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(C,α) Summability of General Dirichlet's Integrals

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As is well known, Taberski has investigated $(C,1)$ summability issue for general Dirichlet's integrals, and proved theorem about sufficient conditions for the uniform convergence of $(C,1)$ means of the general Dirichlet's integrals. We have generalised this theorem to (C,α) $(0 < \alpha < 1)$ means. For this, it was needed to represent Dirichlet's kernel in a convenient form.

In this paper the form of representation of (C,α) kernels is also obtained, that allows you to use second mean value theorem, while integration on the finite intervals. On the other hand, that makes it possible to think about generalisation of theorems on $(C,1)$ means, to (C,α) means.

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1. Introduction

The purpose of the paper represents to improve results on the convergence of the general Dirichlet integrals, obtained previously. As we know, in 1973 Roman Taberski [1] published a paper on the sumability of some trigonometric sums, (see, also, [2]). In this paper he had proved the theorem on the uniform convergence of $(C,1)$ means of the general Dirichlet integrals. Similar theorems for (C,α) means $(0 < \alpha < 1)$ have not previously been investigated. Our goal is exactly to consider this issue. In particular, in the given paper the theorems on sufficient conditions for uniform convergence of (C,α) means of general Dirichlet's integrals are established and proved.

At the beginning, we need some definition.

Definition 1.1: Let, E be a class of functions $f(t)$ that are Lebesgue integrable on every finite interval, and

$$\frac{1}{T}\int_{T}^{T+c}|f(t)|dt = o(1), \quad \text{and} \quad \frac{1}{T}\int_{-T-C}^{-T}|f(t)|dt = o(1),$$

as $T \to \infty$, for all fixed $c > 0$.

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Suppose, $G_n^\alpha(t)$ denotes (C,α) kernels and $\sigma_n^\alpha(x)=\sigma_n^\alpha(x;f)$ denotes (C,α) means for $S[f]$ [3]. Then

$$G_n^\alpha(t)=\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}D_\nu(t)/A_{n,}^\alpha,$$

$$\sigma_n^\alpha(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)G_n^\alpha(t)dt,$$

$$\sigma_n^\alpha(x)-f(x)=\frac{2}{\pi}\int_0^{\pi}\phi_x(t)G_n^\alpha(t)dt.$$

Before moving to main theorems, we emphasize important equalities and inequalities, which will be used in the proof of those theorems, and we will show their correctness.

$$A_n^\alpha=\frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}=\left(\frac{\alpha}{n}+1\right)\left(\frac{\alpha}{n-1}+1\right)\cdots(\alpha+1)$$
$$=\frac{n^\alpha}{\Gamma(\alpha+1)}\left(1+O\left(\frac{1}{n}\right)\right).$$

For all $i\in N$ we have

$$\left|D_i^l(t)\right|=\left|\frac{\sin(2i+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}}\right|\le\frac{(2i+1)\frac{\pi t}{2l}}{2\frac{t}{l}}=\frac{(2i+1)\cdot\frac{1}{2}}{\frac{4}{\pi}}\le n+\frac{1}{2},$$

because, if

$$|t|\le l,\qquad\left|\sin\frac{\pi t}{2l}\right|\ge\frac{|t|}{l}.$$

By the last inequality, for Dirichlet's (C,α) kernels we have

$$\left|G_n^{l,\alpha}(t)\right|\le\frac{1}{\left(2\sin\frac{\pi t}{2l}\right)^{\alpha+1}}\cdot\frac{1}{A_n^\alpha}+\frac{2A_{n+1}^{\alpha-1}}{A_n^\alpha}\cdot\frac{1}{\left(2\sin\frac{\pi t}{2l}\right)^2}$$
$$\leqslant B_\alpha\left(n^{-\alpha}\left(\frac{t}{l}\right)^{-\alpha-1}+n^{-1}\left(\frac{t}{l}\right)^{-2}\right)$$
$$=B_\alpha\left(n^{-\alpha}t^{-(\alpha+1)}l^{\alpha+1}+n^{-1}t^{-2}l^2\right)\le 2B_\alpha h^{-\alpha}\left(\frac{t}{l}\right)^{-\alpha-1},$$

where B_α is constant only depending on α,
if $\frac{nt}{l}\geqslant 1$ and $0<\alpha<1$

$$n\left(\frac{t}{l}\right)^2=\left(n\frac{t}{l}\right)^{1-\alpha}n^\alpha\left(\frac{t}{l}\right)^{\alpha+1}\ge n^\alpha\left(\frac{t}{l}\right)^{\alpha+1}.$$

Also, on $[\delta, l]$ interval we have another inequality

$$\max_{\delta\le t\le l}\left|G_n^{l,\alpha}(t)\right| \le 2B_\alpha n^{-\alpha}\left(\frac{\delta}{l}\right)^{-\alpha-1} = \frac{2B_\alpha l^{\alpha+1}}{\delta^{\alpha+1}}\cdot\frac{1}{n^\alpha} = C_\alpha\left(\frac{l}{n}\right)^\alpha\cdot l.$$

Let's prove correctness of the following inequality. Before that, we have to emphasize, the symbol $\oint$ represents an imaginary part of a complex number.

