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Multilinear Integral Operators in Weighted Function Spaces

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Abstract. Our aim is to discuss the boundedness of multilinear integral operators (multilinear fractional integrals, multisublinear maximal operators etc) in weighted Lebesgue spaces. In particular, we present criteria governing weighted inequalities for these operators. We are also focused on general multisublinear operators generated by quasi-concave functions between weighted Banach function lattices. These operators, in particular, generalize the Hardy–Littlewood and fractional maximal functions playing an important role in Harmonic Analysis. We show that under some general geometrical assumptions on Banach function lattices two-weight weak type and also strong type estimates for these operators are true. To derive the main results the strong type estimate for a variant of multilinear averaging operators is characterized. As special cases the boundedness results for fractional maximal operators in concrete function spaces are provided.

The talk is based on the research carried our jointly with V. Kokilashvili and M. Mastyło.

Key words and phrases: Multilinear fractional integrals, multisublinear maximal operators, Banach function lattices, weighted inequalities.

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1 Introduction

Recently, much attention has been paid to the study of the boundedness of various types of operators between weighted L^p-spaces playing an important role in analysis, in particular, in harmonic analysis and its applications in partial differential equations (PDE). For this purpose the Hardy-Littlewood maximal function defined for any $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$\mathscr{M} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes, has proved to be a tool of great importance. One of the important related operators is the so-called fractional maximal function $\mathscr{M}_\alpha$ $(0 < \alpha < n)$ defined by

$$\mathscr{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n$$

for any $f \in L^1_{loc}(\mathbb{R}^n)$.

It is well-known that $\mathscr{M}_\alpha$ is deeply connected to the Riesz potential operator I_α $(0 < \alpha < n)$, given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n,$$

which play an important role in the theory of Sobolev's embeddings.

Multisublinear maximal operators appeared naturally in connection with multilinear Calderón-Zygmund theory.

$$\mathscr{M}(\overrightarrow{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int\limits_Q |f_i(y_i)| dy_i,$$

A multisublinear maximal operator that acts on the product of m-Lebesgue spaces and is smaller than the m-fold product of the Hardy–Littlewood maximal function was studied by A. K. Lerner, S. Ombrosi, C. Perez, R. H. Torres and R. Trujillo-Gonzalez [6]. It was used to obtain a precise control on multilinear singular integral operators of Calderón-Zygmund type and to build a theory of weights adapted to the multilinear setting.

For the boundedness and other properties of multisublinear fractional maximal operators:

$$\mathscr{M}_\alpha(\overrightarrow{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\alpha/(nm)}} \int\limits_Q |f_i(y_i)| dy_i, \quad 0 \le \alpha < mn.$$

in (weighted) Lebesgue spaces we refer to the papers by K. Moen (2009), G. Pradolini (2010), X. Chen and Q. Xue (2010), V. Kokilashvili, M. Mastylo and A. Meskhi (2014-2015).

2 Preliminaries

Let us recall some definitions and well-known facts regarding the boundedness results of multilinear integral operators in (weighted) Lebesgue spaces.

2.1 Lebesgue space

Let w be a weight, i.e., w is an a.e. positive locally integrable function on $\mathbb{R}^n$ and let $1 \le p < \infty$. We denote by $L^p_w(\mathbb{R}^n)$ the weighted Lebesgue space which is the class of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} \big(|f(x)|w(x)\big)^p dx \right)^{1/p} < \infty.$$

If $w \equiv \mathit{const}$, then we denote $L^p_w(\mathbb{R}^n)$ by $L^p(\mathbb{R}^n)$.

For a weight w on $\mathbb{R}^n$, we denote

$$w(E) := \int\limits_E w(x)dx,$$

where E is a measurable set in $\mathbb{R}^n$.

2.2 Multilinear fractional integrals

Historically, multilinear fractional integrals were introduced in the papers by L. Grafakos (1992), C. Kenig and E. Stein (1999), L. Grafakos and N. Kalton (2001). In particular, they deal with the operator

$$B_\alpha(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt, \quad 0 < \alpha < n,$$

where f and g are defined on $\mathbb{R}^n$.

