of f to $D_2(n)$, $f : D_2(n) \to R^n$, takes 0 to 0 and is therefore of the form $f(y) = A \cdot y + B(y)$, where A is an $n \times n$ matrix, and $B(y)$ is a map $R^n \to R^n$ which is homogeneous of degree 2 in $y \in R^n$, i.e. an n-tuple of quadratic forms B_i.

Assume now that f maps $D_L(n)$ into itself. For $z \in D_L(n)$, the i'th coordinate of $f(z)$ is

$$f_i(z) = \sum_k a_{ik}z_k + B_i(z).$$

Squaring this, only the terms in $(\sum_k a_{ik}z_k)(\sum_l a_{il}z_l)$ survive for degree reasons (using that $z \in D_2(n)$). But using further that $z_k z_l = 0$ for $k \neq l$, only the "diagonal" terms survive, and we get

$$f_i(z)^2 = \sum_k a_{ik}^2 z_k^2 = z_1^2 \sum_k a_{ik}^2. \tag{10}$$

Similarly for $i \neq j$

$$f_i(z)f_j(z) = z_1^2(\sum_k a_{ik}a_{jk}). \tag{11}$$

If now $f(z) \in D_L(n)$ for all $z \in D_L(n)$, we get that the expression in (10) is independent of i, and from the uniqueness assertion in Proposition 2.3 we therefore conclude

$$\sum_k a_{ik}^2 = \sum_k a_{jk}^2 \text{ for all } i, j;$$

and similarly we conclude from (11) that

$$\sum_k a_{ik} a_{jk} = 0 \text{ for } i \neq j.$$

These two equations express that all the rows of the matrix A have the same square norm k, and that they are mutually orthogonal. This implies that the linear map df_x represented by the matrix is conformal, and hence f is conformal at x.

The proof that conformality of f at x implies that f maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(f(x))$ goes essentially through the same calculation, and is omitted. ∎

4. A famous pseudogroup in dimension 2

The content of the present section is partly classical, namely the equivalence of the various ways of describing the notion of holomorphic map from (a region in) the complex plane $C = R^2$ to itself. Synthetic concepts enter essentially in two of the conditions in the Theorem below, namely 1) and 6).

An *almost complex* structure on a general manifold M consists in giving, for each $x \in M$, a map $I_x : \mathcal{M}_1(x) \to \mathcal{M}_1(x)$ with $I_x(x) = x$ and $I_x(I_x(z)) = z'$ for any $z \sim_1 x$; Here, z' denotes the mirror image of z in x, i.e. the affine combination $2x - z$; recall [7] that affine combinations of 1-neighbours make "absolutely" sense, i.e. do not depend on, say, a Riemannian structure. It is clear what it means for a map f to *preserve* such structure at the point x: $f(I_x(z)) = I_{f(x)}(f(z))$.

The manifold R^2 carries a canonical almost-complex structure, given by

$$I_{(x_1, x_2)}(z_1, z_2) = (x_1 - (z_2 - x_2), x_2 + (z_1 - x_1)).$$

Identifying R^2 with the complex plane C, this is just

$$I_x(z) = x + i(z - x).$$

Utilizing the multiplication of the complex plane C, we may consider the set D_C of elements of square zero in C (recalling the fundamental role which the set D of elements of square zero in R plays in SDG). We have, by trivial calculation,

4.1. PROPOSITION. *Under the identification of C with R^2,*

$$D_C = D_L(2).$$

Having D_C, we may mimic the basics of SDG and declare a function $f : C \to C$ to be *complex differentiable* at $x \in C$ if there is a number $f'(x) \in C$ so that

$$f(z) = f(x) + f'(x) \cdot (z - x) \text{ for all } z \text{ with } z - x \in D_C.$$

(The uniqueness of such $f'(x)$, justifying the notation, follows from the general axiom scheme of SDG, applied to $D_1(2)$, the 1-jet classifier in R^2. Note $D_1(2) \subseteq D_L(2)$.) — The notion of course makes sense for functions f which are just defined *locally* around x.

