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Anisotropy of Random Two-Dimensional Composites with Circular Inclusions

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The problem of macroscopic anisotropy of random two-dimensional composite reinforcement with particles is considered. The investigation is based on the exact solution to the Riemann-Hilbert and $\mathbb{R}$-linear problems for a multiply connected domain with circular inclusions, along with approximate analytical formulas for boundary value problems involving harmonic and biharmonic functions. The anisotropy terms start with the coefficient c_2 on f^2 in the power expansion of the effective tensor with respect to the concentration f of circular inclusions. This coefficient c_2 is explicitly expressed by the structural sums e_2 and $e_3^{(1)}$. Universal relations, $e_2 = \pi$ and $e_3^{(1)} = \frac{\pi}{2}$, apply to any macroscopically isotropic composite with the changing parameter f. The deviation of e_2 and $e_3^{(1)}$ for the random location of inclusions is studied. It is demonstrated that elastic composites described by biharmonic functions are more isotropically unstable for geometric perturbations than conductive composites described by harmonic functions.

Keywords: Macroscopic anisotropy, random two-dimensional composite, structural sums.

AMS Subject Classification: 74Q15.

1. Introduction

The problem of macroscopic anisotropy of random composite reinforcement with particles attracts the attention of many physicists and engineers [3, 5, 12]. The anisotropy can be expressed by the algebraic structure of operators describing an anisotropic property [14, 15, 30]. The anisotropy of crystals depends upon the geometrical symmetry of their lattices and yields the classification of crystals reviewed in [30]. The geometric method for crystals leads to boundary value problems for regular periodic structures discussed by analytical and asymptotic methods in [4, 16, 17, 28]. The derived approximate analytical formulas describe the macroscopic anisotropy.

The orientations of the particles can be considered the main factor of anisotropy of dilute composites when interactions among inclusions do not matter. In this case, Eshelby's tensor [7] yields the effective constants up to $O(f^2)$, where f denotes the concentration of inclusions. The term f contains the shape factor of the inclusions, and the term f^2 depends on the location of the inclusions [25].

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We concentrate our attention on two-phase, two-dimensional (2D) composites for which the classic complex analysis methods were developed in the XX century [9, 26, 31]. Boundary value problems for composites and porous media can, in many cases, be considered as the Riemann-Hilbert and $\mathbb{R}$-linear problems [19]. Recent extensions of constructive complex analysis methods to boundary value problems for analytic functions [10, 19] and to generalized analytic functions [1, 11] allow us to apply the obtained results to 2D dispersed composites.

The present paper is based on the exact solution to the Riemann-Hilbert and $\mathbb{R}$-linear problems for a multiply connected domain with circular inclusions [19] and the generalized alternating method of Schwarz [25]. This exact solution can be considered as the representation of physical fields in the form of the generalized Poincaré series for the classic Schottky group. The averaging of fields by the homogenization method [6] gives the effective tensors $\boldsymbol{\varepsilon}_e$ in the form of power series in the concentration of inclusions f with the coefficients c_k depending on the physical constants and the gravitational centers of the inclusions

$$\boldsymbol{\varepsilon}_e = \sum_{k=0}^{\infty} c_k f^k. \tag{1}$$

More precisely, every tensor c_k is a linear combination of structural sums with coefficients expressed through physical constants. In Section 2, we write the effective tensor up to $O(f^3)$ to demonstrate such a linear combination. The coefficient c_2 contains the structural sum e_2 used in [18, 27] to study the anisotropy of heat conduction and $e_3^{(1)}$ used in the plane elasticity problems [8].

The present paper is organized in the following way. Section 2 outlines the lower-order formulas for effective conductivity and Section 3 for elasticity. These formulas are written in the form of the truncated series (1). The anisotropy terms begin with the coefficient c_2 for circular inclusions. It is expressed by the explicitly constructed structural sums e_2 and $e_3^{(1)}$. The universal relations $e_2 = \pi$ and $e_3^{(1)} = \frac{\pi}{2}$ hold for any macroscopically isotropic composite. Random locations of inclusions are generated, and the corresponding structural sums e_2 and $e_3^{(1)}$ are calculated. Statistical analysis demonstrates a lower deviation of e_2 from π than a deviation of $e_3^{(1)}$ from $\frac{\pi}{2}$. This implies that elastic composites described by biharmonic functions are more anisotropically unstable for geometric perturbations than conductive composites described by harmonic functions.