$$\oint \frac{e^{i(n+\frac{1}{2})t}}{A_n^\alpha(2\sin\frac{t}{2})}\left[\frac{1}{(1-e^{-it})^\alpha} - A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{1-e^{-it}} - \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-1}\frac{e^{-i(v+1)t}}{1-e^{-it}}\right]$$
$$= \frac{1}{A_n^\alpha}\frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}\alpha\right)t-\frac{\pi\alpha}{2}\right]}{\left(2\sin\frac{t}{2}\right)^{\alpha+1}} + \frac{\alpha}{(n+1)\left(\sin\frac{t}{2}\right)^2} + \frac{2\theta\alpha(1-\alpha)}{(n+1)(n+2)\left(2\sin\frac{t}{2}\right)^3}.$$

If we take out $e^{\frac{-it}{2}}$ as a multiplier from denominator, it is easy to see

$$\oint \frac{e^{-i\frac{t}{2}}}{2\sin\frac{t}{2}\,(1-e^{-it})}\cdot\frac{A_{n+1}^{\alpha-1}}{A_n^\alpha} = \frac{\alpha}{n+1}\cdot\frac{1}{\left(2\sin\frac{t}{2}\right)^2},$$

and

$$\oint e^{-i\left(-(n+1)t+\frac{\pi}{2}\right)}\cdot e^{-it}\sum_{k=m}^{n}\left(e^{-it}\right)^k = \oint e^{-i\left(-nt+\frac{\pi}{2}\right)}\sum_{k=m}^{n}\left(e^{-it}\right)^k$$
$$= \oint e^{-i\left((m-n)t+\frac{\pi}{2}\right)}\sum_{k=0}^{n-m}\left(e^{-it}\right)^k = \oint e^{\left(-i(m-n)t+\frac{\pi}{2}\right)}\cdot\frac{e^{-i\frac{tn}{2}}\cdot\sin(n-m+1)\frac{t}{2}}{\sin\frac{t}{2}}$$
$$= \oint e^{-i\left((m-\frac{n}{2})t+\frac{\pi}{2}\right)}\frac{\sin(n-m+1)\frac{t}{2}}{\sin\frac{t}{2}}$$
$$= \frac{\sin\left(\left(\frac{n}{2}-m\right)t-\frac{\pi}{2}\right)\sin(n-m+1)\frac{t}{2}}{\sin\frac{t}{2}} =: \mathrm{S}_{m,n}(t).$$

By the Abel's transform for partial sums we have

$$\oint \frac{e^{(n+\frac{1}{2})t}}{1-e^{-it}}\sum_{v=n+1}^{m} e^{-i(v+1)t}A_{v+1}^{\alpha-2} = \frac{1}{2\sin\frac{1}{2}}\left[\sum_{v=n+1}^{m-1} S_{n+1,v}(t)A_{v+2}^{\alpha-3} + S_{n+1,m}(t)A_{m+1}^{\alpha-2}\right],$$

where $-1<\alpha<1,\ m\ge n$. Besides,

$$\lim_{m\to\infty} S_{m,n+1}(t)\frac{A_{m+1}^{\alpha-2}}{A_n^\alpha} = \lim_{m\to\infty}\frac{\sin\left(\left(\frac{n}{2}-m\right)t-\frac{\pi}{2}\right)\sin(n-m+1)\frac{t}{2}}{\sin\frac{t}{2}}$$
$$\times\frac{(m+1)^{\alpha-2}}{n^\alpha}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)} = 0.$$

At the same time

$$-\sum_{v=n+1}^{m-1} A_{v+2}^{\alpha-3} \le \sum_{v=n+1}^{m-1} \sin\frac{t}{2} S_{n+1,v}(t) A_{v+2}^{\alpha-3} \le \sum_{v=n+1}^{m-1} A_{v+2}^{\alpha-3}$$

$$-\sum_{v=n+3}^{m+1} A_{v}^{\alpha-3} \le \sum_{v=n+3}^{m+1} \sin\frac{t}{2} S_{n+1,v-2}(t) A_{v}^{\alpha-3} \le \sum_{v=n+3}^{m+1} A_{v}^{\alpha-3}$$

$$-\left(A_{m+1}^{\alpha-2} - A_{n+2}^{\alpha-2}\right) \le \sum_{v=n+3}^{m+1} \sin\frac{t}{2} S_{n+1,v-2}(t) A_{v}^{\alpha-3} \le \left(A_{m+1}^{\alpha-2} - A_{n+2}^{\alpha-2}\right).$$

Correspondingly, there exists θ, $|\theta| \le 1$ such that

$$\sum_{v=n+3}^{m+1} \sin\frac{t}{2} S_{n+1,v-2}(t) A_{v}^{\alpha-3} = \frac{\theta}{\sin\frac{t}{2}} \left(A_{m+1}^{\alpha-2} - A_{n+2}^{\alpha-2}\right). \tag{1}$$

If we take a limit as $m \to \infty$ in an equality (1), we will get

$$\sum_{v=n+3}^{\infty} S_{n+1,v-2}(t) A_{v}^{\alpha-3} = -\frac{\theta}{\sin\frac{t}{2}} A_{n+2}^{\alpha-2}.$$