In the mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p_1, p_2, q < \infty$, then B_α is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

The latter boundedness follows from the pointwise estimate

$$B_\alpha(f,g)(x) \le I_\alpha(f^r)^{1/r} I_\alpha(g^s)^{1/s},$$

where $r = p_1/p$, $s = p_2/p$, $f, g \ge 0$ and I_α is the Riesz potential operator. In this case $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. This inequality follows from the Hölder inequality. Consequently, applying again Hölder's inequality we have

$$\|B_\alpha(f,g)\|_{L^q(\mathbb{R}^n)} \le \|[I_\alpha(f^r)]^{1/r}[I_\alpha(g^s)]^{1/s}\|_{L^q(\mathbb{R}^n)}$$

$$\le \left(\int\limits_{\mathbb{R}^n} I_\alpha(f^r)^q dx \right)^{\frac{1}{qr}} \left(\int\limits_{\mathbb{R}^n} I_\alpha(g^s)^q dx \right)^{\frac{1}{qs}}$$

$$\leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

In the latter inequality we used Sobolev inequalities ($L^p \to L^q$ boundedness).

As a tool to understand B_α, the operators

$$\mathscr{I}_\alpha(\overrightarrow{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} d\overrightarrow{y},$$

where $x \in \mathbb{R}^n$, $0 < \alpha < nm$, $\overrightarrow{f} := (f_1, \cdots, f_m)$, $\overrightarrow{y} := (y_1, \cdots, y_m)$, were studied as well. The corresponding maximal operator, as we mentioned above, is given by

$$\mathscr{M}_\alpha(\overrightarrow{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\alpha/(nm)}} \int\limits_Q |f_i(y_i)| dy_i, \quad 0 \leq \alpha < mn.$$

It can be immediately checked that

$$\mathscr{I}_\alpha(\overrightarrow{f})(x) \geq c_n \mathscr{M}_\alpha(\overrightarrow{f})(x), \quad x \in \mathbb{R}^n, \quad f_i \geq 0, \quad i = 1, \cdots, m,$$

for the positive constant c_n depending only on n.

In the sequel the following notation will be used:

$$\overrightarrow{p} := (p_1, \cdots, p_m), \quad \overrightarrow{w} = (w_1, \cdots, w_m), \quad \overrightarrow{f} = (f_1, \cdots, f_m),$$

where p_i are constants ($0 < p_i < \infty$), f_i are functions and w_i are weights on Euclidean space.

It will be also assumed that

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}, \quad 1 < p_i < \infty, \ i = 1, \cdots, m.$$

2.3 Vector Muckenhoupt class. The one-weight problem

Definition (Muckenhoupt type condition). Let $1 < p_i < \infty$ for $i = 1, \cdots, m$. Let w_i be weights on $\mathbb{R}^n$, $i = 1, \cdots, m$. We say that $\overrightarrow{w} \in A_{\overrightarrow{p}}(\mathbb{R}^n)$ (or simply $\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$\sup_Q \left(\frac{1}{|Q|}\int\limits_Q \prod_{i=1}^{m} w_i^{p/p_i}\right)^{1/p} \prod_{i=1}^{m} \left(\frac{1}{|Q|}\int\limits_Q w_i^{1-p_i'}\right)^{1/p_i'} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. When $p_i = 1$, $\left(\frac{1}{|Q|}\int\limits_Q w_i^{1-p_i'}\right)^{1/p_i'}$ is understood as $\left(\inf_Q w_i\right)^{-1}$.

In the linear case ($m = 1$) the class $A_{\vec{p}}$ coincides with the well- known Muckenhoupt class A_p.

It is well-known (see [6]) that if $1 < p_i < \infty$ with $p > 1$, then the one–weight inequality

$$\left(\int_{\mathbb{R}^n} \left(\left|\mathscr{M}(\vec{f})(x)\right| \prod_{i=1}^m w_i(x)\right)^p dx\right)^{1/p} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \left(|f_i(y_i)| w_i\right)^{p_i} dy_i\right)^{1/p_i},$$

holds if and only if $\vec{w} \in A_{\vec{p}}(\mathbb{R}^n)$.