2. An asymptotic formula for the effective conductivity

The dependence of the effective constants on shapes and locations is clearly seen in [25] and [24, formula (17)]. We now write an approximate analytical formula for the effective conductivity tensor (permittivity) $\boldsymbol{\varepsilon}_e$ for two-dimensional two-phase composites.

Let a composite be represented by a periodicity cell Q formed by two fundamental translation vectors ω_1 and ω_2 on the complex plane $\mathbb{C}$

$$Q = \left\{ z = t_1\omega_1 + t_2\omega_2 \in \mathbb{C} : -\frac{1}{2} < t_1, t_2 < \frac{1}{2} \right\}. \tag{2}$$

Without loss of generality the periods ω_1 and ω_2 satisfy the conditions $\omega_1 > 0$, Im $\omega_2 > 0$. Let the area of Q be normalized to unity. Then $\omega_1 \text{Im } \omega_2 = 1$. For definiteness, we consider the hexagonal lattice with

$$\omega_1 = \frac{\sqrt{2}}{\sqrt[4]{3}}, \quad \omega_2 = \frac{\sqrt{2}}{\sqrt[4]{3}} \frac{1 + \mathrm{i}\sqrt{3}}{2}. \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The considered doubly periodic structure can be represented by a plane torus denoted by $\mathcal{Q}$.

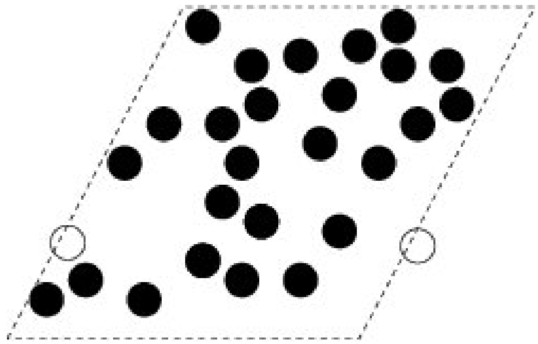


Figure 1. The periodicity cell Q of disks representing the plane torus. Two white disks on the opposite sides of Q are identified.

Consider N mutually disjoint simply connected domains D_k with Lyapunov's boundaries L_k in $\mathcal{Q}$. Let a_k be the center of gravity of D_k. Let $D = Q \backslash \cup_{k=1}^{N} (D_k \cup L_k)$ denote the complement of all the closures of D_k to Q. An example of disks D_k is shown in Figure 1. Let the conductivity of D be normalized to the unity. Introduce the piece-wise conductivity function

$$\varepsilon(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in D, \\ \varepsilon_1, & \mathbf{x} \in D_k \quad (k = 1, 2, \ldots, N) \end{cases} \tag{4}$$

and the dimensionless contrast parameter

$$\varrho = \frac{\varepsilon_1 - 1}{\varepsilon_1 + 1}. \tag{5}$$

The parameter $\varepsilon_1 \geq 0$ represents the steady heat conduction and may be complex when referring to the dielectric permittivity.

Let a domain D_k be fixed. Introduce the singular integral defined in [31]

$$\mathcal{J}_k = \frac{1}{\pi} \int_{D_k} \int_{D_k} \frac{1}{(z - \zeta)^2} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \quad \text{and} \quad \mathcal{J} = \sum_{k=1}^{N} \mathcal{J}_k, \tag{6}$$

where $z = x_1 + \mathrm{i}x_2$ and $\zeta = \xi_1 + \mathrm{i}\xi_2$. One can see that $\mathcal{J}_k$ does not depend on the translation $D_k \mapsto D_k + w$.