4.2. THEOREM. *Let $f : R^2 \to R^2$ be a (local) orientation preserving diffeomorphism. Let $x \in R^2 = C$. Then the following conditions are equivalent:*

1) f maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(f(x))$
2) f is conformal at x
3) f satisfies Cauchy-Riemann equations at x
4) f preserves almost complex structure at x.
Also the following conditions are equivalent, and they imply 1)-4):
5) f is complex-differentiable at x
6) f maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(f(x))$, and f preserves affine combinations of x and z for any $z \sim_L x$.
Finally, if 1)-4) hold for all x, 5) and 6) hold for all x.

(Note that 6) says that f is *harmonic* at x, in the sense of Remark at the end of Section 2.)

PROOF. The equivalence of 1) and 2) is already in Theorem 3.2, and this in turn is, as we have seen, equivalent to conformality of the linear df_x. But conformal orientation preserving 2×2 matrices are of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{12}$$

Since the entries of the matrix for df_x are $\partial f_i / \partial x_j$, this form (12) of the matrix therefore expresses that the Cauchy-Riemann equations hold at x, i.e. is equivalent to 3). On the other hand, a simple calculation with 2×2 matrices gives that a matrix commutes with the matrix $I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for the almost complex structure iff it has the above "Cauchy-Riemann" form (12).

Now assume 5). If f is complex differentiable at x, we prove that condition 1) holds at x as follows. Let $z \sim_L x$. Then $(z - x)^2 = 0$ Proposition 4.1, and by complex differentiability

$$f(z) - f(x) = f'(x)(z - x), \tag{13}$$

so since the right hand side has square zero, then so does the left hand side, but again by Proposition 4.1, this means that $f(z) \sim_L f(x)$, proving 1), and hence also the first part of 6). But also, if f is complex-differentiable at x, f preserves affine combinations of the form $tx + (1 - t)z$ for $z \in \mathcal{M}_L(x)$; this follows from (13), since $z - x \in D_C(x)$ by Proposition 4.1, so also the second part of 6) is proved. Conversely, if 6) holds, f is conformal at x by Theorem 3.2, so df_x is if the form (12). Then $f'(x) = a + ib$ will serve as the complex derivative; for, since f preserves affine combinations of x and z for $z \sim_L x$, we have the first equality sign in

$$f(z) = f(x) + df_x(z - x) = f(x) + f'(x) \cdot (z - x)$$

for such z, i.e. for $z - x \in D_L(2) = D_C$.

Finally, assume 1)-4) hold for all x. Then we may differentiate the Cauchy-Riemann equations for $f = (f_1, f_2)$ by $\partial/\partial x_1$ and $\partial/\partial x_2$ and compare, arriving in the standard way to $\Delta f_1 \equiv 0$ and $\Delta f_2 \equiv 0$. From the "Taylor expansion" in Proposition 2.6, applied to f_1, we conclude that $f_1(z) = f_1(x) + (df_1)_x(z - x)$ for $z \sim_L x$, and similarly for f_2, so $f(z) = f(x) + df_x(z - x)$ for such z, i.e. for $z - x \in D_C$. Since df_x is given by a conformal matrix (12), by 2), this proves that $a + ib$ will serve as the complex derivative of f at x. ∎

5. Support of the Laplacian

We arrived at $D_L(n)$ from the geometric side, namely as the $\sim_L$-neighbours of 0 in the Riemannian manifold $M = R^n$; the differential operator Δ was then seen to provide the top term in the Taylor expansion of functions defined on $D_L(n)$.

Here, we briefly indicate how to arrive at $D_L(n)$ from the algebraic side, starting with $\Delta = \sum \partial^2/\partial x_i^2$. More precisely, we consider Δ as a *distribution* at $0 \in R^n$. So Δ is the linear map

$$k[X_1, ..., X_n] \to R$$

given by

$$f \mapsto \sum_i \frac{\partial^2 f}{\partial x_i^2}(0). \tag{14}$$

The algebraic concept that will give $D_L(n)$ out of this data is the notion of *coalgebra*, and *subcoalgebra*, as in [9]. If we let A denote the algebra $k[X_1, ..., X_n]$, then the distribution Δ of (14) factors

