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On the Zeroes and Extrema of Generalised Clausen Functions

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Generalised Clausen functions arise in many areas of research in mathematics and physics. Despite their importance in such different areas as Hodge theory, number theory and quantum field theories, many properties still remain to be investigated. In particular, there seems to be no account on the location of their zeroes and extreme values available in the literature. However, it is well known that generalised Clausen functions may be expressed in terms of Bernoulli polynomials in some special cases. In the following, we will take advantage of this relationship in order to extend known results about Bernoulli polynomials to the generalised Clausen functions.

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1. Definitions and some properties

The Clausen function $Cl_2(\theta)$ given by

$$Cl_2(\theta) = -\int_0^\theta \ln\left|2\sin\left(\frac{x}{2}\right)\right| dx = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2}$$

was introduced in 1832 by Thomas Clausen [1] and is but one of a whole class of related functions.

Definition 1.1: Standard Clausen Functions
The standard Clausen functions may be defined as:

$$\begin{cases} Cl_{2n}(\theta) := \sum_{k=1}^{\infty} \frac{sin(k\theta)}{k^{2n}} \\ Cl_{2n-1}(\theta) := \sum_{k=1}^{\infty} \frac{cos(k\theta)}{k^{2n-1}} \\ Sl_{2n}(\theta) := \sum_{k=1}^{\infty} \frac{cos(k\theta)}{k^{2n}} \\ Sl_{2n-1}(\theta) := \sum_{k=1}^{\infty} \frac{sin(k\theta)}{k^{2n-1}} \end{cases},$$

for $n \in \mathbb{N}^* = \{1, 2, \ldots\}$.

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This definition can be generalised to non-integer and even complex values of the order.

Definition 1.2: Generalised Clausen Functions
The generalised Clausen functions may be defined through:

$$\begin{cases} S_\nu(\theta) := \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^\nu} \\ C_\nu(\theta) := \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^\nu} \end{cases},$$

for $\Re(\nu) > 1$ and extended to all of the complex plane through analytic continuation.

Some basic properties of the generalised Clausen functions may be derived directly from the definitions.

Property *1.3***:** Periodicity
As the sine and cosine functions are periodic with period 2π, the generalised Clausen functions exhibit the same periodicity:

$$\begin{cases} S_\nu(\theta) = S_\nu(\theta + 2n\pi) \\ C_\nu(\theta) = C_\nu(\theta + 2n\pi) \end{cases},$$

where $n \in \mathbb{Z}$.

Property *1.4***:** Symmetries
As the sine and cosine functions are odd, respectively even functions, we have for the generalised Clausen functions:

$$\begin{cases} S_\nu(\theta) = -S_\nu(-\theta) \\ C_\nu(\theta) = C_\nu(-\theta) \end{cases}.$$

Furthermore, we thus have

$$\begin{cases} S_\nu(\theta) = -S_\nu(2\pi - \theta) \\ C_\nu(\theta) = C_\nu(2\pi - \theta) \end{cases}.$$

Property *1.5***:** Derivatives with respect to the argument
As $\frac{d}{d\theta}\sin(k\theta) = k\cos(k\theta)$ and $\frac{d}{d\theta}\cos(k\theta) = -k\sin(k\theta)$, we obtain

$$\begin{cases} \frac{d}{d\theta} S_\nu(\theta) = C_{\nu-1}(\theta) \\ \frac{d}{d\theta} C_\nu(\theta) = -S_{\nu-1}(\theta) \end{cases}.$$

Some other properties are also easily obtained, i.e. the relationship between the generalised Clausen functions and Bernoulli polynomials, respectively polylogarithms.

Property *1.6***:** Relationship to the Bernoulli polynomials on the unit interval
The generalised Clausen functions of integer order are closely related to the Bernoulli

polynomials on the unit interval. Indeed, using the Fourier series representation of the Bernoulli polynomials, it is easy to see that for

$$\begin{cases} \nu \text{ odd: } S_\nu(\theta) = \frac{(-1)^{\frac{\nu}{2}+\frac{1}{2}}(2\pi)^\nu}{2\nu!} B_\nu\left(\frac{\theta}{2\pi}\right) \\ \nu \text{ even: } C_\nu(\theta) = \frac{(-1)^{\frac{\nu}{2}-1}(2\pi)^\nu}{2\nu!} B_\nu\left(\frac{\theta}{2\pi}\right) \end{cases},$$

where $0 \leq \frac{\theta}{2\pi} \leq 1$ respectively $0 \leq \theta \leq 2\pi$.

Property *1.7***:** Relationship to polylogarithms
The generalised Clausen functions may be expressed using polylogarithms:

$$\begin{cases} S_\nu(\theta) = \Im\left(\sum_{k=1}^\infty \frac{e^{i\theta}}{k^\nu}\right) = \Im\left(\mathrm{Li}_\nu\left(e^{i\theta}\right)\right) = \frac{1}{2}i\left(\mathrm{Li}_\nu(e^{-i\theta}) - \mathrm{Li}_\nu(e^{i\theta})\right) \\ C_\nu(\theta) = \Re\left(\sum_{k=1}^\infty \frac{e^{i\theta}}{k^\nu}\right) = \Re\left(\mathrm{Li}_\nu\left(e^{i\theta}\right)\right) = \frac{1}{2}\left(\mathrm{Li}_\nu(e^{-i\theta}) + \mathrm{Li}_\nu(e^{i\theta})\right) \end{cases}.$$

2. The zeroes of the Bernoulli polynomials

The quest for the location of the zeroes and extrema of the Bernoulli polynomials in the unit interval has been of considerable interest in the past.

In 1920, Nørlund [8] showed that

$$\begin{cases} \nu \text{ odd: } B_\nu\left(\frac{\theta}{2\pi}\right) \text{ has three zeroes at } 0, \frac{1}{2}, 1 \text{ in } [0,1] \\ \nu \text{ even: } B_\nu\left(\frac{\theta}{2\pi}\right) \text{ has two zeroes } r_\nu \text{ and } 1-r_\nu \text{ in } [0,1] \end{cases}$$

and that the zeroes r_ν lie in $[\frac{1}{6}, \frac{1}{4}]$, $\forall \nu \geq 2$ and $\nu \in \mathbb{N}$. Several years later, Lense [5] proved that the zeroes r_ν are strictly increasing towards $\frac{1}{4}$ as $\nu \to \infty$, $\forall \nu \geq 2$ and $\nu \in \mathbb{N}$. Then, in 1940, Lehmer [4] proves that for ν even

$$\frac{1}{4} - \frac{1}{2\pi}2^{-\nu} < r_\nu < \frac{1}{4}, \forall \nu \geq 6$$

and checks explicitly that this is also true for $\nu = 2, 4$. Furthermore he proves the asymptotic formula:

$$r_\nu \sim \frac{1}{4} - \frac{1}{2\pi}\left(2^{-\nu} - 4^{-\nu} + 4 \cdot 6^{-\nu} + \mathcal{O}\left(8^{-\nu}\right)\right), \forall \nu \geq 6 \text{ and } \nu \text{ even.}$$

Finally, the problem is satisfactorily solved by Ostrowski in his 1960 paper [9], where he gives, for ν even, the bounds

$$r_\nu \sim \frac{1}{4} - \frac{1}{2\pi}\left(2^{-\nu} - 4^{-\nu} + 4 \cdot 6^{-\nu} - \frac{17}{6}8^{-\nu} - 4 \cdot 10^{-\nu} - 4 \cdot 12^{-\nu} + 13p \cdot 14^{-\nu}\right),$$

where $0 < p < 1$, $\nu \geq 2$ and also proves that r_ν increases monotonically with ν.

3. An empirical result

In a recent paper [3], Filothodoros et al. notice that the zeroes θ_{2n-1} of the Clausen functions $Cl_{2n-1}(\theta)$ occur close to rational multiples of π and give a formula for the approximate location of these zeroes respectively the extrema of the Clausen functions $Cl_{2n}(\theta)$ for $n \in \mathbb{N}^* = \{1, 2, \ldots\}$.