The complex static moment of D_k is defined by the formula

$$s_{qk} = \textstyle\int_{D_k} (z - a_k)^q \mathrm{d}x_1 \mathrm{d}x_2 \quad (q = 0, 1, \ldots). \tag{7}$$

In particular, $s_{0k} = |D_k|$, where $|D_k|$ denotes the area of D_k. Let D_k^* denote the domain D_k rotated about a_k on the angle θ in the clockwise direction. Then, $s_{qk}^* = \exp(\mathrm{i}\theta q) s_{qk}$. This formula corrects the sign in the formula (89) from [25] and the final formula for the effective tensor. It is convenient to use dimensionless complex static moments

$$s_{qk}^{(0)} = s_{qk} s_{0k}^{-\frac{q}{2}-1}, \tag{8}$$

for which $s_{0k}^{(0)} = 1$. The integral (6) changes after the rotation as follows

$$\mathcal{J}_k^* = \exp(2\mathrm{i}\theta)\mathcal{J}_k. \tag{9}$$

Let $\wp(z)$ be the elliptic Weierstrass function constructed by the periods (3) [2]. It is convenient to use Eisenstein functions related to Weierstrass functions by simple relations [32]

$$E_2(z) = \wp(z) + S_2, \quad E_j(z) = \frac{(-1)^j}{(j-1)!} \frac{d^{j-2}}{^{j-2}} \wp(z), \quad j = 2, 3, \ldots \tag{10}$$

In the hexagonal array under consideration, the lattice sum S_2 equals π. Define the values

$$E_{j,km} = E_j(a_k - a_m), \text{ for } a_m \neq a_k, \quad E_{j,kk} = 0,$$
$$(m, k = 1, 2, \ldots, N) \quad j = 2, 3, \ldots \tag{11}$$

Let g_2 and g_4 be the invariants of $\wp(z)$ [2]. Using $\wp''(z) = 12\wp^2(z) - g_2$ and the relation (10), we get $E_{4,km} = 2(E_{2,km} - S_2)^2 - \frac{1}{6}g_2$ and $E_{5,km} = E_{3,km}(E_{2,km} - S_2)$. In the case of hexagonal lattice, $E_{4,km} = 2(E_{2,km} - \pi)^2$ and $E_{5,km} = E_{3,km}(E_{2,km} - \pi)$ since $g_2 = 0$.

Introduce the following sums

$$\mathcal{L} = \frac{1}{\pi} \sum_{k,m=1}^{N} |D_k||D_m| E_{2,km}, \tag{12}$$

$$\mathcal{V}^{(1)} = \frac{2}{\pi} \sum_{k,m=1}^{N} |D_k||D_m| E_{3,mk} (s_{1k}^{(0)} |D_k|^{\frac{1}{2}} - s_{1m}^{(0)} |D_m|^{\frac{1}{2}}), \tag{13}$$

$$\begin{gathered}\mathcal{V}^{(2)} = \tfrac{3}{\pi} \textstyle\sum_{k,m=1}^{N} |D_k||D_m| E_{4,mk} \times \\ \left(|D_k| s_{2k}^{(0)} + |D_m| s_{2m}^{(0)} - 2\left(|D_k||D_m|\right)^{\frac{1}{2}} s_{1m}^{(0)} s_{1k}^{(0)}\right)\end{gathered} \tag{14}$$

and

$$\mathcal{V}^{(3)} = \frac{4}{\pi} \sum_{k,m=1}^{N} |D_k||D_m| E_{5,mk} \times \\ \left[|D_k|^{\frac{3}{2}} s_{3k}^{(0)} - |D_m|^{\frac{3}{2}} s_{3m}^{(0)} + 3\left(|D_k|^{\frac{1}{2}} |D_m| s_{1k}^{(0)} s_{2m}^{(0)} - |D_m|^{\frac{1}{2}} |D_k| s_{2k}^{(0)} s_{1m}^{(0)} \right) \right]. \tag{15}$$

Define the value

$$\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)}. \tag{16}$$

The formula (91) from [25] is slightly corrected here, where the effective tensor has the form

$$\boldsymbol{\varepsilon}_e = (1 + 2\varrho f) I + 2\varrho^2 \sum_{k=1}^{N} \begin{pmatrix} \text{Re } \mathcal{J}_k & -\text{ Im } \mathcal{J}_k \\ -\text{Im } \mathcal{J}_k & -\text{ Re } \mathcal{J}_k \end{pmatrix} \\ +2\varrho^2 \begin{pmatrix} \text{Re } \mathcal{L} & -\text{ Im } \mathcal{L} \\ -\text{Im } \mathcal{L} & 2f^2 - \text{Re } \mathcal{L} \end{pmatrix} + 2\varrho^2 \begin{pmatrix} \text{Re } \mathcal{V} & -\text{ Im } \mathcal{V} \\ -\text{Im } \mathcal{V} & -\text{ Re } \mathcal{V} \end{pmatrix} + O(|\varrho|^3 f^4). \tag{17}$$

