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On Superalgebras and their Moduli Space

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There is a nice geometric picture of the moduli spaces of associative, Lie, Leibniz algebras of a given dimension, at least in low dimensions. In this talk I show this pattern for Lie superalgebras which are of high interest in physics. There is a stratification of the moduli space by projective orbifolds and singleton algebras. The link between them are defined by jump deformations.

Keywords: Lie superalgebra, Cohomology, Deformation, Moduli space.

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1. Introduction

Superalgebras are mainly used for algebraic description of **supersymmetry** which is a fundamental notion in physics, like in string theory, quantizing relativistic field theories, condensed matter physics, atomic physics, nuclear physics etc.The notion first appeared in deformation theory (late 50's, Frölicher, Nijenhaus [7], Gerstenhaber [8]). In physics (G.L. Stavraki 1966, [14]) it is fundamental to describe transformations (symmetries) connecting *bosons* as commuting variables, and *fermions* as anticommuting variables. Mathematicians in the early 70's (Berezin [2, 3], G.Kac [9], etc.) worked out the precise mathematical background. It made possible to relate the two elementary partices in a symmetry: fermions (with half-integer spin) and bosons (particles with integer value spin). This mathematical framework is based on *group transformations*. The theory rapidly growed, as an attempt to provide a self-consistent quantum theory which unifies all particles and forces in nature.

The first superalgebra appeared in physics. It is the *supersymmetry algebra* (SUSY) [1]

$$\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$$

where $\mathfrak{L}_0$ contains the *Poincaré algebra*, $\mathfrak{L}_1$ consists of spinors of the *Lorentz group*, on which there is an anticommuting relation with values in the even part. So when represented as symmetries of physical fields, they transform fermions to bosons, and conversely, so define a "supersymmetry".

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2. Definitions

2.1. *Superalgebras*

For details see [10–13].

Let A be an algebra. We call A $\mathbb{Z}_2$*-graded* if it can be decomposed into a direct sum of subspaces $A = A_{\bar{0}} \oplus A_{\bar{1}}$ for which

$$A_{\bar{i}}A_{\bar{j}} \subset A_{\bar{i+j}}$$

The elements of $A_{\bar{0}}$ we call *even*, the elements of $A_{\bar{1}}$ *odd*. The degree of an element of A is either 0 or 1, depending where it belongs. We call it a *superalgebra*.

There is a bracket operation in the superalgebra A:

$$[a, b] = ab - (-1)^{deg(a)deg(b)}ba \qquad \text{"}\textit{anticommutator}\text{"}$$

An associative superalgebra also satifies the following identity:

$$[a, bc] = [a, b]c + (-1)^{deg(a)deg(b)}b[a, c] \qquad \text{"}\textit{super associativity}\text{"}$$

Example 1. $\mathbb{N}$-graded space: $\mathbb{N}$*-graded ring of polynomials*: the subspaces are homogeneous elements of degree n.

Example 2. End(V), where $V = V_{\bar{0}} + V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) vector space. The space of endomorphisms, End(V) has the induced $\mathbb{Z}_2$-grading:

$$\text{End}(V) = \text{End}_{\bar{0}}V + \text{End}_{\bar{1}}V.$$

and it becomes an associative superalgebra. Denote them by $l(V) = l(m, n)$, where $m = \text{dim}V_0, n = \text{dim}V_1$. They are the *matrix superalgebras.*

2.2. *Lie superalgebras*

Definition 2.1: A *superalgebra* $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an operation $[,]$ which satifies

(1) $[a, b] = -(-1)^{deg(a)deg(b)}[b, a]$ (*anticommutativity*)
(2) $[a, [b, c]] = [[a, b], c] + (-1)^{deg(a)deg(b)}[b, [a, c]]$ (*super Jacobi identity*)

The dimension of $\mathfrak{g}$ is denoted by $m|n$.

Examples

1. Lie algebras.
2. An associative superalgebra with the induced bracket becomes a Lie superalgebra. The super-Jacobi identity follows from the super-associativity of the bracket. So End$V = l(m, n)$ becomes a Lie superalgebra.

Special classes of Lie superalgebras

- *Solvable Lie superalgebra:* if the derived series terminates in 0:

$$L \leq [L, L] \leq [[L, L], [L, L]] \leq ...$$

- *Nilpotent Lie superalgebra:* if its lower central series terminates in 0:

$$L \leq [L, L] \leq [L, [L, L]] \leq ...$$

 Nilpotent superalgebras are obviously solvable.

- *Semisimple Lie superalgebra*: its solvable radical (maximal solvable ideal) is trivial.

 If a Lie superalgebra is not solvable, then the quotient by the solvable radical is semisimple.

