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About the Numerical Solutions of Two Nonlinear Integro-Differential Equations

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In this work we consider the issues of the approximate solutions and the results of numerical computations for the following two practical problems: 1. Non-linear initial-boundary value problem for the J. Ball dynamic beam. 2. Non-linear initial-boundary value problem for the Kirchhoff dynamic string.

A mathematical model is formulated for an initial-boundary value problem associated with the J. Ball integro-differential equation, which serves as a mathematical description of the dynamic state exhibited by a beam. The solution to this problem is approximated through a combination of the Galerkin method, a stable symmetrical difference scheme, and the Jacobi iteration method. Our aim is to present an approximate solution to a problem, specifically focusing on the numerical results obtained from the initial-boundary value problem pertaining to a specific iron beam. Notably, the effective viscosity of the material is considered to be dependent on its velocity.

We consider the numerical algorithm for the Kirchhoff type inhomogeneous integro-differential equation describing the string oscillation. The algorithm has been approved by tests and the results of calculations is presented in tables and graphs.

The presented article is a direct continuation of the articles [1]-[4] and [8]-[9] that consider the construction of algorithms and their corresponding numerical computations for the approximate solution of nonlinear integro-differential equations for the J. Ball dynamic beam (see [1]-[4]) and for the Kirchhoff dynamic string (see [8]-[9]).

Keywords: Nonlinear dynamic beam equation, J. Ball equation, Galerkin method, Implicit symmetric difference scheme, Jacobi iterative method, Iron beam, Numerical realization, Kirchhoff string wave equation, Galerkin's method, Crank-Nicolson difference scheme, Picard iteration process, Test results.

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1. Statement of the Problem 1

Let us consider the nonlinear equation

$$
\begin{aligned}
&u_{tt}(x,t) + \delta u_t(x,t) + \gamma u_{xxxxt}(x,t) + \alpha u_{xxxx}(x,t) \\
&- \left(\beta + \kappa \int_0^L u_x^2(x,t)\,dx\right) u_{xx}(x,t) - \sigma \left(\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right) \\
&\times u_{xx}(x,t) = f(x,t), \qquad 0 < x < L, \qquad 0 < t \le T,
\end{aligned}
\tag{1}
$$

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with the initial boundary conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \tag{2}$$

$$u(0,t) = u(L,t) = 0, \quad u_{xx}(0,t) = u_{xx}(L,t) = 0. \tag{3}$$

In the given context, let $\alpha, \gamma, \kappa, \sigma, \beta$, and δ be constants, where the first four are positive numbers. Furthermore, consider the functions $u^0(x) \in W_2^2(0,L)$ and $u^1(x) \in L_2(0,L)$, satisfying the conditions $u^0(0) = u^1(0) = u^0(L) = u^1(L) = 0$. The right-hand side function $f(x,t)$ belongs to $L_2((0,L) \times (0,T))$. We assume the existence of a solution $u(x,t) \in W_2^2((0,L) \times (0,T))$ for problem (1)-(3).

The present article serves as a direct continuation of previous works [1]-[4], which focused on developing algorithms and performing corresponding numerical computations for approximating solutions to nonlinear integro-differential equations of the Timoshenko type. In this particular study, we address an initial-boundary value problem associated with the J. Ball integro-differential equation, which characterizes the dynamic state of a beam (see [5]). To approximate the solution, we employ the Galerkin method, a stable symmetric difference scheme, and the Jacobi iteration method. The algorithms proposed in [2]-[3] have been validated through various tests. Additionally, this article, along with [4], presents an approximate solution to a practical problem. Specifically, we provide numerical results for the initial-boundary value problem concerning an iron beam, which are presented in a tabular form.

The physical model utilized by J. Ball in his publication [5] is derived from the Handbook of Engineering Mechanics, authored by E. Mettler (see [6]). In this model, the corresponding initial-boundary value problem for the integro-differential equation governing the behaviour of a beam (denoted as equation (1)) is formulated. The constants α, γ, κ, σ, β, and δ present in the problem are defined as follows:

$$\alpha = \frac{E \cdot I}{\rho}, \ \ \beta = \frac{E \cdot A \cdot \Delta}{L \cdot \rho}, \ \ \gamma = \frac{\eta \cdot I}{\rho}, \ \ \kappa = \frac{E \cdot A}{2L \cdot \rho}, \ \ \sigma = \frac{A\eta}{L \cdot \rho}.$$

Here, E denotes Young's modulus, A represents the cross-sectional area, η signifies the effective viscosity, I stands for the cross-sectional second moment of area, ρ corresponds to the mass per unit length in the reference configuration, L symbolizes the length of the beam, Δ signifies the extension or change in the beam length, and δ refers to the coefficient of external damping.