If we take into account that

$$\frac{A_{n+2}^{\alpha-2}}{A_{n}^{\alpha}} = \frac{(\alpha-1)\alpha}{(n+1)(n+2)},$$

we will have

$$\oint -\frac{e^{i\left(n+\frac{1}{2}\right)t}}{A_{n}^{\alpha}\left(2\sin\frac{t}{2}\right)} \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-2} \frac{e^{i(v+1)t}}{1-e^{-it}} = \frac{2\theta(\alpha-1)\alpha}{(n+1)(n+2)\left(2\sin\frac{t}{2}\right)^{3}}, \quad |\theta| \le 1.$$

Let $\frac{\pi t}{l} := t$, then

$$\oint -\frac{e^{i\left(n+\frac{1}{2}\right)\frac{\pi t}{l}}}{A_{n}^{\alpha}\left(2\sin\frac{\pi t}{2l}\right)} \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-2} \frac{e^{i(v+1)\frac{\pi t}{l}}}{1-e^{-i\frac{\pi t}{l}}} = \frac{2\theta(\alpha-1)\alpha}{(n+1)(n+2)\left(2\sin\frac{\pi t}{2l}\right)^{3}}, \quad |\theta| \le 1.$$

Taking into account that

$$\frac{1}{\left(1-e^{-it}\right)^{\alpha}} = \frac{1}{e^{-i\frac{t\alpha}{2}+i\frac{\pi\alpha}{2}}\left(2\sin\frac{t}{2}\right)^{\alpha}} = \frac{e^{i\left(\frac{t\alpha}{2}-\frac{\pi\alpha}{2}\right)}}{\left(2\sin\frac{t}{2}\right)^{\alpha}},$$

we will get

$$\oint \frac{e^{i(n+\frac{1}{2})t}}{2A_n^\alpha \sin\frac{1}{2}t}\left(1-e^{-it}\right)^{-\alpha} = \frac{e^{i(n+\frac{1}{2})t}}{2A_n^\alpha \sin\frac{1}{2}t}\cdot\frac{e^{i\left(\frac{t\alpha}{2}-\frac{\pi\alpha}{2}\right)}}{\left(2\sin\frac{t}{2}\right)^\alpha}$$
$$= \frac{1}{A_n^\alpha}\frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}\alpha\right)t-\frac{1}{2}\pi\alpha\right]}{\left(2\sin\frac{1}{2}t\right)^{\alpha+1}}$$

that ends the proof.

Therefore, if we denote $\frac{\pi t}{l} := t$ we get

$$\oint \frac{e^{i(n+\frac{1}{2})\frac{\pi t}{l}}}{A_n^\alpha(2\sin\frac{\pi t}{2l})}\left[\frac{1}{1-e^{-i\frac{\pi t}{l})^\alpha}} - A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)\frac{\pi t}{l}}}{1-e^{-i\frac{\pi t}{l}}} - \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-1}\frac{e^{-i(v+1)\frac{\pi t}{l}}}{1-e^{-i\frac{\pi t}{l}}}\right]$$
$$= \frac{1}{A_n^\alpha}\frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}\alpha\right)\frac{\pi t}{l}-\frac{\pi\alpha}{2}\right]}{\left(2\sin\frac{\pi t}{2l}\right)^{\alpha+1}} + \frac{\alpha}{(n+1)\left(\sin\frac{\pi t}{2l}\right)^2} + \frac{2\theta\alpha(1-\alpha)}{(n+1)(n+2)\left(2\sin\frac{\pi t}{2l}\right)^3}.$$

Later we will show that

$$G_n^\alpha(t) =$$
$$\oint \left\{\frac{e^{i(n+\frac{1}{2})t}}{A_n^\alpha\left(2\sin\frac{1}{2}t\right)}\left[\frac{1}{(1-e^{-it})^\alpha} - A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{1-e^{-it}} - \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)t}}{1-e^{-it}}\right]\right\}. (2)$$

Before this, lets prove

$$G_n^\alpha(t) = \frac{1}{2A_n^\alpha\sin\frac{1}{2}t}\oint\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}e^{i(\nu+\frac{1}{2})t} = \oint\frac{e^{i(n+\frac{1}{2})t}}{2A_n^\alpha\sin\frac{1}{2}t}\sum_{\nu=0}^{n}A_\nu^{\alpha-1}e^{-i\nu t}$$
$$= \oint\left\{\frac{e^{i(n+\frac{1}{2})t}}{2A_n^\alpha\sin\frac{1}{2}t}\left[\left(1-e^{-it}\right)^{-\alpha} - \sum_{\nu=n+1}^{\infty}A_v^{\alpha-1}e^{-i\nu t}\right]\right\}$$
$$= \frac{1}{A_n^\alpha}\frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}\alpha\right)t-\frac{1}{2}\pi\alpha\right]}{\left(2\sin\frac{1}{2}t\right)^{\alpha+1}} + \frac{2\theta\alpha}{n\left(2\sin\frac{1}{2}t\right)^2} \quad (|\theta|\leqslant 1),$$

the last equality of the sequences of equalities.