For $\nu = 2n - 1$ (and thus odd), they conjecture that

$$r_\nu \approx \frac{1}{4} - \frac{5}{4}\frac{1}{4^{\frac{\nu+3}{2}} - (-1)^{\frac{\nu+3}{2}}}, \text{ for } \nu \geq 1,$$

with $r_\nu = \frac{\theta_\nu}{2\pi}$. Obviously, we have $r_1 = \frac{1}{6}$ and

$$\lim_{\nu\to\infty}\left(\frac{1}{4} - \frac{5}{4}\frac{1}{4^{\frac{\nu+3}{2}} - (-1)^{\frac{\nu+3}{2}}}\right) = \frac{1}{4}$$

Rewriting as

$$r_\nu \approx \frac{1}{4} - \frac{5}{32}\left(\frac{1}{1 - (-\frac{1}{4})^{\frac{\nu+3}{2}}}\right)2^{-\nu}$$

and noting that $\frac{1}{1-(-\frac{1}{4})^{\frac{\nu+3}{2}}} \leq \frac{16}{15}$ for $\nu \geq 1$, so that $\frac{5}{32}\left(\frac{1}{1-(-\frac{1}{4})^{\frac{\nu+3}{2}}}\right) \leq \frac{5}{32}\cdot\frac{16}{15} = \frac{1}{6}$, the similarity to Lehmer's expression above becomes apparent.

The expression given by Filothodoros et al. presents a major inconvenience as it may not be applied to non integer values of the parameter n. We therefore propose to modify it slightly as follows,

$$r_\nu \approx \frac{1}{4} - \frac{5}{4}\frac{1}{4^{\frac{\nu+3}{2}} - \sin(\frac{\pi}{2}\nu)}, \text{ for } \nu \geq 1,$$

producing the same results for integer ν, but not limited to integer values.

4. Generalised Clausen functions: Asymptotics for the zeroes and extrema

Lemma 4.1: *The generalised Clausen functions $C_\nu(\theta)$, $S_\nu(\theta)$ and their derivatives with respect to order converge absolutely and uniformly $\forall\theta \in \mathbb{R}$ and $\nu > 1$.*

Proof: As

$$\begin{cases} S_\nu(\theta) = \sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^\nu} \leq \sum_{k=1}^{\infty}\frac{|\sin(k\theta)|}{k^\nu} \leq \sum_{k=1}^{\infty}\frac{1}{k^\nu} \\ C_\nu(\theta) = \sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^\nu} \leq \sum_{k=1}^{\infty}\frac{|\cos(k\theta)|}{k^\nu} \leq \sum_{k=1}^{\infty}\frac{1}{k^\nu} \end{cases}$$

and

$$\begin{cases} -\frac{\partial S_\nu(\theta)}{\partial \nu} = \sum_{k=1}^{\infty} \frac{\ln(k)\sin(k\theta)}{k^\nu} \leq \sum_{k=1}^{\infty} \frac{|\ln(k)\sin(k\theta)|}{k^\nu} \leq \sum_{k=1}^{\infty} \frac{ln(k)}{k^\nu} \\ -\frac{\partial C_\nu(\theta)}{\partial \nu} = \sum_{k=1}^{\infty} \frac{\ln(k)\cos(k\theta)}{k^\nu} \leq \sum_{k=1}^{\infty} \frac{|\ln(k)\cos(k\theta)|}{k^\nu} \leq \sum_{k=1}^{\infty} \frac{ln(k)}{k^\nu} \end{cases},$$

the lemma is proved. □

Lemma 4.2: *$\lim_{\nu\to\infty} S_\nu(\theta) = \sin(\theta)$ and $\lim_{\nu\to\infty} C_\nu(\theta) = \cos(\theta)$.*

Proof: Trivial, as $\lim_{\nu\to\infty} k^{-\nu} = 0, \forall k \neq 1$. □

Proposition 4.3: *In the interval $[0,\pi]$, the generalised Clausen function $S_\nu(\theta)$ with $\nu > 0$ has no other zeroes than $\theta = 0$ and $\theta = \pi$.*

Proof: Provided by Gergő Nemes [7].

We have

$$\begin{aligned} S_\nu(\theta) &= \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^\nu} = \Im\left(\sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k^\nu}\right) = \Im\left(\mathrm{Li}_\nu(e^{i\theta})\right) \\ &= \frac{1}{\Gamma(\nu)}\Im\left(\int_0^{+\infty} \frac{t^{\nu-1}}{e^{t-i\theta}-1}dt\right) = \frac{\sin(\theta)}{\Gamma(\nu)}\int_0^{+\infty} \frac{t^{\nu-1}e^t}{e^{2t}-2e^t\cos(\theta)+1}dt, \end{aligned}$$

using

$$\frac{1}{e^{t-i\theta}-1} = \frac{e^t\cos(\theta)-1+i\sin(\theta)}{e^{2t}-2e^t\cos(\theta)+1}.$$

Now, as $e^{2t} - 2e^t\cos(\theta) + 1 \geq e^{2t} - 2e^t + 1 = (e^t - 1)^2 > 0$ and $t^{\nu-1}e^t > 0$, for $t > 0$, the integral is strictly positive. As $\Gamma(\nu) > 0$ for $\nu > 0$ and $\sin(\theta) > 0$ for $\theta \in]0,\pi[$, the assumption follows. □

Proposition 4.4: *In the interval $]0,\pi[$, the generalised Clausen function $C_\nu(\theta)$ with $\nu > 0$ is strictly decreasing.*

Proof: Using the same approach as in the preceeding proposition, write

$$\begin{aligned} C_\nu(\theta) &= \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^\nu} = \Re\left(\sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k^\nu}\right) = \Re\left(\mathrm{Li}_\nu(e^{i\theta})\right) \\ &= \frac{1}{\Gamma(\nu)}\Re\left(\int_0^{+\infty} \frac{t^{\nu-1}}{e^{t-i\theta}-1}dt\right) = \frac{1}{\Gamma(\nu)}\int_0^{+\infty} \frac{t^{\nu-1}\left(e^t\cos(\theta)-1\right)}{e^{2t}-2e^t\cos(\theta)+1}dt := f(\theta) \end{aligned}$$

Then,

$$\begin{aligned} &f(\theta_2) - f(\theta_1) = \\ &\quad \frac{1}{\Gamma(\nu)}\left(\cos(\theta_2)-\cos(\theta_1)\right)\int_0^{+\infty} \frac{t^{\nu-1}e^t\left(e^{2t}-1\right)}{\left(e^{2t}-2e^t\cos(\theta_1)+1\right)\left(e^{2t}-2e^t\cos(\theta_2)+1\right)}dt \end{aligned}$$

Now, since $e^{2t} - 2e^t\cos(\theta_i) + 1 \geq (e^t - 1)^2 > 0$,$e^{2t} - 1 > 0$, $e^t > 0$ and $t^{\nu-1}e^t > 0$, for $t > 0$, the integral is strictly positive. Thus, as $\Gamma(\nu) > 0$ for $\nu > 0$ and $\cos(\theta)$ is decreasing in $]0, \pi[$, the assumption follows. $\square$

Corollary 4.5: *The generalised Clausen function $C_\nu(\theta)$ with $\nu > 1$ has a single zero in the interval $[0, \pi]$ and reaches at the endpoints of the interval its extrema given by*

$$C_\nu(0) = \zeta(\nu) > 0, \text{ and } C_\nu(\pi) = (2^{1-\nu} - 1)\zeta(\nu) < 0.$$

Proof: We have

$$C_\nu(0) = \sum_{k=1}^{\infty} \frac{\cos(0)}{k^\nu} = \sum_{k=1}^{\infty} \frac{1}{k^\nu} = \zeta(\nu) > 0$$

and

$$C_\nu(\pi) = \sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^\nu} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^\nu} = (2^{1-\nu} - 1)\zeta(\nu) < 0.$$

As, by proposition 4.4, $C_\nu(\theta)$ is strictly decreasing in $]0, \pi[$, the proof is complete. $\square$

Proposition 4.6: *The derivative $\frac{\partial C_\nu(\theta)}{\partial \nu}$ of the generalised Clausen function $C_\nu(\theta)$ with respect to order is positive in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$ for $\nu > 2.837756935\ldots$*

Proof: Write

$$\begin{aligned}
\frac{\partial C_\nu(\theta)}{\partial \nu} &= -\sum_{k=1}^{\infty} \frac{\ln(k)\cos(k\theta)}{k^\nu} \\
&= -\frac{\ln(2)}{2^\nu}\cos(2\theta) - \frac{\ln(3)}{3^\nu}\cos(3\theta) - \sum_{k=4}^{\infty} \frac{\ln(k)\cos(k\theta)}{k^\nu} \\
&\geq -\frac{\ln(2)}{2^\nu}\cos(2\theta) - \frac{\ln(3)}{3^\nu}\cos(3\theta) - \sum_{k=4}^{\infty} \frac{\ln(k)}{k^\nu}, \text{ as } |\cos(k\theta)| \leq 1 \\
&= -\frac{\ln(2)}{2^\nu}\cos(2\theta) - \frac{\ln(3)}{3^\nu}\cos(3\theta) + \zeta'(\nu) + \frac{\ln(2)}{2^\nu} + \frac{\ln(3)}{3^\nu} \\
&= 2\frac{\ln(2)}{2^\nu}\sin(\theta)^2 + 2\frac{\ln(3)}{3^\nu}\sin\left(\frac{3\theta}{2}\right)^2 + \zeta'(\nu).
\end{aligned}$$