2. The numerical realization of the Problem 1

To approximate the solutions to initial-boundary value problems (1)-(3), a collection of programs was developed within the Maple software environment. Subsequently, several numerical experiments were conducted to facilitate this approximation process. The purpose of this paper is to present an approximate solution to a practical problem. Specifically, the tables in this paper illustrate the results obtained from numerical computations of the initial-boundary value problem concerning an iron beam.

Based on the observed numerical experiments, it is evident that as the effective viscosity, denoted by η, increases (or decreases), the corresponding numerical values of the displacement function, $u(x,t)$, for specific values of x and t exhibit a decreasing (or increasing) trend. Specifically, when considering the case of velocity-dependent effective viscosity, an increase in velocity leads to a decrease in viscosity, resulting in amplified deflections (or bending) of the beam. Furthermore, for a fixed value of η, the numerical values of the displacement function for a given x tend to increase as time t progresses. Notably, the numerical values of the displacement function at a particular t exhibit symmetry with respect to the midpoint of the beam, located at $x = L/2$.

3. Statement of Problem 2

Consider the nonlinear inhomogeneous equation

$$w_{tt}(x,t) - \left(\lambda + \frac{2}{\pi}\int_0^\pi w_x^2(x,t)\,dx\right) w_{xx}(x,t) = f(x,t), \tag{4}$$

$$0 < x < \pi, \quad 0 < t \le T,$$

with the initial boundary conditions

$$\begin{gathered} w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \\ w(0,t) = w(\pi,t) = 0, \end{gathered} \tag{5}$$

$$0 \le x \le \pi, \quad 0 \le t \le T.$$

Here $\lambda > 0$ and T are given constants, while $f(x,t), w^0(x), w^1(x)$ are given functions.

The equation (4), when $f(x,t) = 0$, is proposed by Kirchhoff [7] in 1876. It is a generalization of D'Alembert string's oscillation model with equation $w_{tt} = c^2 w_{xx}$. Many authors researched the homogeneous equation, corresponding to (4) and its generalizations in terms of solvability.

Here we will generalize the numerical algorithm offered in [8]-[9] for the approximate solution of problem (4), (5) for the case $f(x,t) = 0$. Then we solve test examples using this algorithm and present the results in tables and graphs.

4. Test examples Problem 2

Here we present results of calculations of two test examples.

Example 4.1 Let $T = 1$, $\lambda = 0.4$,

$$f(x,t) = 6t\sin 2x + (\lambda + 1 + 4t^6)(\sin x + 4t^3 \sin 2x),$$

$w^0(x) = \sin(x)$, $w^1(x) = 0$. The exact solution is the function $w(x,t) = \sin x + t^3 \sin 2x$. The algorithm is applied with $n = 5, M = 20$ and $\tau_m = 0.05$. The number of iterations is $k = 9$. The error is $\Delta_n^k = 0.0744789237$. The results are presented below in tables and graphs.

Example 4.2 Let $T = 1$, $\lambda = 1.0$,

$$f(x,t) = \left[\left(\frac{x}{\pi}\right)^2 \sin x - \left(\lambda + \frac{e^{2t}-1}{2t}\right)\left(\frac{2t}{\pi}\cos x - \left(1 - \left(\frac{t}{\pi}\right)^2\right)\sin x\right)\right] e^{\frac{1}{\pi}xt},$$

$w^0(x) = \sin x$, $w^1(x) = \frac{1}{\pi} x \sin x$. The exact solution is the function $w(x,t) = e^{\frac{1}{\pi}xt}\sin x$. The algorithm is applied with $n = 5, M = 20$ and $\tau_m = 0.05$. The number of iterations is $k = 10$. The error is $\Delta_n^k = 0.0441088504$. The results are presented below in tables and graphs.

If we increase the values of parametres n and M, the error improves. Namely, if we take $n = 12$ and $M = 160$ in example 4.1, the error is $\Delta_n^k = 0.0093695833$. If we take $n = 12$ and $M = 80$ in example 4.2, the error is $\Delta_n^k = 0.0096361646$. Based on the obtained results, it can be concluded that the numerical algorithm for solving problem (4), (5) is effective.

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