As is well known, Abel's transform formula for partial sums is the following equation

$$\sum_{k=m}^{N} a_k b_k = \sum_{k=m}^{N-1} A_k(b_k - b_{k+1}) + A_n b_n \quad A_k = \sum_{i=m}^{N} a_i.$$

By the partial sum formula for the geometric progression, we get

$$\sum_{k=m}^{N}\left(e^{-it}\right)^k = e^{-imt}\sum_{k=0}^{N-m}\left(e^{-it}\right)^k = e^{-imt}\frac{e^{-i(N-m+1)t}-1}{e^{-it}-1}$$
$$= e^{-imt}e^{-i\frac{(N-m)}{2}t}\frac{\sin(N-m+1)t}{\sin\frac{t}{2}} = e^{-i\frac{(N+m)t}{2}}\frac{\sin(N-m+1)t}{\sin\frac{t}{2}}.$$

And the by Abel's transform, we have

$$\oint e^{i\left(n+\frac{1}{2}\right)t}\sum_{V=n+1}^{N}A_V^{\alpha-1}e^{-ivt} = \oint\Bigg[-\sum_{V=n+1}^{N-1}e^{-i(V-n)\frac{t}{2}}\cdot\frac{\sin(V-n)\frac{t}{2}}{\sin\frac{t}{2}}\cdot A_{V+1}^{\alpha-2}$$
$$+e^{-i(N-n)\frac{t}{2}}\cdot\frac{\sin(N-n)\frac{t}{2}}{\sin\frac{t}{2}}\cdot A_N^{\alpha-1}\Bigg]$$
$$= \sum_{V=n+1}^{N-1}\frac{\sin(V-n)\frac{t}{2}\cdot\sin(V-n)\frac{t}{2}}{\sin\frac{t}{2}}A_{V+1}^{\alpha-2} - \frac{\sin^2(n-N)\frac{t}{2}}{\sin^2\frac{t}{2}}A_N^{\alpha-1}. \tag{3}$$

At the same time,

$$-\sum_{V=n+1}^{N-1}A_{V+1}^{\alpha-2} \le \sum_{V=n+1}^{N-1}\sin^2(V-n)\frac{t}{2}A_{V+1}^{\alpha-2} \le \sum_{V=n+1}^{N-1}A_{V+1}^{\alpha-2}.$$

Besides, if we take into account that

$$\sum_{V=n+1}^{N}A_{V+1}^{\alpha-2} = \sum_{V=n+2}^{N}A_V^{\alpha-2} = A_N^{\alpha-1}-A_{n+1}^{\alpha-1},$$

we will have

$$-\left(A_N^{\alpha-1}-A_{n+1}^{\alpha-1}\right) \le \sum_{V=n+1}^{N-1}\sin^2(V-n)\frac{t}{2}A_{V+1}^{\alpha-2} \le \left(A_N^{\alpha-1}-A_{n+1}^{\alpha-1}\right).$$

Correspondingly, $\exists\theta, |\theta|\le 1$ such that

$$\sum_{V=n+1}^{N-1}\sin^2(V-n)\frac{t}{2}A_{V+1}^{\alpha-2} = \theta\left(A_N^{\alpha-1}-A_{n+1}^{\alpha-1}\right). \tag{4}$$

If we take a limit as $N\to\infty$ in equality (3), and take into account equality (4) we will get

$$\oint e^{i\left(n+\frac{1}{2}\right)t}\sum_{V=n+1}^{\infty}A_V^{\alpha-1}e^{-iVt} = -\frac{2\cdot\theta}{2\cdot\sin\frac{t}{2}}\cdot A_{n+1}^{\alpha-1}.$$

At the same time,

$$\frac{A_{n+1}^{\alpha-1}}{A_n^{\alpha}} = \frac{\alpha}{n+1}.$$

Therefore,

$$-\frac{1}{A_n^{\alpha}2\sin\frac{t}{2}}\oint e^{\left(in+\frac{1}{2}\right)t}\sum_{V=n+1}^{\infty}A_V^{\alpha-1}e^{-iVt} = \frac{2\theta\alpha}{(n+1)\left(2\sin\frac{t}{2}\right)^2}.$$

When $N\to\infty$, by the boundness of $\sin^2(n-N)\frac{t}{2}$ and $A_N^{\alpha-1}\to 0$, the last term of equality (3) tends to 0. Therefore, we have gotten the desired equality.

Completely by the same way we will get

$$G_n^{l,\alpha}(t) = \frac{1}{2A_n^{\alpha}\sin\frac{1}{2}\frac{\pi t}{l}}\oint\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}e^{i\left(\nu+\frac{1}{2}\right)t} = \oint\frac{e^{i\left(n+\frac{1}{2}\right)\frac{\pi t}{l}}}{2A_n^{\alpha}\sin\frac{1}{2}\frac{\pi t}{l}}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-1}e^{-i\nu\frac{\pi t}{l}}$$
$$= \oint\left\{\frac{e^{i\left(n+\frac{1}{2}\right)\frac{\pi t}{l}}}{2A_n^{\alpha}\sin\frac{1}{2}\frac{\pi t}{l}}\left[\left(1-e^{-i\frac{\pi t}{l}}\right)^{-\alpha}-\sum_{\nu=n+1}^{\infty}A_v^{\alpha-1}e^{-i\nu\frac{\pi t}{l}}\right]\right\}$$
$$= \frac{1}{A_n^{\alpha}}\frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}\alpha\right)\frac{\pi t}{l}-\frac{1}{2}\pi\alpha\right]}{\left(2\sin\frac{1}{2}\frac{\pi t}{l}\right)^{\alpha+1}} + \frac{2\theta\alpha}{n\left(2\sin\frac{1}{2}\frac{\pi t}{l}\right)^2}\quad (|\theta|\leqslant 1).$$