Now, for $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$ the function $f(\theta, \nu) := \frac{\ln(2)}{2^\nu}\sin(\theta)^2 + \frac{\ln(3)}{3^\nu}\sin\left(\frac{3\theta}{2}\right)^2$ reaches its minimum in one of the endpoints of the interval, with

$$f\left(\frac{\pi}{3}, \nu\right) = \frac{3}{4} \cdot 2^{-\nu}\ln(2) + 3^{-\nu}\ln(3),$$

and

$$f\left(\frac{\pi}{2}, \nu\right) = 2^{-\nu}\ln(2) + \frac{1}{2} \cdot 3^{-\nu}\ln(3).$$

Finally, as $\zeta'(\nu)$ is negative and monotonically increasing and both $f\left(\frac{\pi}{3},\nu\right)$ and $f\left(\frac{\pi}{2},\nu\right)$ are positive and monotonically decreasing, we have

$$2f\left(\frac{\pi}{3},\nu\right)+\zeta'(\nu)=2\left(\frac{3}{4}\cdot 2^{-\nu}\ln(2)+3^{-\nu}\ln(3)\right)+\zeta'(\nu)\geq 0, \forall\nu\geq 2.836317129\ldots$$

and

$$2f\left(\frac{\pi}{2},\nu\right)+\zeta'(\nu)=2\left(2^{-\nu}\ln(2)+\frac{1}{2}\cdot 3^{-\nu}\ln(3)\right)+\zeta'(\nu)\geq 0, \forall\nu\geq 2.837756935\ldots,$$

so that $\frac{\partial C_\nu(\theta)}{\partial\nu}\geq 0$ for all $\nu\geq 2.837756935\ldots$, thus proving the assertion. □

Remark *4.7***:** Using the method of the preceeding proposition, the limiting value for ν may, in principle, be improved by considering more terms of the Fourier series. However, this soon becomes impracticable, as the number of terms to be considered increases rapidly, i.e. in order to refine the bound to $\nu\geq 2$, some forty terms would be necessary.

Theorem 4.8: *The sequence of the zeroes $\theta_r(\nu)$ of the generalised Clausen functions $C_\nu(\theta)$ is increasing with ν for $\nu\geq 2.837756935\ldots$ and $\lim_{\nu\to\infty}\theta_r(\nu)=\frac{\pi}{2}$.*

Proof: We will prove the theorem following Lense's method [5]. By property 1.5 and lemma 4.1,

$$C_\nu(\theta)=\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^\nu}$$

and

$$\frac{\partial}{\partial\nu}C_\nu(\theta)=-\sum_{k=1}^{\infty}\frac{\ln(k)\cos(k\theta)}{k^\nu}$$

converge absolutely and uniformly $\forall\theta\in\mathbb{R}$ and $\nu>1$, while

$$\frac{d}{d\theta}C_\nu(\theta)=-S_{\nu-1}(\theta)=-\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^{\nu-1}}$$

converges absolutely and uniformly $\forall\theta\in\mathbb{R}$ and $\nu>2$. By corollary 4.5, every $C_\nu(\theta)$ has a single zero $\theta_r(\nu)$ in $[0,\pi]$ and at these zeroes, we have

$$\frac{d\theta_r(\nu)}{d\nu}=-\frac{\frac{\partial C_\nu(\theta_r(\nu))}{\partial\nu}}{\frac{\partial C_\nu(\theta_r(\nu))}{\partial\theta_r(\nu)}}=-\frac{-\sum_{k=1}^{\infty}\frac{\ln(k)\cos(k\theta_r(\nu))}{k^\nu}}{-\sum_{k=1}^{\infty}\frac{\sin(k\theta_r(\nu))}{k^{\nu-1}}}$$

by implicit differentiation and using the fact that $\sum_{k=1}^{\infty}\frac{\sin(k\theta_r(\nu))}{k^{\nu-1}}>0$ in $]0,\pi[$ by proposition 4.3. By proposition 4.6, the derivative $\frac{\partial}{\partial\nu}C_\nu(\theta)$ is positive in the interval

$[\frac{\pi}{3}, \frac{\pi}{2}]$ for $\nu \geq 2.837756935\ldots$, so that

$$\frac{d\theta_r(\nu)}{d\nu} \geq 0$$

under the given conditions. Finally, $\theta_r(2.837756935\ldots) = 1.436065269\ldots \in [\frac{\pi}{3}, \frac{\pi}{2}]$ and $\lim_{\nu\to\infty} C_\nu(\theta) = \cos(\theta)$ by lemma 4.2. Noting that $\cos\left(\frac{\pi}{2}\right) = 0$, the theorem is proved. □

Corollary 4.9: *The sequence of the maxima $\theta_M(\nu)$ of the generalised Clausen functions $S_\nu(\theta)$ is increasing with ν for $\nu \geq 3.837756935\ldots$ and $\lim_{\nu\to\infty} \theta_M(\nu) = \frac{\pi}{2}$.*

Proof: Using property 1.5,

$$\frac{d}{d\theta} S_\nu(\theta) = C_{\nu-1}(\theta)$$

and theorem 4.8, the proof is immediate. □

Corollary 4.10: *For all $n \in \mathbb{N}^* = \{1, 2, \ldots\}$, the sequence of the zeroes $\theta_r(n)$ of the generalised Clausen functions $C_n(\theta)$ is increasing with n and $\lim_{n\to\infty} \theta_r(n) = \frac{\pi}{2}$.*

Proof: We check that the assertion is true for $n = 1$ and $n = 2$. Indeed, we have

$$\theta_r(1) = \frac{\pi}{3} = 1.047197551\ldots$$

and

$$\theta_r(2) = \pi\left(1 - \frac{1}{\sqrt{3}}\right) = 1.327793289\ldots > \theta_r(1)$$

Using theorem 4.8, the assertion follows. □

Corollary 4.11: *For all $n \in \{2, 3, \ldots\}$, the sequence of the maxima $\theta_M(n)$ of the generalised Clausen functions $S_n(\theta)$ is increasing with n and $\lim_{n\to\infty} \theta_r(n) = \frac{\pi}{2}$.*

Proof: Follows immediately from corollary 4.10 and property 1.5. □

The analysis may be taken further by Lehmer's approach [4], thereby obtaining more precise estimates for the zeroes respectively extrema of the generalised Clausen functions.

Theorem 4.12: *The zeroes $\theta_r(\nu)$ of the generalised Clausen functions $C_\nu(\theta)$ are bounded by $\frac{\pi}{2} - 2^{-\nu}$ from below and by $\frac{\pi}{2}$ from above for all $\nu \geq 4.504983930\ldots$.*

Proof: Write

$$\begin{aligned}
C_\nu(\theta) &= \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^\nu} \\
&= \sum_{k=1}^{\infty} \frac{\cos\left(k\left(\frac{\pi}{2}-\alpha\right)\right)}{k^\nu} \\
&= \sum_{\ell=1}^{\infty} \frac{(-1)^\ell \cos(2\ell\alpha)}{(2\ell)^\nu} + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} \sin((2\ell-1)\alpha)}{(2\ell-1)^\nu} \\
&= \sin(\alpha) - 2^{-\nu}\cos(2\alpha) - 3^{-\nu}\sin(3\alpha) + 4^{-\nu}\cos(4\alpha) + 5^{-\nu}\sin(5\alpha) + \ldots,
\end{aligned}$$

with $\alpha = \frac{\pi}{2} - \theta$. Thus, if $C_\nu(\theta_r(\nu)) = 0$, we have

$$\begin{aligned}
\sin(\alpha_r(\nu)) &= 2^{-\nu}\cos(2\alpha_r(\nu)) + 3^{-\nu}\sin(3\alpha_r(\nu)) - 4^{-\nu}\cos(4\alpha_r(\nu)) - 5^{-\nu}\sin(5\alpha_r(\nu)) + \ldots \\
&< 2^{-\nu} + 3^{-\nu} + 4^{-\nu} + \ldots = \zeta(\nu) - 1,
\end{aligned}$$

as $|\cos(x)| \leq 1$ and $|\sin(x)| \leq 1$. Furthermore, using

$$\begin{cases}
\cos(2x) < 1 \\
\sin(3x) = 3\sin(x)\cos(x)^2 - \sin(x)^3 < 3\sin(x) \\
\cos(4x) = 1 - 8\sin(x)^2\cos(x)^2 > 1 - 8\sin(x)^2 \\
\sin(5x) > 0
\end{cases},$$