Here, I is the unit matrix. One can check that the orders of the terms $\mathcal{J}$, $\mathcal{L}$, $\mathcal{V}^{(1)}$, $\mathcal{V}^{(2)}$, $\mathcal{V}^{(3)}$ hold $O(f)$, $O(f^2)$, $O(f^{\frac{5}{2}})$, $O(f^3)$ and $O(f^{\frac{7}{2}})$, respectively. The tensor $\boldsymbol{\varepsilon}_e$ becomes isotropic up to $O(|\varrho|^3 f^4)$ if Re $\mathcal{J} = 0$, Re $\mathcal{L} = f^2$ and Re $\mathcal{V} = 0$. Macroscopic isotropy also holds for special relations between f, ϱ, $\mathcal{J}$, $\mathcal{L}$ and $\mathcal{V}$.

Consider the case of the same inclusions D_k when $|D_k| = N^{-1} f$, $s_{jk}^{(0)} := s_j^{(0)}$ $(k = 1, 2, \ldots, N)$

$$\boldsymbol{\varepsilon}_e = (1 + 2\varrho f) I + 2\varrho^2 \sum_{k=1}^{N} \begin{pmatrix} \text{Re } \mathcal{J}_k & -\text{ Im } \mathcal{J}_k \\ -\text{Im } \mathcal{J}_k & -\text{ Re } \mathcal{J}_k \end{pmatrix} \\ +\frac{2\varrho^2 f^2}{\pi} \sum_{k,m=1}^{N} \begin{pmatrix} \text{Re } E_{2,km} & -\text{ Im } E_{2,km} \\ -\text{Im } E_{2,km} & -\text{ Re } E_{2,km} \end{pmatrix} \\ +\frac{2\varrho^2 f^3}{\pi} [s_2^{(0)} - (s_1^{(0)})^2] \sum_{k,m=1}^{N} \begin{pmatrix} \text{Re } E_{4,km} & -\text{ Im } E_{4,km} \\ -\text{Im } E_{4,km} & -\text{ Re } E_{4,km} \end{pmatrix} + O(f^3 |\varrho|^3). \tag{18}$$

Rotate the inclusion D_k by the angle θ_k. Let the angles θ_k be uniformly distributed on the segment $(0, 2\pi)$. Then the $\mathcal{J}$-terms in (18) vanishes.

Consider the case when D_k are equal disks with a radius r. Then, (18) for the hexagonal cell Q can be written in the form

$$\boldsymbol{\varepsilon}_e = \frac{1+\varrho f}{1-\varrho f} I + \frac{2\varrho^2 f^2}{\pi} \sum_{k,m=1}^{N} \begin{pmatrix} \text{Re } \wp_{km} & -\text{ Im } \wp_{km} \\ -\text{Im } \wp_{km} & -\text{ Re } \wp_{km} \end{pmatrix} + O(f^3 |\varrho|^3), \tag{19}$$

where

$$\wp_{km} := \wp(a_k - a_m), \text{ for } a_m \neq a_k, \quad \wp_{kk} := 0, \ (m, k = 1, 2, \ldots, N). \tag{20}$$

The first term in (19) is the famous Clausius-Mossotti approximation. The $\mathcal{L}$-term

(12) becomes $\mathcal{L} = \frac{f^2}{\pi} e_2$, where the structural sum is introduced

$$e_2 = \frac{1}{N^2} \sum_{k,m=1}^{N} E_{2,km} \equiv \pi + \frac{1}{N^2} \sum_{k,m=1}^{N} \wp_{km}. \tag{21}$$

The structural sums, including e_2 and their relations to macroscopic anisotropy, were discussed in [20, 27]. A set of relations on the structural sums for macroscopically isotropic composites was established. In particular, the isotropy up to $O(f^3)$ is maintained when $e_2 = \pi$.