2.3. *Geometry of the moduli space*

Conjecture 2.2 (Fialowski-Penkava)
The moduli space of algebras in dimension n (set of algebra structures on $V = \mathbb{C}^n$ modulo the action of the group $GL(V)$) has a natural decomposition into strata, which are parameterized by projective orbifolds of a very simple type, and some exceptional points.

Each *stratum* is of the form $\mathbb{P}^n/G$, where G is a subgroup of the symmetric group Σ_{n+1}, which acts on the complex projective space $\mathbb{P}^n$ by permuting the projective coordinates. An *orbifold structure* locally looks like the quotient space under the linear action of a finite group.

I am going to present the moduli spaces for Lie superalgebras of total dimension ≤ 4. The construction of the superalgebras is based on extensions, see [6].

2.4. *Extensions*

Let L be a solvable superalgebra. Then it has a codimension 1 ideal, so there is an exact sequence

$$0 \to M \to L \to W \to 0,$$

where M is a $\mathbb{Z}_2$-graded ideal, and W is a 1-dimensional algebra (necessarily trivial); the algebra W is either $1|0$ or $0|1$-dimensional. That means that solvable Lie superalgebras of a fixed dimension $m|n$ can be constructed from solvable superalgebras of dimension $m-1|n$ or dimension $m|n-1$.

If L is not solvable, we have an exact sequence of the form

$$0 \to M \to L \to W \to 0,$$

where W is semisimple and M is the solvable radical.

For both cases, we can describe non-semisimple algebras as extensions of either semisimple or trivial superalgebras by solvable ones. For non-solvable superalgebras the situation is more complicated, but in low dimension, there are not so many complex semisimpe Lie superalgebras. In fact, we have one such case in the study of $3|1$-dimensional simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ by a $0|1$-dimensional (trivial) algebra.

2.5. *One parameter deformations of algebras*

Definition 2.3:

Let L be a Lie superalgebra. Its *one-parameter deformation* is a one-parameter family L_t of Lie superalgebras with the superbracket

$$[,]_t = [,] + t\varphi_1 + t^2\varphi_2 + ...$$

where φ_i are L-valued bilinear, supersymmetric functions, called 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^2 L) = C^2(L; L)$, and L_t is a Lie superalgebra for each $t \in \mathbb{C}$.

For more details see [4].

Remark. For L_t being a superalgebra means that it has to satisfy infinitely many super Jacobi identities.

Special deformations

- *deformation of order n*: the super Jacobi identities are satisfied up to order n.
- *Infinitesimal deformation*: deformations up to order 1.
- *Equivalent deformations*: $L_t \sim L'_t$ if there is an isomorphism between them.
- *Nontrivial deformation*: if $[,]_t \nsim [,]_0$ for all t in some punctured neighborhood of $t = 0$.
- *Jump deformation* if $[,]_t \sim [,]_s$ for all nonzero t, s in some punctured neighborhood of 0.
- *Smooth deformation* if $[,]_t \nsim [,]_s$ for $s \neq t$ for small enough s and t.

2.6. *Structure of the moduli space*

The moduli space is glued together by deformations, which determine the elements that one may deform to locally. Exceptional points refine the picture of how this space is glued together. The different strata are connected by jump deformations.

So deformation theory plays a crucial role in describing the geometry of the moduli space.

2.7. *Cohomology and deformations*

Consider cohomology with adjoint coefficients of Lie superalgebras. Denote the i-th cohomology space by $H^i(L; L)$. The cohomology spaces are also $\mathbb{Z}_2$ graded, and they inherit the grading of the algebra: $dim(H^i) = m|n$.

Meaning of adjont cohomology in low dimensions:

- - $H^0(L; L)$ is the *center* of the algebra;
- - $H^1(L; L)$ is the space of nontrivial *derivations* $\delta : L \to L$ with degree being $s \in \mathbb{Z}_2$, s.t.

$$\delta(ab) = (\delta a)b + (-1)^{|s||a|}a(\delta b).$$

- - $H^2(L; L)$ classifies the *infinitesimal deformations*;
- - H^3 gives the *obstructions* to extending an infinitesimal deformation.

Remark. For deformations, only the even part of the cohomology counts, because we do not consider deformations with a graded commutative base.

3. 1|1-dimensional Lie superalgebras

From now on, let us denote for a Lie superlagebra the basis of the even part by $e_1, ..., e_n$, the odd part of the superalgebra by $f_1, f_2, ..., f_m$.

3.1. *Classification*

For ordinary 2-dimensional Lie algebras (which are the 2|0-dimensional Lie superalgebras), there is only one nontrivial element:

$$[e_1, e_2] = e_2$$

This Lie algebra is solvable but not nilpotent.

Lie superalgebra of dimension 2|0 does not exist.