Definition (Muckenhoupt-Wheeden type condition). Let $1 \leq p_i < \infty$ for $i = 1, \cdots, m$. Suppose that $p < q < \infty$. We say that $\vec{w} = (w_1, \cdots, w_m)$ satisfies $A_{\vec{p},q}(\mathbb{R}^d)$ condition ($\vec{w} \in A_{\vec{p},q}$) if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \left(\prod_{i=1}^m w_i\right)^q\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q w_i^{-p_i'}\right)^{1/p_i'} < \infty.$$

If $m = 1$, then this condition coincides with the classical Muckenhoupt-Wheeden condition.

Theorem ([7]). *Let $1 < p_1, \cdots, p_m < \infty$, $0 < \alpha < mn$, $\frac{1}{m} < p < \frac{n}{\alpha}$. Suppose that q is an exponent satisfying the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that w_i are a.e. positive functions on $\mathbb{R}^n$ such that $w_i^{p_i}$ are weights. Then the inequality*

$$\left(\int_{\mathbb{R}^n} \left(\left|\mathscr{N}_\alpha(\vec{f})(x)\right| \prod_{i=1}^m w_i(x)\right)^q dx\right)^{1/q} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \left(|f_i(y_i)| w_i\right)^{p_i} dy_i\right)^{1/p_i},$$

holds, where $\mathscr{N}_\alpha$ is either $\mathscr{I}_\alpha$ or $\mathscr{M}_\alpha$, if and if $\vec{w} \in A_{\vec{p},q}(\mathbb{R}^n)$.

This is a generalization of the Muckenhoupt-Wheeden classical theorem to multilinear case.

3 The two–weight problem for Riesz potentials

Let us recall some well-known results regarding the boundedness of I_α in (weighted) Lebesgue spaces.

The classical Hardy-Littlewood-Sobolev inequality says that if $1 < p < \infty$, $0 < \alpha < n/p$ and $q := \frac{np}{n-\alpha p}$, then there is a positice constant C such that for all $f \in L^p(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

In 1958 E. Stein and G. Weiss established the two–weight inequality for power weights $\left(|x|^\beta, |x|^\gamma\right)$.

In 1972 D. Adams characterized the trace inequality ($L^p \to L^q_v$ boundedness) for the case $p < q$);

In 1984-1989 E. Sawyer established two-weight criteria under the conditions involving the operator itself.

In 1995 V. Maz'ya and I. Verbitsky characterized the trace inequality in the diagonal $(p = q)$ case under the pointwise condition involving the operator itself.

In 1988 M. Gabidzashvili and V. Kokilashvili gave a complete characterization for the L^p_w to L^q_v boundedness under integral–type conditions in the non-diagonal $(p < q)$ case.

Our result regarding the trace inequality characterization for the multilinear fractional integral $\mathscr{I}_\alpha$ and the appropriate fractional maximal operator $\mathscr{M}_\alpha$ reads as follows:

Theorem ([4]). $1 < p_i < \infty$, $i = 1, \cdots, m$. *Assume that* $0 < \alpha < n/p$ *and* $p < q < \infty$. *Let* $\mathscr{N}_\alpha$ *be either* $\mathscr{I}_\alpha$ *or* $\mathscr{M}_\alpha$. *Then the following conditions are equivalent:*

(i) $\|v \mathscr{N}_\alpha(\overrightarrow{f})\|_{L^q(\mathbb{R}^{nk})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}$;

(ii) $v\Big(\{x \in \mathbb{R}^n : |\mathscr{N}_\alpha(\overrightarrow{f})(x)| > \lambda\}\Big)^{1/q} \le \frac{C}{\lambda} \prod_{i=1}^m \Bigg(\int\limits_{\mathbb{R}^{nk}} \Big| f_i(x) \Big|^{p_i} dx \Bigg)^{1/p_i}$;

(iii) $\sup_Q \Bigg(\int\limits_Q v^q(x)(x) dx \Bigg)^{1/q} |Q|^{\alpha - n/p} < \infty$, *where the supremum is taken over all cubes* $Q \subset \mathbb{R}^n$ *with sides parallel to the coordinate axes.*

4 Multilinear maximal operators in weighted Banach lattices

4.1 Banach lattices

A Banach (function) lattice $(X, \|\cdot\|_X)$ on (Ω, Σ, μ) is an ideal of $L^0(\mu)$, which is complete with respect to the norm $\|\cdot\|_X$. We also assume that the support of the space X is Ω ($\mathrm{supp}(X) = \Omega$), that is, there is an element $u \in X$ with $u > 0$ μ-a.e. on Ω.