3. Structural sums for elasticity

We now proceed to consider the plane elastic problem for the same set of disks D_k shown in Figure 1. Let the domains D and D_k $(k = 1, 2, \cdots, n)$ be occupied by elastic materials with the shear moduli and Poisson's ratios μ, ν and μ_k, ν_k, respectively.

Following [8] introduce the elastic contrast parameters using Muskhelishvili's constant $\kappa = \frac{3-\nu}{1+\nu}$

$$\varrho_1 = \frac{\frac{\mu_1}{\mu} - 1}{\frac{\mu_1}{\mu} + \kappa_1}, \quad \varrho_2 = \frac{\kappa \frac{\mu_1}{\mu} - \kappa_1}{\kappa \frac{\mu_1}{\mu} + 1}, \quad \varrho_3 = \frac{\frac{\mu_1}{\mu} - 1}{\kappa \frac{\mu_1}{\mu} + 1} \tag{22}$$

related by the identity

$$\varrho_1 = \frac{\varrho_3}{1 - \varrho_2 + \varrho_3}. \tag{23}$$

The denominator of (23) is equal to the value $\frac{\frac{\mu_1}{\mu} + \kappa_1}{\frac{\mu_1}{\mu} \kappa + 1}$, which is always positive.

Following [8, Appendix A.3] introduce the Eisenstein-Natanzon-Filshtinsky function by the elliptic Weierstrass functions

$$E_3^{(1)}(z) = -\frac{1}{2} \overline{z} \wp'(z) + \frac{1}{6\pi} \wp''(z) + \frac{1}{2\pi} \zeta(z) \wp'(z) + S_3^{(1)}, \tag{24}$$

where the lattice sum $S_3^{(1)} = \frac{\pi}{2}$ for the considered hexagonal cell Q.

The effective shear modulus for macroscopically isotropic 2D elastic composites with equal circular inclusions can be calculated by [8, formula (4.21)]

$$\frac{\mu_e}{\mu} = \frac{1 + \mathrm{Re}\, A}{1 - \kappa \mathrm{Re}\, A}, \tag{25}$$

where the parameter A is expanded into the power concentration series

$$A = \sum_{s=1}^{\infty} A^{(s)} f^k. \tag{26}$$

The first few coefficients were explicitly written in [8]. In particular,

$$A^{(1)} = \varrho_3, \quad A^{(2)} = -\frac{2\varrho_3^2}{\pi} e_3^{(1)}, \tag{27}$$

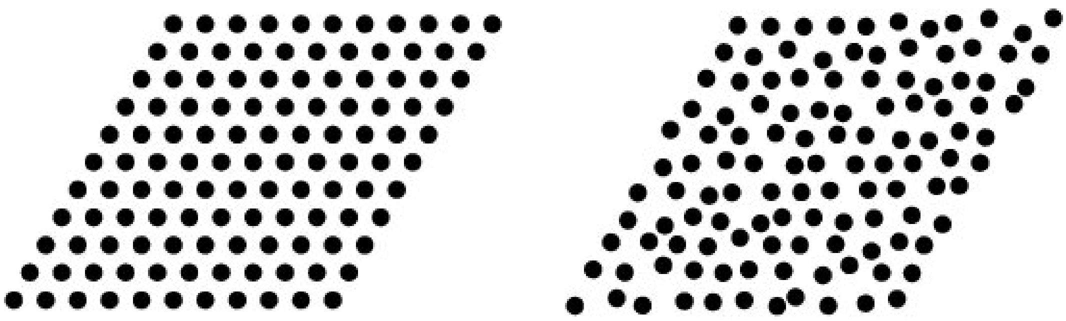
where

$$e_3^{(1)} = \frac{1}{N^2} \sum_{k,m=1}^{N} E_3^{(1)}(a_k - a_m). \tag{28}$$

Similar to the sum (21) it is assumed that $E_3^{(1)}(a_k - a_m) := 0$ if $a_k = a_m$. The coefficients (28) depend on the contrast parameter ϱ_3. The next coefficients $A^{(s)}$ were calculated in symbolic form in [8]. They depend on all the contrast parameters for $s \geq 2$.