for $x < \frac{\pi}{6}$, we have

$$\begin{aligned}
\sin(\alpha_r(\nu)) =& 2^{-\nu}\cos(2\alpha_r(\nu)) + 3^{-\nu}\sin(3\alpha_r(\nu)) - 4^{-\nu}\cos(4\alpha_r(\nu)) - 5^{-\nu}\sin(5\alpha_r(\nu)) + \ldots \\
\leq& 2^{-\nu} + 3^{-\nu}\cdot 3\sin(\alpha_r(\nu)) - 4^{-\nu}\left(1 - 8\sin(\alpha_r(\nu))^2\right) + \sum_{i=6}^{\infty} i^{-\nu} \\
=& 2^{-\nu} + 3^{-\nu}\cdot 3\sin(\alpha_r(\nu)) - 4^{-\nu}\left(1 - 8\sin(\alpha_r(\nu))^2\right) \\
& + \zeta(\nu) - 1 - 2^{-\nu} - 3^{-\nu} - 4^{-\nu} - 5^{-\nu} \\
=& 3^{-\nu}(3\sin(\alpha_r(\nu)) - 1) - 2\cdot 4^{-\nu} + 8\cdot 4^{-\nu}\sin(\alpha_r(\nu))^2 - 5^{-\nu} + \zeta(\nu) - 1
\end{aligned}$$

Now,

$$\begin{aligned}
&\sin(\alpha_r(\nu)) > \alpha_r(\nu) - \frac{\alpha_r(\nu)^3}{6} \\
&\Leftrightarrow \alpha_r(\nu) < \sin(\alpha_r(\nu)) + \frac{\alpha_r(\nu)^3}{6} \\
&\qquad < \sin(\alpha_r(\nu)) + \frac{1}{6}\left(\frac{\pi}{3}\sin(\alpha_r(\nu))\right)^3 = \sin(\alpha_r(\nu))\left(1 + \frac{\pi^3}{162}\sin(\alpha_r(\nu))^2\right),
\end{aligned}$$

so that

$$\Leftrightarrow \frac{\alpha_r(\nu)}{1+\frac{\pi^3}{162}\sin(\alpha_r(\nu))^2} < \sin(\alpha_r(\nu)),$$

as $\alpha_r(\nu) < \frac{\pi}{3}\sin(\alpha_r(\nu))$ for $\alpha_r(\nu) < \frac{\pi}{6}$. Thus,

$$\frac{\alpha_r(\nu)}{1+\frac{\pi^3}{162}\sin(\alpha_r(\nu))^2} < 3^{-\nu}\left(3\sin(\alpha_r(\nu))-1\right) - 2\cdot 4^{-\nu} + 8\cdot 4^{-\nu}\sin(\alpha_r(\nu))^2 - 5^{-\nu} + \zeta(\nu) - 1,$$

respectively,

$$\begin{aligned}
&\frac{\alpha_r(\nu)}{1+\frac{\pi^3}{162}(\zeta(\nu)-1)^2} \\
&< 3^{-\nu}\left(3(\zeta(\nu)-1)-1\right) - 2\cdot 4^{-\nu} + 8\cdot 4^{-\nu}(\zeta(\nu)-1)^2 - 5^{-\nu} + \zeta(\nu) - 1 \\
&= 8\cdot 4^{-\nu}\zeta(\nu)^2 + \left(3^{-\nu+1} - 4^{-\nu+2} + 1\right)\zeta(\nu) - 4\cdot 3^{-\nu} + 6\cdot 4^{-\nu} - 5^{-\nu} - 1,
\end{aligned}$$

using the upper bound $\sin(\alpha_r(\nu)) < \zeta(\nu) - 1$ given previously. Hence

$$\begin{aligned}
\alpha_r(\nu) < \left(1+\frac{\pi^3}{162}(\zeta(\nu)-1)^2\right)&\left(8\cdot 4^{-\nu}\zeta(\nu)^2 + \left(3^{-\nu+1} - 4^{-\nu+2} + 1\right)\zeta(\nu)\right. \\
&\left. -4\cdot 3^{-\nu} + 6\cdot 4^{-\nu} - 5^{-\nu} - 1\right).
\end{aligned}$$

The right hand side of this equation is monotonously decreasing to zero as $\nu \to \infty$, such that

$$\begin{aligned}
-2^{-\nu} + \left(1+\frac{\pi^3}{162}(\zeta(\nu)-1)^2\right)&\left(8\cdot 4^{-\nu}\zeta(\nu)^2 + \left(3^{-\nu+1} - 4^{-\nu+2} + 1\right)\zeta(\nu)\right. \\
&\left. -4\cdot 3^{-\nu} + 6\cdot 4^{-\nu} - 5^{-\nu} - 1\right) = 0
\end{aligned}$$

for $\nu_0 = 4.504983930\ldots$ and is negative for $\nu > \nu_0$. Therefore

$$\begin{aligned}
&\alpha_r(\nu) - 2^{-\nu} < 0 \\
\Leftrightarrow\ &\alpha_r(\nu) < 2^{-\nu}
\end{aligned}$$

for $\nu > \nu_0$, so that

$$\frac{\pi}{2} - 2^{-\nu} \le \theta_r(\nu) \le \frac{\pi}{2}$$

for $\nu > \nu_0$, which completes the proof. □

Corollary 4.13: *The maxima $\theta_M(\nu)$ of the generalised Clausen functions $S_\nu(\theta)$ are bounded by $\frac{\pi}{2} - 2^{-\nu}$ from below and by $\frac{\pi}{2}$ from above for all $\nu \geq 5.504983930\ldots$.*

Proof: Using property 1.5,

$$\frac{d}{d\theta}S_\nu(\theta) = C_{\nu-1}(\theta)$$

and theorem 4.12, the proof is immediate. □

Corollary 4.14: *For all $n \in \{2, 3, \ldots\}$, the zeroes $\theta_r(n)$ of the generalised Clausen functions $C_n(\theta)$ are bounded by $\frac{\pi}{2} - 2^{-n}$ from below and by $\frac{\pi}{2}$ from above.*

Proof: We check that the assertion is true for $n = 2$, $n = 3$ and $n = 4$. Indeed, we have

$$\frac{\pi}{2} - 2^{-2} = 1.320796326\ldots \leq \theta_r(2) = \pi\left(1 - \frac{1}{\sqrt{3}}\right) = 1.327793289\ldots \leq \frac{\pi}{2},$$

$$\frac{\pi}{2} - 2^{-3} = 1.445796326\ldots \leq \theta_r(3) = 1.450345466\ldots \leq \frac{\pi}{2}$$

and

$$\frac{\pi}{2} - 2^{-4} = 1.508296326\ldots \leq \theta_r(4) = 1.510070527\ldots \leq \frac{\pi}{2}$$

Using theorem 4.12, the assertion follows. □

Corollary 4.15: *For all $n \in \{3, 4, \ldots\}$, the maxima $\theta_M(n)$ of the generalised Clausen functions $S_n(\theta)$ are bounded by $\frac{\pi}{2} - 2^{-n}$ from below and by $\frac{\pi}{2}$ from above.*

Proof: Follows immediately from corollary 4.14 and property 1.5. □

Remark *4.16*: More exact asymptotics may be obtained using now an iterative process, such as the one described by Lehmer [4]. However, this is unnecessary at this point, as we are going to investigate upper and lower bounds for the zeroes and extrema in the next section, thereby automatically recovering more precise asymptotics.

5. Generalised Clausen functions: Bounds for the zeroes and extrema

In 1960, Ostrowski gave a very satisfactory solution for the problem under consideration in the case of Bernoulli polynomials of even order [9], resulting in the following theorem.

Theorem 5.1: *(Ostrowski)*
The zeroes of the Bernoulli polynomials of even order are bounded by

$$\theta_r(\nu) \leq \frac{\pi}{2} - \left(2^{-\nu} - 4^{-\nu} + 4 \cdot 6^{-\nu} - \frac{17}{6} 8^{-\nu} - 4 \cdot 10^{-\nu} - 4 \cdot 12^{-\nu}\right),$$

from above, and by

$$\theta_r(\nu) \geq \frac{\pi}{2} - \left(2^{-\nu} - 4^{-\nu} + 4 \cdot 6^{-\nu} - \frac{17}{6} 8^{-\nu} - 4 \cdot 10^{-\nu} - 4 \cdot 12^{-\nu} + 13 \cdot 14^{-\nu}\right),$$

from below, where $\nu \in \{2, 4, 6, \ldots\} = 2\mathbb{N}^*$.