For 1|1-dimensional Lie superalgebras we have two nontrivial ones (we give the nonzero brackets only):

$$d_1 : [e, f] = f, \qquad d_2 : [f, f] = e$$

The algebra d_1 is an extension of the trivial algebra structure on a 0|1 dimensional space by the trivial algebra on a 1|0-dimensional vectorspace. It is unique up to a constant multiplier.

The algebra d_2 is an extension of the trivial algebra on a 1|0-dimensional space by the trivial algebra on a 0|1-dimensional space. It is also unique up to constant multiplier.

3.2. *Moduli space of* 1|1*-dimensional Lie superalgebras*

In the Table below we give the bidimension h_i of the cohomology spaces $H^i(L; L)$.

Algebra	h_0	h_1	h_2	h_3
d_1	0|0	0|0	**0**|0	0|0
d_2	1|0	0|1	**0**|0	0|0

Table 1. Cohomology of 1|1-dimensional complex Lie superalgebras

The even dimension of H^2 is 0 in both cases, so there are no deformations. The moduli space consists of 2 singleton elements. (We omit the trivial algebra structure.)

4. 2|1-dimensional Lie superalgebras

We can identify them by the following extensions: extending the trivial algebra structure on either a 1|0-dimensional algebra by an algebra structure on a 1|1-dimensional space, or extending the trivial $0|1$-dimensional algebra by an algebra structure on a $2|0$-dimensional space.

4.1. *Classification*

There are only two single algebras and a family of algebras on a $2|1$-dimensional vector space.

- d_1 has nontrivial brackets

$$[e_1, e_1] = 4e_2, \quad [e_1, f] = e_1, \quad [e_2, f] = -2e_2.$$

- d_2 has nontrivial brackets $[e_1, e_1] = 4e_2$.
- The projective family $d_3(p : q)$ has nontrivial brackets

$$[e_1, f] = pe_1, \quad [e_2, f] = qe_2.$$

For the solvable projective family $d_3(p : q)$, $d_3(up : uq) \sim d_3(p : q)$ for all nonzero $u \in \mathbb{C}$ and there are no isomorphisms between $d_3(p : q)$ and $d_3(x : y)$, except for the isomorphisms that give rise to the projective description in our notation.

Algebra	h_0	h_1	h_2	h_3
d_1	$0\|0$	$0\|0$	$\mathbf{0}\|0$	$0\|0$
d_2	$2\|0$	$1\|3$	$\mathbf{1}\|1$	$0\|0$
$d_3(p : q)$	$0\|0$	$0\|1$	$\mathbf{1}\|0$	$0\|0$
$d_3(1 : -2)$	$0\|0$	$0\|1$	$\mathbf{2}\|0$	$0\|1$
$d_3(0 : 0) = 0$				

Table 2. Cohomology of $2|1$-Dimensional Complex Lie superalgebras

Remark. As is usually the case, when there is a family of algebras, there are some special values of the parameters $(p : q)$ where the cohomology or even the deformation picture is different than generically. There is a special element $(0 : 0)$, which is called by algebraic geometers generic element of the projective space, but for us the algebra corresponding to $(0 : 0)$ is never generic in behavior. In this case, $d_3(0 : 0)$ is actually the trivial algebra, which has jump deformations to every nontrivial algebra in the modular space.

4.2. *Moduli space of* $2|1$*-dimensional Lie superalgebras*

Arrows show jump deformations, curly arrow shows smooth deformations along the family.

5. $1|2$-dimensional Lie superalgebras

5.1. *Classification*

There is one solvable projective family $d_1(p : q)$ and three singletons in this moduli space.

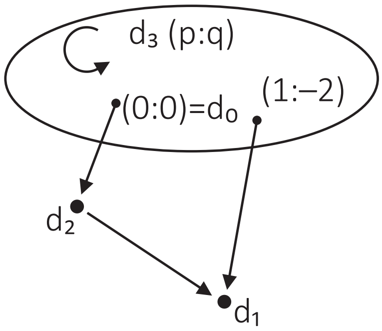

Figure 1. Moduli space of (2|1)-dimensional superalgebras

- $d_1(p:q)$ with nontrivial superbrackets

$$[e, f_2] = pe, \quad [f_1, f_2] = e, \quad [f_1, f_2] = qf_1;$$

- d_2 has nontrivial superbrackets

$$[e, f_2] = e, \quad [f_1, f_2] = f_1;$$

- d_3 has nontrivial superbrackets

$$[e, f_1] = f_2, \quad [e, e] = 2f_2;$$

- d_4 has nontrivial superbracket $[e, e] = 2f_2$.

On the projective family $d_1(p:q)$ there is an action of the symmetric group Σ_2 by permuting the coordinates: $d_1(p:q) \sim d_1(q:p)$ for all $(p:q)$. This means that $d_1(p:q)$ is parametrized by the projective orbifold $\mathbb{P}^1/\Sigma_2$, which is a typical pattern in our moduli spaces.