Remark 1 : The structural sums e_2 and $e_3^{(1)}$ arise from the conditionally convergent Eisenstein series. This necessitates a meticulous approach to utilizing the sums based on asymptotic analysis and homogenization theory. Specifically, the local stress fields and the effective constants can be derived using various formulas that require different summation methods, resulting in either $e_2 = \pi$ and $e_3^{(1)} = \frac{\pi}{2}$ or $e_2 = \pi = e_3^{(1)} = 0$. We refer to [22, 23] for a detailed explanation of this nuanced issue.

4. Computation of structural sums



a) b)

Figure 2. a) Hexagonal array of disks; b) perturbed hexagonal array.

It follows from the previous sections that macroscopic anisotropy can be characterized up to $O(f^3)$ by the structural sums e_2 and $e_3^{(1)}$. The present section is devoted to computer simulations of these structural sums for dispersed random composites. The calculations were carried out in the *Wolfram Mathematica*® environment.

First, the following regular lattice shown in Figure 2a is built

$$Q_{reg} = \left\{ \frac{1}{11} \left(m\omega_1 + n\omega_2 \right) : m, n = 0, \pm 1, \ldots, \pm 5 \right\} \tag{29}$$

Next, random deviations of this lattice are constructed, as an example shown in Figure 2b

$$Q_{random} = \left\{ \frac{1}{11} \left(m\omega_1 + n\omega_2 + r_{n,m} \exp[\mathrm{i}\theta_{n,m}] \right) : m, n = 0, \pm 1, \ldots, \pm 5 \right\}. \quad (30)$$

Here, $r_{n,m}$ and $\theta_{n,m}$ denote the realizations of random variables R and Θ uniformly distributed on the intervals $(0, \varepsilon)$ and $(0, 2\pi)$. The structural sums e_2 and $e_3^{(1)}$ were calculated for a set of points, i.e., for a set of 121 points displayed in Figure 2b, two complex numbers e_2 and $e_3^{(1)}$ were calculated.

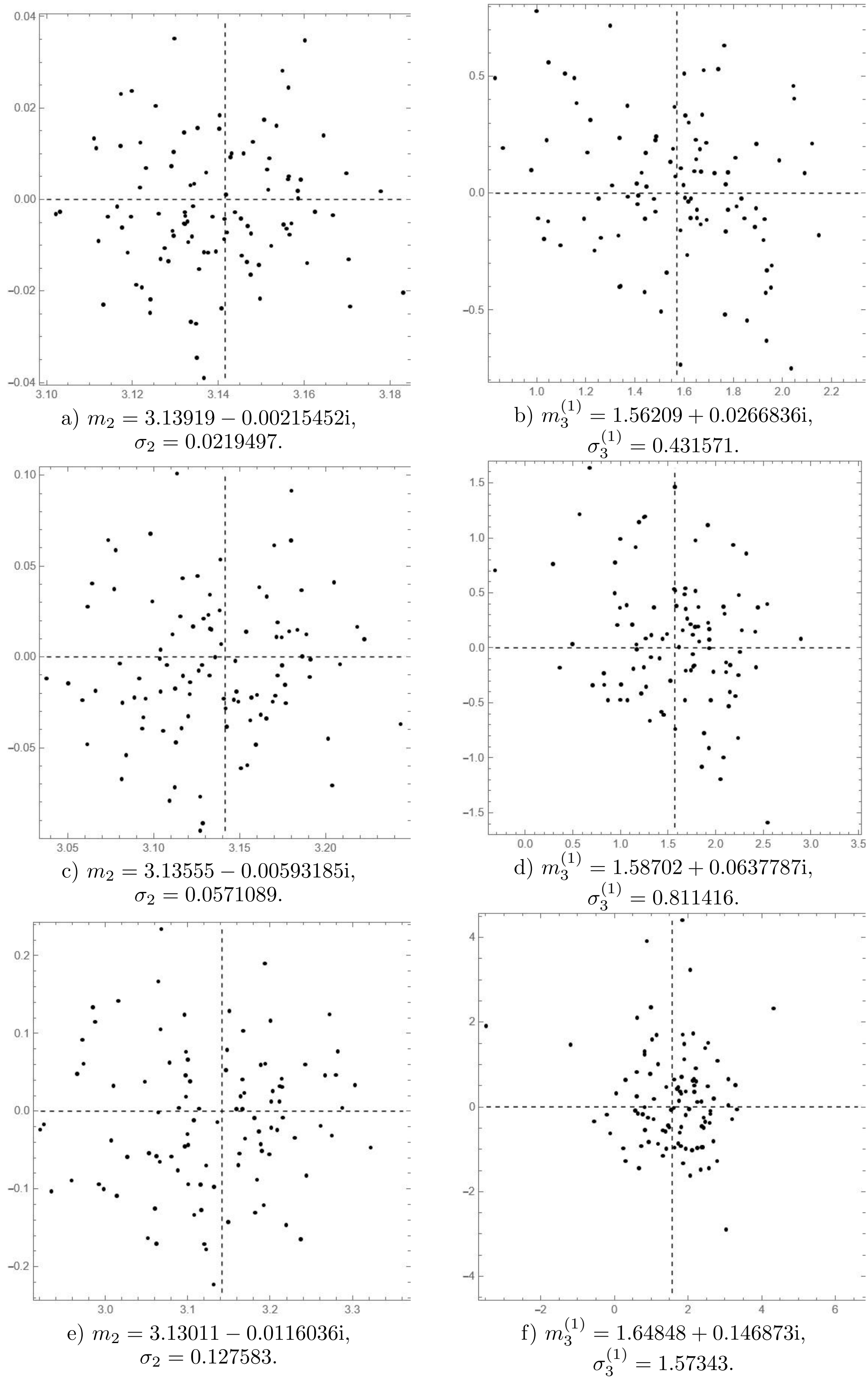
The simulations were performed for $\varepsilon = 0.2,\ 0.3,\ 0.4$. The values e_2 and $e_3^{(1)}$ were calculated for the generated random coordinates by the formulas (21), (28). This procedure was repeated 100 times. The means m_2 and $m_3^{(1)}$, the standard deviations σ_2 and $\sigma_3^{(1)}$ for e_2 and $e_3^{(1)}$, respectively, are presented in Figure 3.