Now, we move on proving correctness of the equality (2). By the equivalent formulation of the Abel transform (where counting index of partial sums begins from one), we have

$$\sum_{v=1}^{m}e^{-ivt} = e^{-it}\sum_{v=0}^{m-1}e^{-ivt} = e^{-it}\frac{\left(e^{-imt}-1\right)}{e^{-it}-1}.$$
$$\sum_{v=n+1}^{m}A_v^{\alpha-1}e^{-ivt} = A_m^{\alpha-1}e^{-it}\frac{\left(e^{-imt}-1\right)}{e^{-it}-1}$$
$$-A_{n+1}^{\alpha-1}e^{-it}\frac{\left(e^{-int}-1\right)}{e^{-it}-1} - \sum_{v=n+1}^{m-1}\left(A_{v+1}^{\alpha-1}-A_v^{\alpha-1}\right)e^{-it}\frac{\left(e^{-ivt}-1\right)}{e^{-it}-1}$$
$$= A_m^{\alpha-1}\frac{e^{-i(m+1)t}}{e^{-it}-1} - A_m^{\alpha-1}\frac{e^{-it}}{e^{-it}-1} - A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{e^{-it}-1}$$
$$+A_{n+1}^{\alpha-1}\frac{e^{-it}}{e^{-it}-1} - \sum_{v=n+1}^{m-1}A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)t}}{e^{-it}-1} + \sum_{v=n+1}^{m-1}A_{v+1}^{\alpha-2}\frac{e^{-it}}{e^{-it}-1}$$
$$= A_m^{\alpha-1}\frac{e^{-i(m+1)t}}{e^{-it}-1} - A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{e^{-it}-1} - \sum_{v=n+1}^{m-1}A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)t}}{e^{-it}-1}$$

$$+\left(-A_m^{\alpha-1}\frac{e^{-it}}{e^{-it}-1}+A_{n+1}^{\alpha-1}\frac{e^{-it}}{e^{-it}-1}+\sum_{v=n+2}^{m}A_v^{\alpha-2}\frac{e^{-it}}{e^{-it}-1}\right).$$

Taking into account the formula

$$A_{n+1}^{\alpha-1}=\sum_{v=0}^{n+1}A_v^{\alpha-2},$$

we will get

$$\sum_{v=n+1}^{m}A_v^{\alpha-1}e^{-ivt}=A_m^{\alpha-1}\frac{e^{-i(m+1)t}}{e^{-it}-1}-A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{e^{-it}-1}-\sum_{v=n+1}^{m-1}A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)t}}{e^{-it}-1}.$$

If we take a limit as $m\to\infty$ in the last equality and take into account, that

$$A_m^{\alpha-1}\frac{e^{-i(m+1)t}}{e^{-it}-1}\to 0,$$

we get

$$\sum_{v=n+1}^{\infty}A_v^{\alpha-1}e^{-ivt}=-A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{e^{-it}-1}-\sum_{v=n+1}^{\infty}A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)t}}{e^{-it}-1}.$$

And then equality (2) is proved. If we replace t with $\frac{\pi t}{l}$ we will get

$$G_n^{l,\alpha}(t)=$$
$$\oint\left\{\frac{e^{i(n+\frac{1}{2})\frac{\pi t}{l}}}{A_n^{\alpha}\left(2\sin\frac{1}{2}\frac{\pi t}{l}\right)}\left[\frac{1}{(1-e^{-it})^{\alpha}}-A_{n+1}^{\alpha-1}\frac{e^{-i(n+1)t}}{1-e^{-i\frac{\pi t}{l}}}-\sum_{v=n+1}^{\infty}A_{v+1}^{\alpha-2}\frac{e^{-i(v+1)\frac{\pi t}{l}}}{1-e^{-i\frac{\pi t}{l}}}\right]\right\}.$$

By repeatedly using Abel's transform, we are able to get more and more precise approximation for (C,α) kernels (refers to the equality (2)).

2. Main results

In this section we will establish and prove two main theorems of an article, on the uniform convergence to some function of (C,α) means of the general Dirichlet integrals.

Later we will need to rate difference between the Cesaro means of an α degree and function $g(x)$. We have (see [2])

$$\sigma_n^{l,\alpha}(x) - g(x) = \frac{1}{2l}\int_{-l+x}^{l-x}(f(x+t)+f(x-t)-2g(x))G_n^{l,\alpha}(t)dt$$
$$+\frac{1}{2l}\int_{-l-x}^{-l+x}(f(x+t)-g(x))G_n^{l,\alpha}(t)dt$$
$$+\frac{1}{2l}\int_{l-x}^{l+x}(f(x-t)-g(x))G_n^{l,\alpha}(t)dt = I_n^{l,\alpha}(x)+U_n^{l,\alpha}(x)+V_n^{l,\alpha}(x).$$