Algebra	h_0	h_1	h_2	h_3
$d_0 = 0$				
$d_1(p:q)$	0|0	0|1	**1**|0	0|0
$d_1(0:0)$	0|1	1|2	**2**|2	2|2
d_2	0|0	0|3	**3**|0	0|0
d_3	1|0	0|2	**0**|2	0|2
d_4	1|1	1|3	**1**|3	1|3

Table 3. Cohomology of 1|2-dimensional complex Lie superalgebras

5.2. *Moduli space and deformations of* 1|2*-dimensional superalgebras*

The generic element in a family always has jump deformation to every other element in the family and it has at least one more cohomology class ($d_1(0:0)$ is nilpotent). Here d_2 jumps to $d_1(1:1)$ and in its smooth neighborhood.

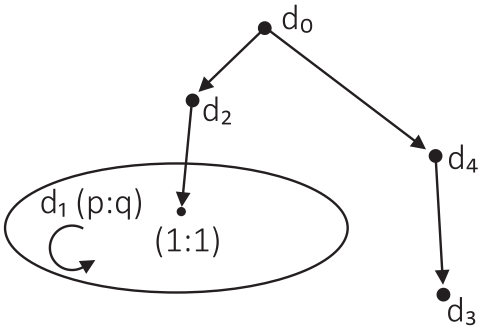

Figure 2. Moduli space of (1|2)-dimensional superalgebras

6. 3|1-dimensional Lie superalgebras

Among these we have a non-nilpotent element, given by an extension of the simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ by a 0|1-dimensional trivial algebra (d_1 in the list). Except of this, all other algebras are solvable and we can get them by extensions of the trivial algebra on the space 1|0 by an algebra structure on a 2|1-dimensional space (four such algebras), and of the trivial 0|1 dimensional space by an algebra structure on a 3|0-dimensional space (three such Lie algebras).

6.1. *Classification of* 3|1*-dimensional Lie superalgebras*

There are four singletons, two nonisomorphic 2-parameter families, one 3-parameter projective family.

d_1 $[e_2,e_3]=f,\ [e_2,f]=e_3,\ [e_3,f]=e_2$

$d_2(p:q)$ $[e_1,e_1]=8e_2,\ [e_1,f]=pe_1,\ [e_2,f]=-2pe_2,$
$[e_3,f]=e_2,\ [e_3,f]=qe_3$

d_3 $[e_1,e_1]=8e_2,\ [e_1,f]=e_1,$
$[e_2,f]=-2e_2,\ [e_3,f]=-2e_3$

d_4 $[e_1,e_1]=8e_2;$

d_5 $[e_2,e_3]=e_2,\ [e_1,f]=e_1$

$d_6(p:q:r)$ $[e_1,f]=pe_1,\ [e_2,f]=qe_2,\ [e_3,f]=e_2+re_3$

$d_7(p:q)$ $[e_1,f]=pe_1,\ [e_2,f]=qe_2,\ [e_3,f]=qe_3.$

Special subfamilies and special points

There are some special subfamilies of $d_6(p:q:r)$. This family is parametrized by $\mathbb{P}^2/\Sigma_2$, where the action of Σ_2 on $\mathbb{P}^2$ is given by interchanging the second two coordinates; there are special $\mathbb{P}^1$s for which the cohomology does not follow the generic pattern. We only included in our Table those subfamilies which generate new deformations.

There are also special points in the families, and again, some of them have new deformations.

Remark. If any element of a family has a deformation to some algebra, then the generic element also has such a defor mation (follows from transitivity of deformations).

Algebra	h_0	h_1	h_2	h_3
d_1	$0\|1$	$0\|1$	$\mathbf{0}\|1$	$1\|1$
$d_2(p:q)$	$0\|0$	$0\|1$	$\mathbf{1}\|0$	$0\|0$
$d_2(0:0)$	$1\|0$	$2\|4$	$\mathbf{4}\|4$	$2\|1$
d_3	$0\|0$	$0\|2$	$\mathbf{2}\|0$	$0\|0$
d_4	$3\|0$	$2\|7$	$\mathbf{5}\|4$	$2\|1$
d_5	$0\|0$	$0\|0$	$\mathbf{0}\|0$	$0\|0$
$d_6(p:q:r)$	$0\|0$	$0\|2$	$\mathbf{2}\|0$	$0\|0$
$d_6(p:q:0)$	$1\|0$	$0\|3$	$\mathbf{3}\|0$	$0\|1$
$d_6(1:-2:0)$	$1\|0$	$0\|3$	$\mathbf{4}\|0$	$0\|3$
$d_6(p:q:-2p)$	$0\|0$	$0\|2$	$\mathbf{3}\|0$	$0\|1$
$d_6(0:1:-1)$	$0\|1$	$1\|2$	$\mathbf{3}\|3$	$4\|4$
$d_6(0:0:0)$	$1\|1$	$3\|5$	$\mathbf{8}\|7$	$9\|9$
$d_7(p:q)$	$0\|0$	$0\|4$	$\mathbf{4}\|0$	$0\|0$
$d_7(1:0)$	$2\|0$	$0\|6$	$\mathbf{6}\|0$	$0\|2$
$d_7(1:-2)$	$0\|0$	$0\|4$	$\mathbf{6}\|0$	$0\|2$
$d_7(0:0)=0$				