$$A \to B \to k,$$

where $A \to B$ is an *algebra* map, and B is finite dimensional (take e.g. $B = A/J$ where J is the ideal generated by monomials of degree ≥ 3). The set A^o of linear maps $A \to k$ having such a factorization property constitute a coalgebra, [9] Proposition 6.0.2. Every element in a coalgebra generates a finite dimensional subcoalgebra, by [9] Theorem 2.2.1. In particular, $\Delta \in A^o$ generates a finite dimensional coalgebra $[\Delta]$ of A^o, and this coalgebra "*is*" $D_L(n)$. More specifically, the dual algebra of $[\Delta]$ is the coordinate ring $\mathcal{O}(D_L(n))$ of $D_L(n)$, i.e. the Weil algebra $\mathcal{O}(D_L(n)) := k[Z_1, ..., Z_n]/I$ considered in the proof of Proposition 2.3, as we shall now argue.

The following "Leibniz rule" for Δ is well known,

$$\Delta(f \cdot g) = \Delta f \cdot g + 2 \sum_i \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} + f \cdot \Delta g.$$

This means that in the coalgebra A^o, we have the following formula for $\psi(\Delta)$ (ψ = the comultiplication of the coalgebra; δ the Dirac distribution "evaluate at 0"):

$$\psi(\Delta) = \Delta \otimes \delta + 2 \sum_i \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_i} + \delta \otimes \Delta, \tag{15}$$

where now Δ, $\partial d/\partial x_i$, δ are viewed as distributions at 0, like in (14), meaning that one evaluates in 0 after application,

$$f \mapsto (\Delta f)(0), \; f \mapsto \frac{\partial f}{\partial x_i}(0), \; f \mapsto f(0).$$

From (15), (and from $\psi(\partial/\partial x_i) = \partial/\partial x_i \otimes \delta + \delta \otimes \partial/\partial x_i$, which expresses the Leibniz rule for $\partial/\partial x_i$) we see that the subcoalgebra $[\Delta]$ generated by Δ is generated as a vector space by the elements

$$\delta, \; \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \; \Delta,$$

and since these are clearly linearly independent, we see that $[\Delta] \subseteq A^o$ is $(n+2)$-dimensional. The dual algebra of $[\Delta]$ is a quotient algebra A/I of A, where I is the ideal of those $f \in A$ which are annihilated by the elements of $[\Delta]$. This ideal I contains $x_i^2 - x_j^2$, and $x_i x_j$ for $i \neq j$. Since the quotient of A by the ideal generated by $x_i^2 - x_j^2$, and $x_i x_j$ for $i \neq j$ is already $(n+2)$-dimensional, as calculated in the proof of Proposition 2.3, it follows that the quotient algebra there is actually the dual of $[\Delta]$.

The idea that a coalgebra like $[\Delta]$ *is* itself an infinitesimal geometric object goes back to Gavin Wraith in the early seventies, [10]. The specific way of generating Weil algebras from differential operators was considered by Emsalem [2] (without coalgebras).

References

[1] S.S. Chern, W.H. Chen and K.S. Lam, Lectures on differential geometry, World Scientific 1999.

[2] J. Emsalem, Géométrie des points épais, Bull. Soc. math. France 106 (1978), 399-416.

[3] A. Kock, Formal manifolds and synthetic theory of jet bundles, Cahiers de Topologie et Géométrie Différentielle 21 (1980), 227-246.

[4] A. Kock, Synthetic Differential Geometry, London Math. Soc. Lecture Notes Series No. 51, Cambridge Univ. Press 1981.

[5] A. Kock, Geometric construction of the Levi-Civita Parallelism, Theory and Appl. of Categories 4 (1998), No. 9.

[6] A. Kock, Volume form as volume of infinitesimal simplices, arXiv:math.CT/0006008, June 2000.

[7] A. Kock, Differential forms as infinitesimal cochains, Journ. Pure Appl. Alg. 154 (2000), 257-264.

[8] R. Lavendhomme, Basic concepts of Synthetic Differential Geometry, Kluwer Texts in the Math. Sciences 13, Kluwer 1996.

[9] M.E. Sweedler, Hopf Algebras, Benjamin 1969.

[10] G.C. Wraith, Talk at Oberwolfach, July 27, 1972.

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