The obtained results demonstrate that the values e_2 and $e_3^{(1)}$ are distributed near the complex numbers $\pi + \mathrm{i}0$ and $\frac{\pi}{2} + \mathrm{i}0$, respectively. The standard deviation of $e_3^{(1)}$ is significantly higher than the standard deviation of e_2 for the same sets of disks. This observation justifies that elastic composites described by biharmonic functions are more isotropically unstable for geometric perturbations than conductive composites described by harmonic functions.

This conclusion is important in the theory of composites and in applied optimal design problems. A three-dimensional extension of the structural sums can be found in [21]. This implies that the method of structural sums can be extended to some general problem of solid and fluid mechanics [13, 29].

It is worth noting that the popular pure numerical methods, e.g., FEM, are limited to a small number of inclusions and insufficient numbers of numerical experiments. If one has 1-3 points in a square of Figure 3 instead of 100, the proper statistical investigations are impossible. Moreover, the physical constants, e.g., the shear moduli and Poisson's ratios, are fixed in the pure numerical methods. This requires an additional number of computations for various parameters.

Contrary to FEM, an ordinary laptop performed our computations in minutes. This stresses the computational effectiveness of the method of structural sums used in the present paper.



a) $m_2 = 3.13919 - 0.00215452i$,
$\sigma_2 = 0.0219497$.

b) $m_3^{(1)} = 1.56209 + 0.0266836i$,
$\sigma_3^{(1)} = 0.431571$.

c) $m_2 = 3.13555 - 0.00593185i$,
$\sigma_2 = 0.0571089$.

d) $m_3^{(1)} = 1.58702 + 0.0637787i$,
$\sigma_3^{(1)} = 0.811416$.

e) $m_2 = 3.13011 - 0.0116036i$,
$\sigma_2 = 0.127583$.

f) $m_3^{(1)} = 1.64848 + 0.146873i$,
$\sigma_3^{(1)} = 1.57343$.

Figure 3. The results of computation of structural sums in the form of 100 points per a square: e_2 in the left column (a), (c), (e); $e_3^{(1)}$ in the right column (b), (d), (f). The parameter $\varepsilon = 0.2$ (first line (a), (b)), $\varepsilon = 0.3$ (second line (c), (d)), $\varepsilon = 0.4$ (third line (e), (f)). The mean values and the standard deviations are written.

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