Table 4. Cohomology of 3|1-dimensional complex Lie superalgebras

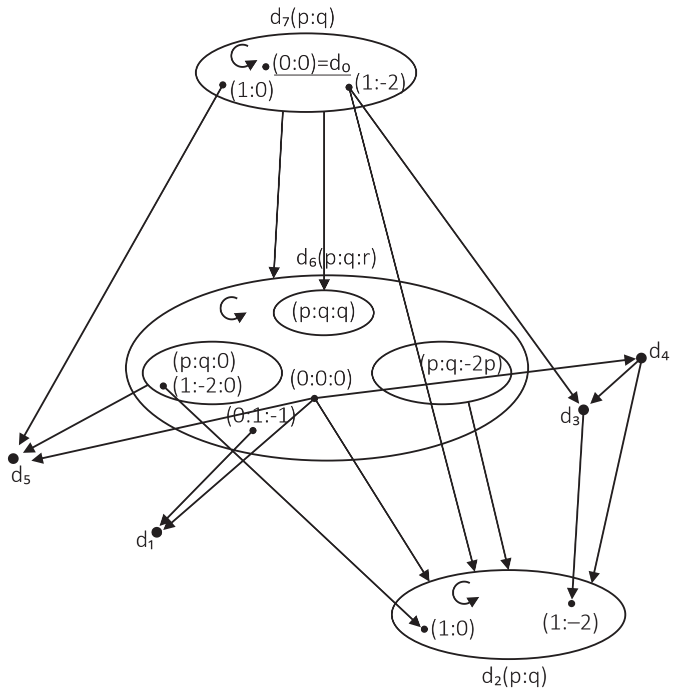

Figure 3. Moduli space of (3|1) dimensional superalgebras

6.2. *Moduli space and deformations of* 3|1*-dimensional superalgebras*

The algebras d_1 and d_5 are rigid.

$d_2(p:q)$ generically only deforms along the family, except $d_2(0:0)$ has jump deformations to all the other elements of the family.

The algebra d_3 jumps to $d_2(1:-2$ and also deforms smoothly in a neighborhood of $d_2(1:-2)$.

d_4 jumps to the entire family $d_2(x:y)$ and to d_3.

The 2-parameter family $d_6(p:q:r)$ generically deforms only along the family, but $d_6(p:q:0)$ also jumps to d_5, and the subfamily $d_6(p:q:-2p)$ also jumps to $d_2(p:q)$ and deforms in a neighborhood of $d_2(p:q)$. If some special points belong to a special subfamily, they take part in every extra deformaton which the subfamily has. In this 2-parameter family we have symmetry in the last 2 coordinates.

The family $d_7(p:q)$ is parameterized by $\mathbb{P}^1$, with no action of the symmetric group. It generically has jump deformations to $d_6(p:q:q)$ and deforms in a neighborhood ot it, and of $d_7(p:q)$. There are 2 special points of this family: $d_7(1:0)$ which also jumps to d_5, and $d_7(1:-2)$, which jumps to the entire family $d_2(x:y)$ and to d_3.

The algebra d_4 deforms to d_3 and to the family $d_2(p:q)$, while d_3 deforms to $d_2(1:-2)$.

7. 2|2-dimensional Lie superalgebras

This is the most complicated of the moduli spaces we consider. The algebras are extensions of the trivial algebra structure on the 0|1-dimensional space by an algebra structure on a 2|1-dimensional space and extensions of the trivial algebra on the 1|0-dimensional space by an algebra structure on a 1|2-dimensional space.

7.1. *Classification*

There are six singleton algebras, four 2-parameter projective families and one 3-parameter family.

d_1 $[e_1,f_1]=e_1,\ [e_2,f_1]=e_1+2e_2,\ [e_2,f_2]=e_1+e_2;$

d_2 $[e_1,e_2]=f_1+f_2,\ [e_1,e_1]=4f_1,\ [e_2,e_2]=2f_2;$

d_3 $[e_1,e_2]=f_1+f_1,\ [e_1,e_1]=4f_1,$
$[e_1,f_2]=2e_2,\ [e_2,f_2]=e_2,$
$[f_1,f_2]=-f_1-f_2,\ [e_2,f_1]=-e_2;$

d_4 $[e_1,e_1]=8f_1+2f_2,\ [e_1,e_2]=f_2;$

$d_5(p:q)$ $[e_2,f_1]=e_1,\ [e_1,f_2]=(p-q)e_1,$
$[e_2,f_2]=pe_2,\ [f_1,f_2]=qf_1;$