Proof: For the proof, we would like to refer the interested reader to the original paper [9]. □

Remark *5.2***:** Although Ostrowski's original theorem (and proof) considered only the Bernoulli polynomials of even integer order, it may be applied to all values of $\nu \geq 10$ and therefore to the generalised Clausen functions studied here.

Remark *5.3***:** Using similar techniques, Delange [2] gives a complete asymptotic series for the small positive zeroes of the Bernoulli polynomials in 1991, allowing us, when extended to non-integer values of the order, to determine the location of the zeroes/extrema of the generalised Clausen functions to an arbitrary precision.

At this point, one might be tempted to try to extend, respectively prove the validity of these results for smaller and non-integer values of ν by the same methods, but we would like to propose a slightly different approach in the following.

Lemma 5.4: *The sum* $\mathfrak{S}_0(\nu) := \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell+2)^{\nu}}$ *may be expressed using Riemann's zeta function as:*

$$\mathfrak{S}_0(\nu) := \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell+2)^{\nu}} = 2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu)$$

Proof: As

$$\begin{aligned}\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell+2)^{\nu}} &= 2^{-\nu}\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)^{\nu}} = -2^{-\nu}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\nu}} \\ &= -2^{-\nu}\eta(\nu) = -2^{-\nu}\left(1-2^{1-\nu}\right)\zeta(\nu) = 2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu),\end{aligned}$$

where $\eta(\nu)$ is the Dirichlet eta function (alternating zeta function), the assertion follows. □

Lemma 5.5: *The sums* $\mathfrak{S}_m(\nu) := \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} 2^{m-1}(m+\ell-1)!}{m!\ell!(2\ell+m)^{\nu-1}}$ *are given by the recurrence relation:*

$$\mathfrak{S}_m(\nu) := \frac{(m-2)^2}{m(m-1)}\mathfrak{S}_{m-2}(\nu) - \frac{1}{m(m-1)}\mathfrak{S}_{m-2}(\nu-2),$$

with initial conditions

$$\mathfrak{S}_1(\nu) := \beta(\nu-1),$$

$\beta(\nu)$ *denoting the Dirichlet beta function, and*

$$\mathfrak{S}_2(\nu) := -2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2).$$

Proof: By induction. We have

$$\mathfrak{S}_1(\nu) := \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} 2^{1-1}(1+\ell-1)!}{1!\ell!(2\ell+1)^{\nu-1}} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)^{\nu-1}} = \beta(\nu-1)$$

by the definition of the Dirichlet beta function and

$$\begin{aligned}\mathfrak{S}_2(\nu) : &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{2-1}\,(2+\ell-1)!}{2!\ell!\,(2\ell+2)^{\nu-1}} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\,(\ell+1)}{(2\ell+2)^{\nu-1}} = 2^{1-\nu} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\,(\ell+1)}{(\ell+1)^{\nu-1}} \\ &= 2^{1-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k^{\nu-2}} = 2^{1-\nu}\eta(\nu-2) = 2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2).\end{aligned}$$

Now, writing

$$\mathfrak{S}_{m-2}(\nu) := \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-1}},$$

we have

$$\begin{aligned}&\frac{(m-2)^2}{m\,(m-1)}\mathfrak{S}_{m-2}(\nu) - \frac{1}{m\,(m-1)}\mathfrak{S}_{m-2}(\nu-2) \\ &= \frac{(m-2)^2}{m\,(m-1)} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-1}} \\ &- \frac{1}{m\,(m-1)} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-3}} \\ &= \frac{(m-2)^2}{m\,(m-1)} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-1}} \\ &- \frac{1}{m\,(m-1)} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!\,(2\ell+m-2)^2}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-1}} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, 2^{m-3}\,(m+\ell-3)!}{(m-2)!\ell!\,(2\ell+m-2)^{\nu-1}} \left(\frac{(m-2)^2-(2\ell+m-2)^2}{m\,(m-1)}\right) \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{(\ell}\, 2^{m-3}\,(m+(\ell-1)-2)!}{(m-2)!\,((\ell-1)+1)!\,(2(\ell-1)+m)^{\nu-1}} \left(\frac{(m-2)^2-(2(\ell-1)+m)^2}{m\,(m-1)}\right).\end{aligned}$$

By a change of variable $\ell-1 \to k$, the expression may be written as

$$\begin{aligned}&- \sum_{k=-1}^{\infty} \frac{(-1)^{k}\, 2^{m-3}\,(m+k-2)!}{(m-2)!\,(k+1)!\,(2k+m)^{\nu-1}} \left(\frac{(m-2)^2-(2k+m)^2}{m\,(m-1)}\right) \\ &= - \sum_{k=-1}^{\infty} \frac{(-1)^{k}\, 2^{m-1}\,(m+k-1)!2^{-2}m\,(m-1)}{m!k!\,(2k+m)^{\nu-1}\,(m+k-1)\,(k+1)} \left(\frac{(m-2)^2-(2k+m)^2}{m\,(m-1)}\right)\end{aligned}$$

$$= -\sum_{k=-1}^{\infty} \frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k! (2k+m)^{\nu-1}} \left(\frac{2^{-2} m (m-1) \left((m-2)^2 - (2k+m)^2 \right)}{(m+k-1)(k+1) m (m-1)} \right)$$

$$= -\sum_{k=-1}^{\infty} \frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k! (2k+m)^{\nu-1}} (-1)$$

$$= \sum_{k=-1}^{\infty} \frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k! (2k+m)^{\nu-1}}.$$

Finally, using the fact that for $k=-1$ we have $\frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k!(2k+m)^{\nu-1}} = 0$, and renaming the variable $k \to \ell$;

$$\begin{aligned} \sum_{k=-1}^{\infty} \frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k! (2k+m)^{\nu-1}} &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{m-1} (m+k-1)!}{m!k! (2k+m)^{\nu-1}} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell 2^{m-1} (m+\ell-1)!}{m!\ell! (2\ell+m)^{\nu-1}} \\ &= \mathfrak{S}_m(\nu), \end{aligned}$$

the assumption is proved. □

Remark *5.6***:** The interested reader might notice that there exists a "closed form" for the sums given above. Indeed, we may write

$$\begin{cases} \mathfrak{S}_{2m-1}(\nu) = \frac{1}{(2m-1)!} \sum_{i=1}^{m} (-1)^i 2^{2i} t_{2m-1,2m-1-2i} \beta(\nu - 2i + 1) \\ \mathfrak{S}_{2m}(\nu) = -\frac{2^{m-\nu}}{(2m)!} \sum_{i=1}^{m} (-1)^i t_{2m,2m-2i} \left(2^{i+1-\nu} - 1 \right) \zeta(\nu - 2i) \end{cases},$$

for $m \in \mathbb{N}^* = \{1, 2, \ldots\}$ and where the $t_{n,k}$ are the (unsigned) central factorial numbers (see OEIS sequences A008956 [10] and A008955 [11]).

The next lemma is a well known result, used in diverse areas of mathematics and physics, but we will nevertheless give its proof for the sake of completeness.

Lemma 5.7: *We have*

$$a^{-(b+1)} = \frac{1}{\Gamma(b+1)} \int_0^\infty t^b e^{-at} dt.$$

Proof: Letting $x = at$, we have $dx = adt$, so that

$$\int_0^\infty t^b e^{-at} dt = \int_0^\infty \left(\frac{x}{a} \right)^b e^{-x} \frac{1}{a} dx = \left(\frac{1}{a} \right)^{b+1} \int_0^\infty x^b e^{-x} dx = a^{-(b+1)} \Gamma(b+1),$$

by the definition of the gamma function. Dividing by $\Gamma(b+1)$, the proof is complete. □

The following is another quite elementary lemma that will be needed.