Let X be a Banach lattice. X is called *minimal* if the closed linear span $\{\chi_A; \mu(A) < \infty\}$ is dense in X, where χ_A is the characteristic function of a set A. It is said that X has the *Fatou property* (or X is maximal) if for any $f \in L^0$, $f_n \in X_+$ such that $f_n \uparrow f$ a.e. and $\sup \|f_n\|_X < \infty$, we have that $f \in X$ and $\|f_n\|_X \to \|f\|_X$. We say that X has the *weak Fatou property* whenever if $f_n, f \in X_+$, $f_n \uparrow f$ a.e., then $\|f_n\|_X \to \|f\|_X$.

The *Köthe dual* space X' of a Banach lattice X on (Ω, Σ, μ) is the space of all $f \in L^0(\mu)$ such that $\int_\Omega |fg| \, d\mu < \infty$ for every $g \in X$. It is a Banach lattice on (Ω, Σ, μ) when equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_\Omega |fg| \, d\mu, \quad f \in X'.$$

Let us remark that the Köthe dual X' of X is a maximal Banach lattice on (Ω, μ), as for a number of classical spaces such as Lebesgue spaces L_p, $1 \le p \le \infty$, Orlicz spaces or more general Musielak-Orlicz spaces. It is well known that a Banach lattice X is maximal if and only if $X = X'' := (X')'$ with equality of norms (see, e.g., [3]).

In what follows we will use the following well-known fact that the Köthe dual X' identified in a natural way with a subspace of the Banach dual X^* is a norming subspace, i.e.,

$$\|f\|_X = \sup_{\|g\|_{X'} \le 1} \left| \int_\Omega fg \, d\mu \right|, \quad f \in X,$$

if and only if X has the weak Fatou property (see [3]).

If X is a Banach lattice on (Ω, Σ, μ) and $w \in L^0(\mu)$ is strictly positive a.e., then we define $X(w)$ to be the Banach lattice of all $f \in L^0(\mu)$ such that $fw \in X$, equipped with the norm $\|f\|_{X(w)} = \|fw\|_X$. In what follows we will use the following easily verified formula, which holds with equality of norms

$$X(w)' = X'(w^{-1}).$$

If $T\colon X \to Y$ is a bounded operator between Banach spaces, then we say that T is of *strong type* (or has strong type). Let X be a Banach space and let Y be a Banach lattice on (Ω, μ). Then a map $T\colon X \to L^0(\mu)$ is said to be of *weak type* (X, Y) (or has weak type (X, Y)) if there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$\left\| \chi_{\{\omega \in \Omega;\, |Tx(\omega)| > \lambda\}} \right\|_Y \le C \lambda^{-1} \|x\|_X, \quad x \in X.$$

In what follows if X is a Banach space and Y is a Banach lattice on $(\mathbb{R}^n, \mu)$ and S is a map from a subspace E of X to Y. We put $\|S\|_{X \to Y} := \sup\{\|Sx\|_Y;\, x \in X \cap E,\, \|x\|_X \le 1\}$. If $\|S\|_{X \to Y} < \infty$ and there is no misunderstanding, we say for short that S is a bounded operator from X to Y. Note that in the paper we consider the case $E = \prod_{k=1}^m L^1_{loc}$ and $X = \prod_{k=1}^m X_k$ equipped with the norm $\|(x_1, \ldots, x_m)\|_X := \max_{1 \le k \le m} \|x_k\|_{X_k}$, where $X_1, \ldots, X_m$ are Banach latices on $(\mathbb{R}^n, \mu)$ and $S\colon E \to L^0(\mathbb{R}^n, \mu)$ is a multi(sub)linear operator.