Besides,

$$I_n^{l,\alpha}(x) = \frac{1}{l}\int_0^{l-x}(f(x+t)+f(x-t)-2g(x))G_n^{l,\alpha}(t)dt$$
$$= \frac{1}{l}\left(\int_0^l - \int_{l-x}^l\right)(f(x+t)+f(x-t)-2g(x))G_n^{l,\alpha}(t)dt = J_n^{l,\alpha}(x) - R_n^{l,\alpha}(x).$$

Suppose, $|g(x)| \le k, \quad x \in [a,b]$, then

$$\left|R_n^{l,\alpha}(x)\right| \le \frac{1}{l}\left(\int_{l-x}^l |f(x+t)||G_n^{l,\alpha}(t)|dt + \int_{l-x}^l |f(x-t)||G_n^{l,\alpha}(t)|dt\right.$$
$$\left.+2|g(x)|\int_{l-x}^l |G_n^{l,\alpha}(t)|dt\right) = \frac{1}{2l\sin\frac{\pi(l-x)}{2l}}\left(\int_{l-x}^l |f(x+t)|dt\right.$$
$$\left.+\int_{l-x}^l |f(x-t)|dt + 2k(b-a)\right) = \frac{1}{2\sin\frac{\pi(l-x)}{2l}}\left(\frac{1}{l}\int_l^{l+x}|f(t)|dt\right.$$
$$\left.+\frac{1}{l}\int_{-l+x}^{-l+2x}|f(t)|dt + \frac{2k(b-a)}{l}\right) \rightrightarrows 0,$$

as $l \to \infty$, by $f \in E$.

By the same reason,

$$\left|U_n^{l,\alpha}(x)\right| \le \frac{1}{2l\sin\frac{\pi(l-x)}{2l}}\left(\frac{1}{l}\int_{-l}^{-l+2x}|f(t)|dt + \frac{2k(b-a)}{l}\right) \rightrightarrows 0$$

uniformly on $[a,b]$, as $l \to \infty$.

Similarly, we get

$$V_n^{l,\alpha}(x) \rightrightarrows 0$$

uniformly on $[a,b]$, as $l \to \infty$.

Now we will establish and prove lemmas that help us to prove the main theorems.

Lemma 2.1: *Suppose, that a function f is integrable on every finite interval and*

$$\int_{|t|\geq 1} \frac{|f(t)|}{|t|^{\alpha+1}} < \infty, \quad 0 < \alpha < 1, \tag{5}$$

then

$$\lim_{\frac{l}{n}\to 0} \frac{1}{l} \int_{\delta}^{l} f(x \pm t) G_n^{l,\alpha}(t) dt = 0,$$

for all fixed $\delta < l$, as $l, n \to \infty$, $\frac{l}{n} \to 0$, uniformly in $x \in [a,b]$.

Proof: Condition (5) implies that $\exists \Delta > max\{|a|+1, |b|+1, \delta\}$ such that

$$\left(\int_{-\infty}^{b-\Delta} + \int_{\Delta+a}^{\infty}\right) \frac{|f(t)|}{|t|^{\alpha+1}} dt < \frac{\varepsilon}{2}.$$

Thus, we will have,

$$\begin{aligned}\left|\frac{1}{l}\int_{\Delta}^{l} f(x\pm t) G_n^{l,\alpha}(t) dt\right| &\leq \frac{1}{l}\int_{\Delta}^{l} |f(x\pm t)| \mid G_n^{l,\alpha},(t) \mid dt \\ &\leq \left(\frac{l}{n}\right)^{\alpha} \int_{\Delta}^{l} \frac{|f(x\pm t)|}{|t|^{\alpha+1}} dt < \frac{\varepsilon}{2}.\end{aligned}$$

Besides,

$$\left|\frac{1}{l}\int_{\delta}^{\Delta} f\left(x\pm t\right) G_n^{l,\alpha}(t) dt\right| \leq C_\alpha \left(\frac{l}{n}\right)^{\alpha} < \frac{\varepsilon}{2}, \quad \text{if } \frac{l}{n} < \min\left\{\delta, \left(\frac{\varepsilon}{2C_\alpha}\right)^{\frac{1}{\alpha}}\right\}.$$

Correspondingly, if $\frac{l}{n}$ setisfies the last condition, then for all $\epsilon > 0$

$$\left|\frac{1}{l}\int_{\delta}^{l} f\left(x\pm t\right) G_n^{l,\alpha}(t) dt\right| < \varepsilon.$$

□

Lemma 2.2: *Let $f(t)$ be integrable on every finite interval and function $\frac{f(t)}{t^2}$ has a bounded variation on some intervals $(-\infty, H]$, $[H, +\infty)$ $(H > 0)$, then for all real $a \leq b$, $\delta > 0$ and $0 < \alpha < 1$, we have*

$$\lim_{\frac{l^2}{n^\alpha}\to 0} \frac{1}{l} \int_{\delta}^{l} f(x \pm t) G_n^{l,\alpha}(t) dt = 0,$$

uniformly on $[a,b]$.