$d_6(p:q)$ $[e_1,e_2]=f_1,\ [e_1,e_1]=4f_1,$
$[e_1,f_2]=qe_1+2(p-q)e_2,\ [e_2,f_2]=pe_2,$
$[f_1,f_2]=-(p+q)f_1;$

$d_7(p:q)$ $[e_1,e_1]=4f_1,\ [e_1,f_2]=pe_1+e_2,$
$[e_2,f_2]=qe_2,\ [f_1,f_2]=-2pf_1;$

d_8 $[e_1,e_1]=4f_1,\ [e_1,f_2]=e_1,\ [e_2,f_2]=e_2,\ [f_1,f_2]=-2f_1;$

d_9 $[e_1,e_1]=4f_1;$
$d_{10}(p:q:r)$ $[e_1,f_2]=pe_1,\ [e_2,f_2]=e_1+qe_2,\ [f_1,f_2]=rf_1;$

$d_{11}(p:q)$ $[e_1,f_2]=pe_1,\ [e_2,f_2]=pe_2,\ [f_1,f_2]=qf_1.$

The generic element of each family is nilpotent, and beside those, d_2, d_4 and d_9 are also nilpotent. An interesting phenomena is that $d_5(0:0)\sim d_10(0:0:0)$.

Algebra	h_0	h_1	h_2	h_3
d_1	$0\|0$	$0\|0$	$\mathbf{0}\|0$	$0\|0$
d_2	$2\|0$	$2\|2$	$\mathbf{0}\|2$	$0\|0$
d_3	$0\|0$	$1\|0$	$\mathbf{0}\|1$	$0\|0$
d_4	$2\|0$	$2\|3$	$\mathbf{2}\|3$	$2\|2$
$d_5(p:q)$	$0\|0$	$0\|1$	$\mathbf{1}\|0$	$0\|0$
$d_6(p:q)$	$0\|0$	$0\|1$	$\mathbf{1}\|0$	$0\|0$
$d_7(p:q)$	$0\|0$	$0\|1$	$\mathbf{1}\|0$	$0\|0$
d_8	$0\|0$	$0\|2$	$\mathbf{3}\|0$	$0\|1$
d_9	$2\|1$	$4\|5$	$\mathbf{6}\|6$	$6\|6$
$d_{10}(p:q:r)$	$0\|0$	$0\|2$	$\mathbf{2}\|0$	$0\|0$
$d_{11}(p:q)$	$0\|0$	$0\|4$	$\mathbf{4}\|0$	$0\|0$
$d_{11}(0:0)=0$				

Table 5. Cohomology of 2|2-dimensional complex Lie superalgebras

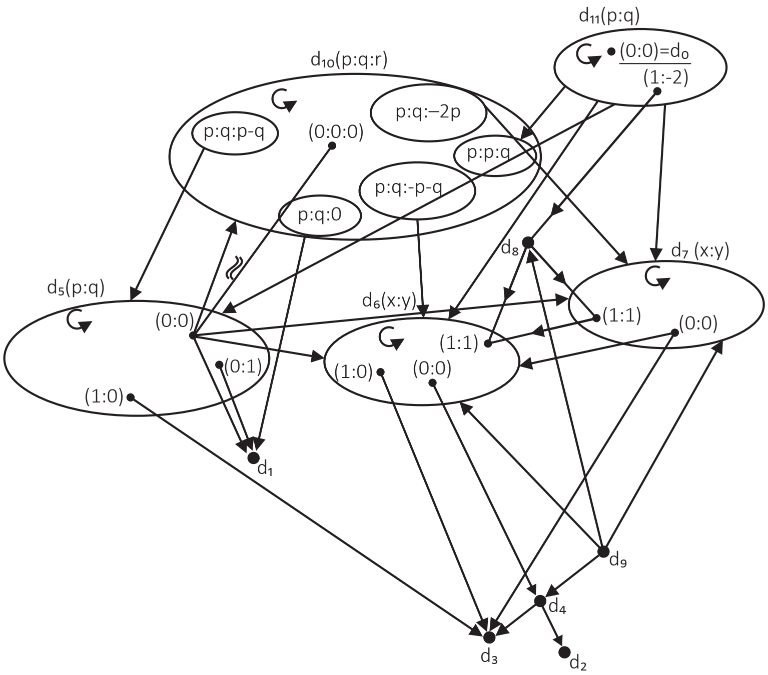

Figure 4. Moduli space of (2|2) dimensional superalgebras

7.2. *Moduli space and deformations of 2|2-dimensional Lie superalgebras*

Here d_1, d_2, d_3 are rigid, d_4 jumps to d_2 and d_3.