Lemma 5.8: *For all $t \in \mathbb{R}$, we have*

$$\frac{\left(2\cosh\left(\frac{t}{\sqrt{m+1}}\right)\right)^{m+1}}{\left(2\cosh\left(\frac{t}{\sqrt{m}}\right)\right)^{m}} \geq 2.$$

Proof: Consider

$$\begin{aligned}\frac{\left(e^{\left(\frac{t}{\sqrt{m+1}}\right)}+e^{\left(-\frac{t}{\sqrt{m+1}}\right)}\right)^{m+1}}{\left(e^{\left(\frac{t}{\sqrt{m}}\right)}+e^{\left(-\frac{t}{\sqrt{m}}\right)}\right)^{m}} &= \frac{e^{\left(t\sqrt{m+1}\right)}\left(1+e^{\left(-\frac{2t}{\sqrt{m+1}}\right)}\right)^{m+1}}{e^{\left(t\sqrt{m}\right)}\left(1+e^{\left(-\frac{2t}{\sqrt{m}}\right)}\right)^{m}} \\ &= \underbrace{\frac{e^{\left(t\sqrt{m+1}\right)}}{e^{\left(t\sqrt{m}\right)}}}_{\geq 1}\underbrace{\left(\frac{1+e^{\left(-\frac{2t}{\sqrt{m+1}}\right)}}{1+e^{\left(-\frac{2t}{\sqrt{m}}\right)}}\right)^{m}}_{\geq 1}\underbrace{\left(1+e^{\left(-\frac{2t}{\sqrt{m+1}}\right)}\right)}_{\geq 1} \\ &\geq 1,\end{aligned}$$

Multiplying both sides by $2 = \frac{2^{m+1}}{2^m}$:

$$\frac{2^{m+1}\left(e^{\left(\frac{t}{\sqrt{m+1}}\right)}+e^{\left(-\frac{t}{\sqrt{m+1}}\right)}\right)^{m+1}}{2^m\left(e^{\left(\frac{t}{\sqrt{m}}\right)}+e^{\left(-\frac{t}{\sqrt{m}}\right)}\right)^{m}} = \frac{\left(2\cosh\left(\frac{t}{\sqrt{m+1}}\right)\right)^{m+1}}{\left(2\cosh\left(\frac{t}{\sqrt{m}}\right)\right)^{m}} \geq 2,$$

thus proving the assertion. □

Proposition 5.9: *For all $m \in \mathbb{N}^*$ and $\nu > 1$, the following holds:*

$$m^{\frac{\nu+1}{2}}\mathfrak{S}_m(\nu) = \frac{1}{2\Gamma(\nu-1)}\int_0^\infty \frac{t^{\nu-2}}{\left(\cosh\left(\frac{t}{\sqrt{m}}\right)\right)^m}dt$$

Proof: Using Abel-summation,

$$\begin{aligned}m^{\frac{\nu+1}{2}}\mathfrak{S}_m(\nu) &= m^{\frac{\nu+1}{2}}\lim_{x\to 1^-}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,2^{m-1}\,(m+\ell-1)!}{m!\ell!\,(2\ell+m)^{\nu-1}}x^{\ell} \\ &= \lim_{x\to 1^-}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\,2^{m-1}\,(m+\ell-1)!}{m!\ell!\,(2\ell+m)^{\nu-1}}m^{\frac{\nu+1}{2}}x^{\ell} \\ &= 2^{m-1}\lim_{x\to 1^-}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(m+\ell-1)!}{(m-1)!\ell!}\frac{m^{\frac{\nu+1}{2}}}{m\,(2\ell+m)^{\nu-1}}x^{\ell} \\ &= 2^{m-1}\lim_{x\to 1^-}\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{m+\ell-1}{\ell}\frac{m^{\frac{\nu+1}{2}-1}}{(2\ell+m)^{\nu-1}}x^{\ell}\end{aligned}$$

$$= 2^{m-1} \lim_{x\to 1^-} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} m^{\frac{\nu-1}{2}} (2\ell+m)^{1-\nu} x^\ell$$

$$= 2^{m-1} \lim_{x\to 1^-} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} \left(m^{-\frac{1}{2}}\right)^{1-\nu} (2\ell+m)^{1-\nu} x^\ell$$

$$= 2^{m-1} \lim_{x\to 1^-} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} \left(\frac{2\ell+m}{\sqrt{m}}\right)^{1-\nu} x^\ell$$

$$= 2^{m-1} \lim_{x\to 1^-} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\frac{2\ell+m}{\sqrt{m}}t} dt \cdot x^\ell$$

$$= 2^{m-1} \lim_{x\to 1^-} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\sqrt{m}t} e^{-\frac{2\ell}{\sqrt{m}}t} dt \cdot x^\ell,$$

using lemma 5.7. Now, writing $e^{-\frac{2\ell}{\sqrt{m}}t} = \left(e^{-\frac{2t}{\sqrt{m}}}\right)^\ell$, and interchanging integration and summation:

$$m^{\frac{\nu+1}{2}} \mathfrak{S}_m(\nu)$$

$$= 2^{m-1} \lim_{x\to 1^-} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\sqrt{m}t} \sum_{\ell=0}^{\infty} (-1)^\ell \binom{m+\ell-1}{\ell} \left(e^{-\frac{2t}{\sqrt{m}}}\right)^\ell x^\ell dt$$

$$= 2^{m-1} \lim_{x\to 1^-} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\sqrt{m}t} \sum_{\ell=0}^{\infty} \binom{m+\ell-1}{\ell} \left(-xe^{-\frac{2t}{\sqrt{m}}}\right)^\ell dt$$

$$= 2^{m-1} \lim_{x\to 1^-} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\sqrt{m}t} \left(1+xe^{-\frac{2t}{\sqrt{m}}}\right)^{-m} dt,$$

where we used Newton's generalised binomial theorem. Thus

$$\begin{aligned}
m^{\frac{\nu+1}{2}} \mathfrak{S}_m(\nu) &= 2^{m-1} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} e^{-\sqrt{m}t} \left(1+e^{-\frac{2t}{\sqrt{m}}}\right)^{-m} dt \\
&= 2^{m-1} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} \left(e^{\frac{t}{\sqrt{m}}}\right)^{-m} \left(1+e^{-\frac{2t}{\sqrt{m}}}\right)^{-m} dt \\
&= 2^{m-1} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} \left(e^{\frac{t}{\sqrt{m}}}+e^{-\frac{t}{\sqrt{m}}}\right)^{-m} dt \\
&= 2^{m-1} \frac{1}{\Gamma(\nu-1)} \int_0^\infty t^{\nu-2} 2^{-m} \left(\cosh\left(\frac{t}{\sqrt{m}}\right)\right)^{-m} dt \\
&= \frac{1}{2\Gamma(\nu-1)} \int_0^\infty \frac{t^{\nu-2}}{\left(\cosh\left(\frac{t}{\sqrt{m}}\right)\right)^m} dt
\end{aligned}$$

as stated. $\square$

Corollary 5.10: *For all $m \in \mathbb{N}^* = \{1, 2, \dots\}$ and $\nu > 1$,*

$$m^{\frac{\nu+1}{2}} \mathfrak{S}_m(\nu) \geq (m+1)^{\frac{\nu+1}{2}} \mathfrak{S}_{m+1}(\nu)$$

Proof: Using proposition 5.9 and lemma 5.8 the corollary follows immediately. □

Proposition 5.11: *The generalised Clausen functions $C_\nu(\theta)$ may alternatively be written as*

$$C_\nu(\theta) = \sum_{m=0}^{\infty} \mathfrak{S}_m(\nu) \left(\sin(\alpha)\right)^m$$

with $\alpha = \frac{\pi}{2} - \theta$.

Proof: Using the properties of the Chebyshev polynomials of the first kind $T_\ell(x)$, write

$$\begin{aligned} C_\nu(\theta) &= \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^\nu} = \sum_{k=1}^{\infty} \frac{\cos\left(k\left(\frac{\pi}{2} - \alpha\right)\right)}{k^\nu} \\ &= \sum_{\ell=1}^{\infty} \frac{(-1)^\ell \cos(2\ell\alpha)}{(2\ell)^\nu} + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} \sin\left((2\ell-1)\alpha\right)}{(2\ell-1)^\nu} \\ &= \sum_{\ell=1}^{\infty} \frac{T_\ell\left(\sin(\alpha)\right)}{\ell^\nu} \end{aligned}$$

with $\alpha = \frac{\pi}{2} - \theta$. Now, the coefficient $\mathfrak{s}_m$ of x^m in the Chebyshev polynomial of the first kind of order $T_{2k+m}(x)$ is given by (see [6]):

$$\mathfrak{s}_m = (-1)^k 2^{m-1} (2k+m) \frac{(m+k-1)!}{k!m!},$$

for $2k + m > 0$. Summing now over all ℓ, we have

$$\begin{aligned} C_\nu(\theta) &= \sum_{\ell=1}^{\infty} \frac{T_\ell\left((\sin(\alpha)\right)}{\ell^\nu} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell+2)^\nu} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell 2^{1-1} (1+\ell-1)!}{1!\ell!\,(2\ell+1)^{\nu-1}} \sin(\alpha) \\ &\quad + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell 2^{2-1} (2+\ell-1)!}{2!\ell!\,(2\ell+2)^{\nu-1}} \left(\sin(\alpha)\right)^2 + \dots \\ &= \mathfrak{S}_0(\nu) + \mathfrak{S}_1(\nu)\sin(\alpha) + \mathfrak{S}_2(\nu)\left(\sin(\alpha)\right)^2 + \dots \\ &= \sum_{m=0}^{\infty} \mathfrak{S}_m(\nu) \left(\sin(\alpha)\right)^m, \end{aligned}$$

as stated. □

Proposition 5.11 can now be used together with corollary 5.10 to obtain bounds for the location of the zeroes of the generalised Clausen functions $C_\nu(\theta)$.