4.2 Maximal and avaraging operators

Let $\mathscr{B}$ denote the family of all cubes in $\mathbb{R}^n$ with edges parallel to the coordinate axes.

We denote by $\mathscr{P}$ the set of all increasing functions $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$.

For an m tuple $\vec{\varphi} := (\varphi_1, \dots, \varphi_m) \in \mathscr{P}^m$ and subfamily $\bar{Q} = \{Q_i\}$ in $\mathscr{B}$, we define the multilinear averaging operator $T_{\bar{Q}}$ and the maximal operator $\mathscr{M}_{\vec{\varphi}}$ by

$$T_{\bar{Q}}\vec{f} = \sum_i \left(\prod_{k=1}^{m} \frac{1}{\varphi_k(|Q_i|)} \int_{Q_i} f_k \, d\mu \right) \chi_{Q_i},$$

and

$$\mathscr{M}_{\vec{\varphi}}\vec{f}(x) = \sup_{Q \ni x} \prod_{k=1}^{m} \frac{1}{\varphi_k(|Q|)} \int_Q f_k \, d\mu, \quad x \in \mathbb{R}^n$$

respectively, where $\vec{f} = (f_1, \dots, f_m) \in \prod_{k=1}^m L^1_{loc}$. Note that if $\varphi_j(t) = t$ for every $t \geq 0$ and each $1 \leq j \leq m$, we obtain the multisublinear Hardy-Littlewood maximal operator $\mathscr{M}$.

4.3 Mutlilinear G-property

A pair (X, Y) of Banach lattices on $(\mathbb{R}^n, \mu)$ is said to have the property $G(\mathscr{B})$ $\big((X, Y) \in G(\mathscr{B})$ for short $\big)$ if there is a constant $C_1 = C_1(\mathscr{B}, X, Y)$ such that

$$\sum_i \|x\chi_{Q_i}\|_X \|y\chi_{Q_i}\|_{Y'} \leq C_1 \|x\|_X \|y\|_{Y'}, \quad (x, y) \in X \times Y$$

for any family $\{Q_i;\, Q_i \in \mathscr{B}\}$ of disjoint cubes. If the above inequality holds for any family $\{Q_i\}$ of pairwise disjoint Lebesgue measurable sets, then we write $(X, Y) \in G$.

We need to define also a mutlilinear variant of $G(\mathscr{B})$-property. Let $X_1, \dots,$ X_m, Y be Banach lattices on $(\mathbb{R}^n, \mu)$. We write $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$ if there exists a constant $C_0 = C_0(\mathscr{B}, X_1, \dots, X_m, Y)$ such that for any family $\{Q_i;\, Q_i \in \mathscr{B}\}$ of disjoint cubes,

$$\sum_i \|x_1\chi_{Q_i}\|_{X_1} \cdots \|x_m \chi Q_i\|_{X_m} \|y\chi_{Q_i}\|_{Y'} \leq C_0 \|x_1\|_{X_1} \cdots \|x_m\|_{X_m} \|y\|_{Y'} \tag{4.3.1}$$

holds for all $x_j \in X_j$ $(1 \leq j \leq m)$ and $y \in Y'$.

If estimate (4.3.1) holds for any family $\{Q_i\}$ of pairwise disjoint Lebesgue measurable sets, then we write $(X_1, \dots, X_m, Y) \in G^{(m)}$. For example, if $X_1 = L^{p_1}, \dots, X_m = L^{p_m}$ and $Y = L^r$ with $1 \leq p_1, \dots, p_m, r < \infty$, then $(X_1, \dots, X_m, Y) \in G^{(m)}$ provided that $1/p_1 + ... + 1/p_m + 1/r' \geq 1$, where $1/r + 1/r' = 1$.

It is easy to see that if $X_1, \dots, X_m$ and Y are Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_{k_1}, \dots, X_{k_n}, Y) \in G^{(n)}$ with $1 \leq k_j < m$ for $1 \leq j \leq n$, then $(X_1, \dots, X_m, Y) \in G^{(m)}$.