Proof: By the Jordan's theorem every function of bounded variation can be presented as the difference of two nonnegative and nonincreasing functions. Hence,

$$\frac{f(t)}{t^2} = f_1(t) - f_2(t),\ t \in [H, +\infty)$$

where f_1, f_2 are nonnegative and nonincreasing functions. Thus,

$$\frac{1}{l}\int_\Delta^l f(x+t) G_n^{l,\alpha}(t)dt = \frac{1}{l}\int_\Delta^l \frac{f(x+t)}{(x+t)^2}\cdot (x+t)^2 G_n^{l,\alpha}(t)dt$$
$$= \frac{1}{l}\int_\Delta^l \left(f_1(x+t) - f_2(x+t)\right)(x+t)^2 G_n^{l,\alpha}(t)dt,$$

where $\Delta \geq \delta,\ a + \Delta \geq H$. By the second mean value theorem, we get

$$\frac{1}{l}\int_\Delta^l f_j(x+t)(x+t)^2 G_n^{l,\alpha}(t)dt = f_j(x+\Delta)\cdot \frac{1}{l}\int_\Delta^\xi (x+t)^2 G_n^{l,\alpha}(t)dt$$
$$= f_j(x+\Delta)\cdot \frac{1}{l}\int_\Delta^l \left(x^2 + 2xt + t^2\right) G_n^{l,\alpha}(t)dt,$$

where

$$G_n^{l,\alpha}(t) = \frac{1}{A_n^\alpha}\cdot \frac{\sin\left(\left(n + \frac{1}{2} + \frac{\alpha}{2}\right)\frac{\pi t}{l} - \frac{\pi\alpha}{2}\right)}{\left(2\sin\frac{\pi t}{2l}\right)^{\alpha+1}} + \frac{2\theta\alpha}{n\left(2\sin\frac{\pi t}{2l}\right)^2},\ |\theta| \leqslant 1.$$

Also, we have

$$\left|\frac{1}{l}\int_\delta^l G_n^{l,\alpha}(t)dt\right| \leq M_1(\alpha, \Delta)\left(\frac{l}{n}\right)^{\alpha+1} + \frac{l}{n}\left(\frac{1}{l} + \frac{1}{\delta}\right)2\theta\alpha \to 0,$$

as $l, n \to \infty,\ \frac{l}{n} \to 0$.

Furthermore,

$$\left|\frac{1}{l}\int_\Delta^l t G_n^{l,\alpha}(t)dt\right| \leq \frac{l^\alpha}{A_n^\alpha\cdot 2^{\alpha+1}}\cdot \int_\Delta^l \frac{1}{t^\alpha}dt + \frac{l\cdot 2\theta\alpha}{n\cdot 4}\int_\Delta^l \frac{1}{t}dt$$
$$\leq \frac{\Gamma(\alpha+1)}{2^{\alpha+1}}\cdot\left(\frac{l}{n}\right)^\alpha\cdot\frac{1}{(1-\alpha)}\left(l^{1-\alpha} + \Delta^{1-\alpha}\right) + \frac{\theta\alpha}{2}\cdot\frac{l}{n}\left(ln(l) + \ln(\Delta)\right) \to 0,$$

as $\frac{l^2}{n^\alpha} \to 0$, and

$$\left|\frac{1}{l}\int_\Delta^l t^2 G_n^{l,\alpha}(t)\right| \leq \frac{l^\alpha}{A_n^\alpha 2^{\alpha+1}}\cdot\int_\Delta^l t^{1-\alpha}dt + \frac{l}{n}\cdot\frac{2\theta\alpha}{4}\int_\Delta^l 1\cdot dt$$
$$= \frac{\Gamma(\alpha+1)}{2^{\alpha+1}}\cdot\left(\frac{l}{n}\right)^\alpha\cdot\frac{1}{2-\alpha}\left(l^{2-\alpha} + \Delta^{2-\alpha}\right) + \frac{l}{n}\cdot\frac{\theta\alpha}{2}\cdot(1+\Delta) \to 0,$$

as $l, n \to \infty$ and $\frac{l^2}{n^\alpha} \to 0$.

Similarly, the correctness of the lemma in the case of $f(x-t)$ will be proved, if we replace in the previous transformations $f(x+t)$ by $f(x-t)$.

The convergence to 0 of a term

$$\frac{1}{l}\int_{\delta}^{\Delta} f(x+t)G_n^{l,\alpha}(t)dt,$$

can be proved by the same way as in the case of lemma 1. □

Theorem 2.3: *If conditions given in lemma* 1 *are setisfied,* $f \in E$, $0<\alpha<1$, *and*

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2g(x)|dt = 0,\ \forall x\in[a,b], \tag{6}$$

then

$$\sigma_n^{l,\alpha}(x) \rightrightarrows g(x)$$

on $[a,b]$, *as* $l,n\to\infty$, $\frac{l}{n}\to 0$.

Proof: Taking into account the results that hade been obtained before lemma 1, it is easy to see, that to prove the theorem it is enough to estimate $J_n^{l,\alpha}(x)$.