Theorem 5.12: *In first approximation, the zeroes $\theta_r(\nu)$ of the generalised Clausen functions $C_\nu(\theta)$ are bounded as*

$$\arccos\left(\frac{-2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu)}{\zeta\left(\frac{\nu+1}{2}\right)\beta(\nu-1)}\right) \geq \theta_r(\nu) \geq \arccos\left(\frac{-2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu)}{\beta(\nu-1)}\right).$$

A three term approximation leads to the improved bounds

$$\arccos\left(-\frac{1}{2}\,\frac{-\beta(\nu-1)+\sqrt{(\beta(\nu-1))^2+4\,2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2)\,2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu)}}{2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2)}\right)$$

$$\geq \theta_r(\nu),$$

$$\theta_r(\nu) \geq \arccos\left(\frac{1}{2}\,\frac{-\beta(\nu-1)+\sqrt{(\beta(\nu-1))^2+4\,2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2)\,2^{-\nu}\left(2^{1-\nu}-1\right)\zeta(\nu)}}{2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta(\nu-2)}\right),$$

Proof: By proposition 5.11,

$$\begin{aligned}C_\nu(\theta) &= \sum_{m=0}^{\infty}\mathfrak{S}_m(\nu)\left(\sin(\alpha)\right)^m = \mathfrak{S}_0(\nu)+\mathfrak{S}_1(\nu)\sin(\alpha)+R_1(\nu)\\ &= \mathfrak{S}_0(\nu)+\mathfrak{S}_1(\nu)\sin(\alpha)+\mathfrak{S}_2(\nu)\left(\sin(\alpha)\right)^2+R_2(\nu),\end{aligned}$$

with

$$R_1(\nu) = \sum_{m=2}^{\infty}\mathfrak{S}_m(\nu)\left(\sin(\alpha)\right)^m$$

respectively

$$R_2(\nu) = \sum_{m=3}^{\infty}\mathfrak{S}_m(\nu)\left(\sin(\alpha)\right)^m.$$

Now, $R_1(\nu)$ and $R_2(\nu)$ may be bounded as

$$0 \leq R_1(\nu) = \sum_{m=2}^{\infty}\mathfrak{S}_m(\nu)\left(\sin(\alpha)\right)^m \leq \mathfrak{S}_1(\nu)\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)\sin(\alpha)$$

and

$$0 \leq R_2(\nu) = \sum_{m=3}^{\infty}\mathfrak{S}_m(\nu)\left(\sin(\alpha)\right)^m \leq \mathfrak{S}_2(\nu)\left(2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)-1\right)\sin(\alpha)^2$$

using corollary 5.10, so that

$$\mathfrak{S}_0(\nu) + \mathfrak{S}_1(\nu)\sin(\alpha) \leq C_\nu(\theta)$$

$$\leq \mathfrak{S}_0(\nu) + \mathfrak{S}_1(\nu)\sin(\alpha) + \mathfrak{S}_1(\nu)\left(\zeta\left(\frac{\nu+1}{2}\right) - 1\right)\sin(\alpha)$$

and similarly

$$\begin{aligned}&\mathfrak{S}_0(\nu) + \mathfrak{S}_1(\nu)\sin(\alpha) + \mathfrak{S}_2(\nu)\left(\sin(\alpha)\right)^2 \leq C_\nu(\theta)\\ &\quad \leq \mathfrak{S}_0(\nu) + \mathfrak{S}_1(\nu)\sin(\alpha) + \mathfrak{S}_2(\nu)2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right) - 1\right)\left(\sin(\alpha)\right)^2.\end{aligned}$$

Using now the fact that

$$\mathfrak{S}_0(\nu) = 2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu),$$

$$\mathfrak{S}_1(\nu) = \beta(\nu - 1)$$

and

$$\mathfrak{S}_2(\nu) = -2^{1-\nu}\left(2^{3-\nu} - 1\right)\zeta(\nu - 2),$$

bounds for $\sin(\alpha_r(\nu))$ (and thus for $\theta_r(\nu)$) may be obtained by solving these inequalities when $C_\nu(\theta_r(\nu)) = 0$. In first approximation, we have

$$\frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\zeta\left(\frac{\nu+1}{2}\right)\beta(\nu - 1)} \leq \sin(\alpha_r(\nu)) \leq \frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\beta(\nu - 1)},$$

respectively

$$\begin{aligned}&\frac{\pi}{2} - \arcsin\left(\frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\zeta\left(\frac{\nu+1}{2}\right)\beta(\nu - 1)}\right)\\ &\geq \theta_r(\nu) \geq \frac{\pi}{2} - \arcsin\left(\frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\beta(\nu - 1)}\right)\\ &\Leftrightarrow \arccos\left(\frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\zeta\left(\frac{\nu+1}{2}\right)\beta(\nu - 1)}\right) \geq \theta_r(\nu) \geq \arccos\left(\frac{-2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}{\beta(\nu - 1)}\right).\end{aligned}$$

Using the second approximation, one obtains

$$\begin{aligned}&-\frac{1}{2}\,\frac{-\beta(\nu - 1) + \sqrt{(\beta(\nu - 1))^2 + 4\,2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right) - 1\right)2^{1-\nu}\left(2^{3-\nu} - 1\right)\zeta(\nu - 2)\,2^{-\nu}\left(2^{1-\nu} - 1\right)\zeta(\nu)}}{2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right) - 1\right)2^{1-\nu}\left(2^{3-\nu} - 1\right)\zeta(\nu - 2)}\\ &\qquad \leq \sin(\alpha_r(\nu)),\end{aligned}$$

$$\sin\left(\alpha_r\left(\nu\right)\right) \leq \frac{1}{2}\,\frac{-\beta\left(\nu-1\right)+\sqrt{\left(\beta\left(\nu-1\right)\right)^2+4\,2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)2^{-\nu}\left(2^{1-\nu}-1\right)\zeta\left(\nu\right)}}{2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)},$$

respectively

$$\arccos\left(-\frac{1}{2}\,\frac{-\beta\left(\nu-1\right)+\sqrt{\left(\beta\left(\nu-1\right)\right)^2+4\,2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)2^{-\nu}\left(2^{1-\nu}-1\right)\zeta\left(\nu\right)}}{2^{\frac{\nu+1}{2}}\left(\zeta\left(\frac{\nu+1}{2}\right)-1\right)2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)}\right) \geq \theta_r\left(\nu\right),$$

$$\theta_r\left(\nu\right) \geq \arccos\left(\frac{1}{2}\,\frac{-\beta\left(\nu-1\right)+\sqrt{\left(\beta\left(\nu-1\right)\right)^2+4\,2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)2^{-\nu}\left(2^{1-\nu}-1\right)\zeta\left(\nu\right)}}{2^{1-\nu}\left(2^{3-\nu}-1\right)\zeta\left(\nu-2\right)}\right),$$

which completes the proof. □

With the help of the preceeding theorem 5.12, proposition 4.6 may be improved slightly.

Proposition 5.13: *The derivative* $\frac{\partial C_\nu(\theta)}{\partial \nu}$ *of the generalised Clausen function* $C_\nu(\theta)$ *with respect to order is positive at the zeroes* $\theta_r\left(\nu\right)$ *of* $C_\nu(\theta)$ *for* $\nu > 2.396613412\ldots$*.*

Proof: Write

$$\begin{aligned}
\frac{\partial C_\nu(\theta)}{\partial \nu} &= -\sum_{k=1}^{\infty}\frac{\ln(k)\cos(k\theta)}{k^\nu} = -\sum_{k=2}^{\infty}\frac{\ln(k)\cos(k\theta)}{k^\nu}\\
&= -\ln(n)\sum_{k=2}^{\infty}\frac{\cos(k\theta)}{k^\nu} - \sum_{k=2}^{\infty}\frac{\ln(\frac{k}{n})\cos(k\theta)}{k^\nu}\\
&= -\ln(n)\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^\nu} + \ln(n)\cos(\theta) - \sum_{k=2}^{\infty}\frac{\ln(\frac{k}{n})\cos(k\theta)}{k^\nu}\\
&= -\ln(n)C_\nu(\theta) + \ln(n)\cos(\theta) - \sum_{k=2}^{\infty}\frac{\ln(\frac{k}{n})\cos(k\theta)}{k^\nu}.
\end{aligned}$$