4.4 Morrey spaces

In what follows we will work with a variant of Morrey spaces. For a given $\varphi \in \mathscr{P}$ we denote by M_φ the space of all $f \in L^0(\mathbb{R}^n, \mu)$ such that

$$\sup_{Q \in \mathscr{B}} \frac{1}{\varphi(|Q|)} \int_Q |f| \, d\mu < \infty.$$

It is easy to verify that M_φ is a Banach lattice on $(\mathbb{R}^n, \mu)$ with the Fatou property when equipped with the norm

$$\|f\|_{M_\varphi} = \sup_{Q \in \mathscr{B}} \frac{1}{\varphi(|Q|)} \int_Q |f| \, d\mu.$$

Now under some conditions we give a characterization of the boundedness of the multilinear averaging operator $T_{\bar{Q}}$ from the product of weighted Banach lattices to weighted Banach lattices.

4.5 The boundedness of the multilinear averaging operator $T_{\bar{Q}}$

Our result regarding the two-weight boundedness of the operator $T_{\bar{Q}}$ is the following statement:

Theorem ([5]). *Let $X_1(w_1), \dots, X_m(w_m)$, $Y(v)$ be weighted Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$. Suppose that Y has the weak Fatou property. Then the multilinear averaging operator $T_{\bar{Q}}$ generated by $\vec{\varphi} = (\varphi_1, \dots, \varphi_m) \in \mathscr{P}^m$ is uniformly bounded with respect to a subfamily $\bar{Q} = \{Q_i\}$ of $\mathscr{B}$ from $X_1(w_1) \times \cdots \times X_m(w_m)$ to $Y(v)$, i.e., the inequality*

$$\sup_{\bar{Q}} \|T_{\bar{Q}}\|_{X_1(w_1) \times \cdots \times X_m(w_m) \to Y(v)} < \infty$$

holds if and only if $(w_1, \dots, w_m, v) \in A_{\vec{\varphi}}(X_1, \dots, X_m, Y)$, i.e.,

$$C_1 := \sup_{Q \in \mathscr{B}} \|v\chi_Q\|_Y \prod_{k=1}^m \frac{1}{\varphi_k(|Q|)} \left\| w_k^{-1} \chi_Q \right\|_{X_k'} < \infty.$$

4.6 Some examples of Banach lattices

Now we show general examples of Banach lattices $X_1, \dots, X_m$, Y such that $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$. To do this we recall that a Banach lattice X on (Ω, μ) is said to be *p-convex* $(1 < p \le \infty)$, respectively, *q-concave* $(1 \le q < \infty)$, if there exists a constant $C > 0$ such that

$$\Big\| \Big(\sum_{k=1}^n |x_k|^p \Big)^{1/p} \Big\|_X \le C \Big(\sum_{k=1}^n \|x_k\|_X^p \Big)^{1/p},$$

respectively,

$$\Big(\sum_{k=1}^{n}\|x_k\|_X^q\Big)^{1/q} \le C\Big\|\Big(\sum_{k=1}^{n}|x_k|^q\Big)^{1/q}\Big\|_X,$$

for any choice of elements $x_1, \dots, x_n$ in X and $n \in \mathbb{N}$. If in the above definitions elements $x_1, \dots, x_n$ are pairwise disjoint, then X is said to be satisfy an upper p-estimate and lower q-estimate, respectively. Clearly, p-convexity implies upper p-estimate, and q-concavity implies lower q-estimate of a Banach lattice X.

It is easy to check that if X satisfies a lower p-estimate, then the Köthe dual X' satisfies an upper p'-estimate. This immediately gives the following observation: if $X_1, \dots, X_m$, Y are Banach lattices on $(\mathbb{R}^n, \mu)$ such that X_k satisfies a lower p_k for each $1 \le k \le m$ and Y satisfies an upper q-estimate with $1/p_1 + \dots + 1/p_m + 1/q' \ge 1$, then $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$.

Applying the well–known results on p-convex and q-concave Orlicz spaces based on the above remark we obtain concrete general examples of Banach lattices for which we have $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$.