The family $d_5(p:q)$ is parameterized by $\mathbb{P}^1$, with no action of the symmetric group. Generically, they deform along the family, but $d_5(1:0)$ deforms also to d_3, and $d_5(0,1)$ to d_1. $d_5(0:0)$ is isomorphic to $d_10(0:0:0)$.

The family $d_6(p:q)$ is parameterized by $\mathbb{P}^1/\Sigma_2$, Σ_2 permutes the coordinates. Generically, a point has deformations in a neighborhood of the point. Some exceptional points are $d_6(1:0)$, which jumps to d_3, and $d_6(0,0)$ which jumps to d_4 and d_2.

The family $d_7(p:q)$ is parameterized by $\mathbb{P}^1$, with no action of a symmetric group. Generically, elements deform along this family. Special points are $d_7(1:1)$, which deforms in the neighborhood of $d_6(1:1)$, but does not jump to it. The element $d_7(0:0)$ jumps to d_3, $d_6(x:y)$ except $(0:0)$, and to $d_7(x:y)$ except $(0:0)$.

The 2-parameter family $d_{10}(p:q:r)$ is parameterized by $\mathbb{P}^2/\Sigma_2$, where Σ_2 permutes the first two coordinates. Generically points deform along the family. There are special subfamilies with unusual deformation picture: $d_{10}(p:q:p-q)$ deforms in a neighborhood of $d_5(p:p-q)$, but does not jump there; $d_{10}(p:q:0)$ jumps to d_1, $d_{10}(p:q:-2p)$ jumps to $d_7(p:q)$, $d_{10}(p:q:-p-q)$ jumps to $d_6(p:q)$ and deforms in a neighborhood of this point. There are also special points, which I did not mark in the Figure.

The family $d_{11}(p:q)$ jumps to $d_10(p:p:q)$, $d_5(p:q)$, $d_6(p:q)$, $d_7(p:q)$, the point $d_{11}(1:-2)$ jumps to d_8, and $d_{11}(0:0)$ is the trivial algebra.

8. 1|3-dimensional Lie superalgebras

Consider a 1|0-dimensional and a 0|3-dimensional vectorspace. There is no nontrivial 0|3-dimensional superalgebra, so the algebras arising in this manner are given by the Jordan decomposition of 3×3 matrices. We get 3 cases

$$\begin{bmatrix} p & 1 & 0\\ 0 & q & 1\\ 0 & 0 & r\end{bmatrix},\quad \begin{bmatrix} p & 0 & 0\\ 0 & p & 1\\ 0 & 0 & q\end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The fourth is the zero matrix and gives the trivial algebra.

The other possible decomposition is by the 0|1-dimensional space and a 1|2-dimensional space.

8.1. *Classification of* 1|3*-dimensional Lie superalgebras*

We get 4 singletons, one 2-parameter, and one 3-parameter family.

$d_1(p:q:r)\ \ [e,f_3]=pe,\ [f_1,f_3]=e+qf_1,\ [f_2,f_3]=f_1+rf_2;$

$d_2(p:q)\ \ [e,f_3=pe,\ [f_1,f_3]=pf_1,\ [f_2,f_3]=f_1+qf_2;$

$d_3\ \ [e,f_3]=4e,\ [f_1,f_3]=f_1,\ [f_2,f_3]=f_2;$

$d_4\ \ [e,e]=4f_3,\ [f_1,f_2]=f_3;$

$d_5\ \ [e,f_1]=f_3,\ [e,e]=4f_3;$

$d_6\ \ [e,e]=4f_3$

8.2. *Moduli space and deformations of* 1|3*-dimensional Lie superalgebras*

The family of algebras $d_1(p:q:r)$ is parameterized by $\mathbb{P}^2/\Sigma_3$, where Σ_3 acts by permuting the coordinates. There are a lot of special subfamilies and special points, for which the cohomology is not generic. However, none of these special cases, except the generic point $d_1(0:0:0)$ give rise to any extra deformations. The generic element $d_1(0:0:0)$ has jump deformation to all elements in the family except itself.