Setting $n = 3$, we have

$$\begin{aligned}
\frac{\partial C_\nu(\theta)}{\partial \nu} &= -\ln(3)C_\nu(\theta) + \ln(3)\cos(\theta) - \sum_{k=2}^{\infty}\frac{\ln(\frac{k}{3})\cos(k\theta)}{k^\nu}\\
&= -\ln(3)C_\nu(\theta) + \ln(3)\cos(\theta) - \frac{\ln(\frac{2}{3})\cos(2\theta)}{2^\nu} - \sum_{k=3}^{\infty}\frac{\ln(\frac{k}{3})\cos(k\theta)}{k^\nu}\\
&= -\ln(3)C_\nu(\theta) + \ln(3)\cos(\theta) + \frac{\ln(\frac{3}{2})\cos(2\theta)}{2^\nu} - \sum_{k=4}^{\infty}\frac{\ln(\frac{k}{3})\cos(k\theta)}{k^\nu}.
\end{aligned}$$

Furthermore

$$\sum_{k=4}^{\infty} \frac{\ln(\frac{k}{3})\cos(k\theta)}{k^{\nu}} \le \sum_{k=4}^{\infty} \frac{\ln(\frac{k}{3})}{k^{\nu}} \le \int_{k=3}^{\infty} \frac{\ln(\frac{k}{3})}{k^{\nu}} = \frac{3^{1-\nu}}{(\nu-1)^2},$$

so that

$$\frac{\partial C_\nu(\theta)}{\partial \nu} \ge -\ln(3)C_\nu(\theta) + \ln(3)\cos(\theta) + \frac{\ln(\frac{3}{2})\cos(2\theta)}{2^{\nu}} - \frac{3^{1-\nu}}{(\nu-1)^2}.$$

Thus, at the zeroes $\theta_r(\nu)$ of $C_\nu(\theta)$,

$$\left.\frac{\partial C_\nu(\theta)}{\partial \nu}\right|_{\theta_r(\nu)} \ge \ln(3)\cos(\theta) + \frac{\ln(\frac{3}{2})\cos(2\theta)}{2^{\nu}} - \frac{3^{1-\nu}}{(\nu-1)^2}.$$

Now, $\ln(3)\cos(\theta) + \frac{\ln(\frac{3}{2})\cos(2\theta)}{2^{\nu}} - \frac{3^{1-\nu}}{(\nu-1)^2}$ is monotonically decreasing from $\ln(3) + \frac{\ln(\frac{3}{2})}{2^{\nu}} - \frac{3^{1-\nu}}{(\nu-1)^2}$ in $[0, \frac{\pi}{2}]$, as

$$\begin{aligned}
&\frac{\partial}{\partial \theta}\left(\ln(3)\cos(\theta) + \frac{\ln(\frac{3}{2})\cos(2\theta)}{2^{\nu}} - \frac{3^{1-\nu}}{(\nu-1)^2}\right)\\
&= -\ln(3)\sin(\theta) - 2^{1-\nu}\ln(\frac{3}{2})\sin(2\theta)\\
&= -2^{-\nu}\sin(\theta)\left(4\ln(\frac{3}{2})\cos(\theta) + 2^{\nu}\ln(3)\right) \le 0
\end{aligned}$$

for all $\theta \in [0, \frac{\pi}{2}]$ and $\nu \ge \frac{\ln(\frac{4\ln(\frac{3}{2})}{\ln(3)})}{\ln(2)}$ and therefore has a single zero

$$\theta_r'(\nu) = \arccos\left(\frac{-\ln(3)2^{\nu} + \sqrt{(\ln(3))^2\, 2^{2\nu} + 8\left(\ln\left(\frac{3}{2}\right)\right)^2 + 8\frac{\ln\left(\frac{3}{2}\right)3^{1-\nu}2^{\nu}}{(\nu-1)^2}}}{4\ln\left(\frac{3}{2}\right)}\right)$$

in this interval, such that $\left.\frac{\partial C_\nu(\theta)}{\partial \nu}\right|_{\theta_r(\nu)} \ge 0$, $\forall \theta \in [0, \theta_r'(\nu)]$. Using now the upper bound from the three term approximation in theorem 5.12 and noting that this upper bound is less than $\theta_r'(\nu)$ for all $\nu > 2.396613412\ldots$ (Figure 1), the proposition is proved.

□

Remark *5.14***:** Again, using the method of the preceeding proposition, the limiting value for ν may be improved by considering more terms of the expansion (i.e. increasing n), but, once more, the number of terms to be considered increases rapidly.

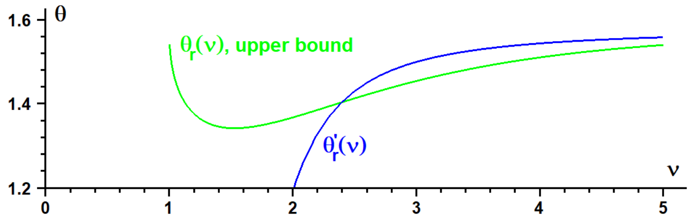

Figure 1. Positivity of $\frac{\partial C_\nu(\theta)}{\partial \nu}$ for $\nu > 2.396613412\ldots$

6. Conclusions

In the present work, several known results about the Bernoulli polynomials have been applied and extended to the generalised Clausen functions of arbitrary real orders $\nu \geq 1$. In the course, improved bounds and asymptotics for the zeroes and extrema have been obtained (Figure 2) and a few new tracks for the study of this class of functions are shown. Finally, we would like to emphasise the importance of further research devoted both to the derivatives of polylogarithms with respect to order, as well as to other functions of a related form.

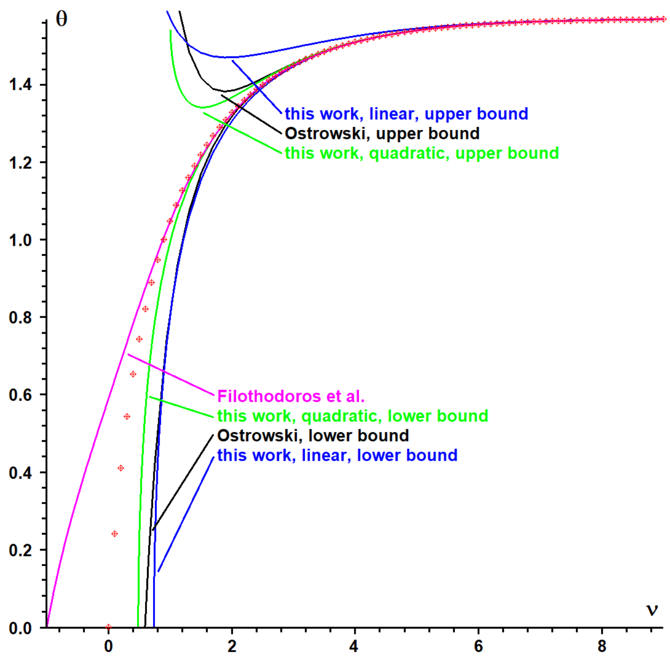

Figure 2. Comparison of the different bounds and exact locations

References

[1] T. Clausen, *Über die Function* $\sin\varphi + \frac{1}{2^2}\sin 2\varphi + \frac{1}{3^2}\sin 3\varphi +$ *etc.*, J. für die Reine und Angew. Math., **8** (1832), 298-300

[2] H. Delange, *Sur les zéros réels des polynômes de Bernoulli*, Ann. Inst. Fourier, **41**, 2 (1991), 267-309

[3] E.G. Filothodoros, A.C. Petkou and N.D. Vlachos, *The fermion-boson map for large d*, Nuclear Physics *B*, **941** (2019), 195-224

[4] D.H. Lehmer, *On the Maxima and Minima of Bernoulli Polynomials*, Am. Math. Monthly, **47**, 8 (1940), 533-538
[5] J. Lense, *Über die Nullstellen der Bernoullischen Polynome*, Monatsh. f. Mathematik und Physik, **41** (1934), 188-190
[6] R.J. Mathar, in *The On-Line Encyclopedia of Integer Sequences*, Sequence A008310, `https://oeis.org/A008310`, 2007
[7] G. Nemes, *Alfréd Rényi Institute of Mathematics*, Personal communication, 2021
[8] N.E. Nørlund, *Mémoire sur les polynômes de Bernoulli*, Acta Math., **43** (1920), 121-196
[9] A.M. Ostrowski, *On the zeros of Bernoulli polynomials of even order*, Enseign. Math., **6** (1960), 27-46
[10] *The On-Line Encyclopedia of Integer Sequences*, Sequence A008956, `https://oeis.org/A008956`
[11] *The On-Line Encyclopedia of Integer Sequences*, Sequence A008955, `https://oeis.org/A008955`.