We have

$$\begin{aligned} J_n^{l,\alpha}(x) &= \frac{1}{l}\int_0^l (f(x+t)+f(x-t)-2g(x))G_n^{l,\alpha}(t)dt \\ &= \frac{1}{l}\left(\int_0^{\frac{l}{n}} + \int_{\frac{l}{n}}^{\delta} + \int_{\delta}^{l}\right)(f(x+t)+f(x-t)-2g(x))G_n^{l,\alpha}(t)dt \\ &= A_n^{l,\alpha}(x)+B_n^{l,\alpha}(x)+C_n^{l,\alpha}(x). \end{aligned}$$

Let's assume

$$\begin{aligned} \varphi_x(t) &:= f(x+t)+f(x-t)-2g(x), \\ \lambda(t) &:= \int_0^t |\varphi_x(u)|\,du. \end{aligned}$$

By the condition (6), $\forall\varepsilon>0,\ \exists\delta>0$ such that for $0<t\le\delta$ we have

$$\lambda(t) < \varepsilon\cdot t.$$

Thus

$$A_n^{l,\alpha}(x) \le \left|\frac{1}{l}\int_0^{\frac{l}{n}}\left|G_n^{l,\alpha}(t)\right|d\lambda(t)\right| \le \frac{2n}{l}\lambda\left(\frac{l}{n}\right) < 2\varepsilon.$$

Besides,

$$B_n^{l,\alpha}(x) \leq \frac{1}{l}\int_{\frac{l}{n}}^{\delta}\left|G_n^{l,\alpha}(t)\right| d\lambda(t) \leq \left(\frac{l}{n}\right)^{\alpha}\cdot 2B_\alpha \int_{\frac{l}{n}}^{\delta}\frac{1}{t^{\alpha+1}}d\lambda(t)$$
$$= 2B_\alpha\left(\frac{l}{n}\right)^{\alpha}\left[\frac{1}{\delta^{\alpha+1}}\lambda(\delta) - \frac{l\left(\frac{l}{n}\right)}{\left(\frac{l}{n}\right)^{\alpha+1}} + (\alpha+2)\int_{\frac{l}{n}}^{\delta}\frac{1}{t^{\alpha+2}}\lambda(t)dt\right]$$
$$< 2B_\alpha\left(\frac{l}{n}\right)^{\alpha}\cdot\varepsilon\left[\frac{1}{\delta^{\alpha}} + \left(\frac{n}{l}\right)^{\alpha} + (\alpha+1)\int_{\frac{l}{n}}^{\delta}\frac{1}{t^{\alpha+1}}dt\right]$$
$$= 2B_\alpha\left(\frac{l}{n}\right)^{\alpha}\cdot\varepsilon\left[\frac{1}{\delta^{\alpha}} + \left(\frac{n}{l}\right)^{\alpha} + (\alpha+1)\frac{1}{\alpha}\left(\frac{1}{\delta^{\alpha}} + \left(\frac{n}{l}\right)^{\alpha}\right)\right] < D_\alpha\varepsilon.$$

Now, for the estimation of $C_n^{l,\alpha}(x)$, we use the second mean value theorem and we get

$$\left|g(x)\cdot\frac{1}{l}\int_{\delta}^{l} G_n^{l,\alpha}(t)dt\right|$$
$$= \left|g(x)\frac{1}{l}\int_0^l\left(\frac{\sin\left(\left(n+\frac{1}{2}+\frac{\alpha}{2}\right)\frac{\pi t}{l} - \frac{\pi\alpha}{2}\right)}{A_n^\alpha\left(2\sin\frac{\pi t}{2l}\right)^{\alpha+1}} + \frac{2\theta\alpha}{n\left(2\sin\frac{\pi t}{2l}\right)^2}\right)dt\right|$$
$$=\mid g(x)\cdot\frac{1}{l}\cdot\frac{1}{\left(2\sin\frac{\pi\delta}{2l}\right)^{\alpha+1}}\int_{\delta}^{\xi}\frac{\Gamma(\alpha+1)\sin\left(\left(n+\frac{1}{2}+\frac{\alpha}{2}\right)\frac{\pi t}{l}-\frac{\pi\alpha}{2}\right)}{n^\alpha}dt$$
$$+g(x)\cdot\frac{1}{l}\cdot\frac{1\cdot l^2}{4n\left(2\sin\frac{\pi\delta}{2l}\right)^2}\int_{\delta}^{l}\frac{2\theta\alpha}{t^2}dt\right| \leq M_1(\alpha,\delta)\left(\frac{l}{n}\right)^{\alpha+1}\mid g(x)\mid$$
$$+|g(x)|\frac{l}{n}\cdot\left(\frac{1}{l}+\frac{1}{\delta}\right)\cdot 2\theta\alpha \to 0, \quad \text{as } l,n\to\infty \text{ and } \frac{l}{n}\to 0.$$

Thus, we have $C_n^{l,\alpha}(x) < \varepsilon$. Hence, the theorem is proved.

□

Theorem 2.4: *Let the conditions given in lemma 2 be satisfied. Besides, let* $0 < \alpha \leq 1$,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2g(x)|dt = 0,\ \forall x\in[a,b], \tag{7}$$

and $f \in E$, *then*

$$\sigma_n^{l,\alpha}(x) \rightrightarrows g(x)$$

on $[a,b]$, *as* $l,n\to\infty$, $\frac{l^2}{n^\alpha}\to 0$.

Proof: The proof is analogous to the previous theorem, if we take into account that $\frac{l^2}{n^\alpha} \to 0 \implies \frac{l}{n} \to 0$. The only difference is that, estimation of the corresponding integral on $[\delta, l]$ interval happens by lemma 2, instead of lemma 1. $\square$

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