The family $d_2(p:q)$ is parameterized by $\mathbb{P}^1$, with no action of a symmetric group. Generically an element $d_2(p:q)$ has a jump deformation to $d_1(p:p:q)$, and smooth

Algebra	h_0	h_1	h_2	h_3
$d_1(p:q:r)$	$0\|0$	$0\|0$	$\mathbf{2}\|0$	$0\|0$
$d_2(p:q)$	$0\|0$	$0\|4$	$\mathbf{4}\|0$	$0\|0$
d_3	$0\|0$	$0\|8$	$\mathbf{8}\|0$	$0\|0$
d_4	$1\|0$	$0\|4$	$\mathbf{0}\|8$	$0\|12$
d_5	$1\|1$	$1\|5$	$\mathbf{1}\|9$	$1\|13$
d_6	$1\|2$	$2\|7$	$\mathbf{3}\|12$	$4\|17$

Table 6. Cohomology of 1|3-dimensional complex Lie superalgebras

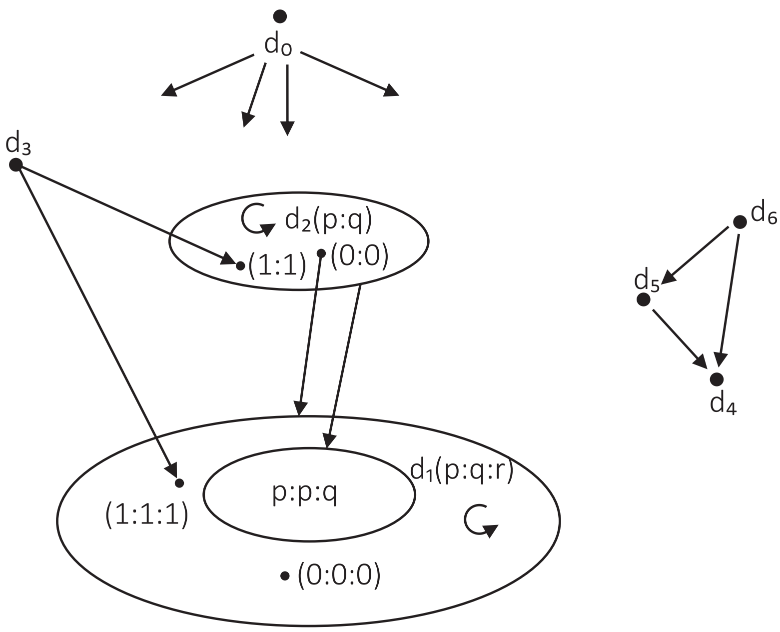

Figure 5. Moduli space of 1|3-dimensional superalgebras

deformations in the neighborhoods of $d_1(p:p:q)$ and $d_2(p:q)$. Again, there are special points, but no extra deformations, because the elements already deform to everything theu could. The exception is $d_2(0:0)$, which has jump deformations to $d_1(x:y:z)$ for all $(x:y:z)$ and $d_2(x:y)$ for all $(x:y)$ except $(0:0)$.

The algebra d_3 has jump deformations to $d_1(1:1:1)$ and $d_2(1:1)$, as well as smooth deformations in a neighborhood of these points. The description of the deformation picture of the first three algebras corresponds exactly to the description of the moduli space of 3×3 matrices with the action given by conjugation by an element in $\mathbf{G}L(3,\mathbb{C})$ and multiplicaton by a nonzero complex number.

The algebra d_4 is rigid. The algebra d_5 has jump deformation to d_4, and d_6 has jump deformations to d_4 and d_5.

References

[1] J. Bagger, J. Wess, *Supersymmetry and Supergravity*, Pronceton Univ. Press, 2-nd ed. 1992
[2] F.A. Berezin, *General concept of quantization*, Comm. Math. Physics **40**, 2 (1975), 153-174
[3] F.A. Berezin, *Introduction to Superanalysis*, Springer, Math. Physics and Applied Math. 1987
[4] A. Fialowski, *Deformations of Lie algebras*, Sbornyik USSR, **127** (1985), 476-482
[5] A. Fialowski, M. Penkava, *Moduli spaces of low dimensional Lie superalgebras*, Int. Journal of Math., **32**, 9 (2021), 2150059

[6] A. Fialowski, M. Penkava, *Extensions of (super) Lie algebras*, Communications in Contemporary Mathematics **11** (2009), 709-737
[7] A. Frölicher, T. Kobayashi, A. Nijenhuis, *Deformation theory of complex manifolds*, Book, Seattle, Washington University, 1953
[8] M. Gerstenhaber, *On the deformations of Rings and Algebras*, Annales of Math. **79** (1964), 59-103
[9] G. Kac, *Arithmetic properties of ring groups*, Funct. Anal. Appl. **6** (1972), 158-160
[10] V. Kac, *Lie superalgebras*, Advances in Mathematics **26**, 1 (1977), 8-96
[11] V. Kac, *The Story of Superalgebras*, In: Concise Encyclopedia of Supersymmetry. Springer, Dordrecht, 2004
[12] V. Kac, *Some algebras related to quantum field theory*, in: Ith All-Union Algebra Coll. Kishinev, 1971
[13] M. Scheunert, *Theory of Lie Superalgebras. An introduction*, Lecture Notes in Mathematics, vol. 716, Springer Verlag, 1979
[14] G. Stavraki, *Some Non-Local Model of Field Self Interactions and the Algebra of Field Operators*, Proc. HEP School, Yalta (Ukraine), 1966.