4.7 Weak type inequality for $\mathscr{M}_{\vec{\varphi}}$

Below we state and prove a theorem which gives a characterization of the generalized weak type inequality for the maximal multisublinear operator $\mathscr{M}_{\vec{\varphi}}$ from the product of weighted Banach lattices to the weighted Banach lattice satisfying the $G^{(m)}(\mathscr{B})$ property. In what follows if $E_1, \dots, E_m$ are Banach spaces and F is a Banach lattice on (Ω, ν). Then a mapping $T\colon E_1 \times \dots \times E_m \to L^0(\mu)$ is said to be of weak type $(E_1, \dots, E_m, F)$ if there is a positive constant c such that

$$\sup_{\lambda>0} \lambda \,\|\chi_{\{\omega\in\Omega;\, |T(x_1,\dots,x_n)(\omega)|>\lambda\}}\|_F \le \|x_1\|_{E_1} \cdots \|x_m\|_{E_m}$$

for all $(x_1, \dots, x_m) \in E_1 \times \dots \times E_m$.

Our result regarding the weak type inequality reeds as follows:

Theorem ([5]). *Let $X_1(w_1), \dots, X_m(w_m)$, $Y(v)$ be weighted Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathscr{B})$. Then the multisublinear operator $\mathscr{M}_{\vec{\varphi}}$ generated by $\vec{\varphi} = (\varphi_1, \dots, \varphi_m) \in \mathscr{P}^m$ is of weak type $(X_1(w_1), \dots, X_m(w_m), Y(v))$ if and only if $(w_1, \dots, w_m, v) \in A_{\vec{\varphi}}\ (X_1, \dots, X_m, Y)$, where the latter condition is defined in the previous theorem.*

In the linear case this statement was derived by E. I. Berezhnoi [1].

4.8 The two-weight boundedness of $\mathscr{M}_{\vec{\varphi}}$.

In the remaining part of the paper, we investigate the boundedness of a bisublinear maximal operator $\mathscr{M}_{\vec{\varphi}}$. We need some definitions. If $\varphi \in \mathscr{P}$ is such that exists $C \geq 1$ with

$$\varphi(s+t) \leq C\left(\varphi(s)+\varphi(t)\right), \quad s,t>0, \tag{4.8.1}$$

then we write $\varphi \in \widetilde{\mathscr{P}}$. Note that the condition (4.8.1) implies that $\varphi(t)/t \leq C\varphi(s)/s$ for all $0<s<t$.

Since φ is non-decreasing, the function $\widetilde{\varphi}$ given by $\widetilde{\varphi}(t) := \inf_{s>0}(1+t/s)\varphi(s)$ for $t>0$ and $\widetilde{\varphi}(0)=0$ is concave on $[0,\infty)$ and satisfies $C^{-1}\varphi(t) \leq \widetilde{\varphi}(t) \leq 2\varphi(t)$ for all $t \geq 0$ and so, in particular, $\widetilde{\varphi}$ is a quasi-concave function on $[0,\infty)$, i.e., $\widetilde{\varphi} \in \mathscr{P}$ and $t \mapsto t/\widetilde{\varphi}(t)$ is a non-decreasing function on $(0,\infty)$.

In what follows we will use the following simple observation: for any $\varphi \in \widetilde{\mathscr{P}}$, then there exist $\gamma, \alpha \in (0,1)$ such that for all $s,t>0$

$$\frac{\varphi(s)}{\varphi(t)} \leq \gamma \quad \textit{implies}\ \frac{s}{t} \leq \alpha. \tag{4.8.2}$$

Theorem ([**5**]). *Let $\vec{\varphi} = (\varphi_1, \varphi_2) \in \widetilde{\mathscr{P}} \times \widetilde{\mathscr{P}}$ and let X_1 and Y be minimal Banach lattices on $(\mathbb{R}^n, \mu)$, where Y has the Fatou property. Let $(X_1, Y) \in G$. Suppose that the Hardy-Littlewood maximal operator $\mathscr{M}$ is bounded in the weighted Banach lattice $X_1(w_1)$. Then the $\mathscr{M}_{\vec{\varphi}}$ is bounded from $X_1(w_1) \times M_{\varphi_2}$ to $Y(v)$ if and only if $(w_1, v) \in A_{\varphi_1}(X_1, Y)$.*

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