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Variational Principles in Coupled Strain Gradient Elasticity

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In recent years, generalized continuum models have been increasingly used again to solve tasks that cannot be analyzed using classical approaches. Here we report on some variational principles of the generalized theories, which show that the classical theories can be extended elementarily. The paper is a review article of the state of art in this field and some contributions of the authors during the last year. With respect to this some theorems are indicated - the proofs can be taken from our papers cited below.

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1. Motivation

The classical theory of elasticity allows the solution of numerous initial boundary value problems. At the same time, special problems, such as concentrated loads or corner problems, cannot be solved within the framework of the classical theory of elasticity. Other approaches such as generalized continua [1] or peridynamic considerations [2] allow better (more correct) solutions. At the same time, the question arises as to how accurate such theories are in comparison to the classical theory of elasticity. One way of testing the exactness is to formulate the analogies of the classical variational principles.

The first gradient elasticity theories were established by F. and E. Cosserat [3] creating the polar media. Firstly the micro-rotations and the associated coupled stresses in the motion equations were introduced by Hellinger [4] who drew attention to the problem of asymmetric stresses of the Cosserat medium. A strain gradient model for fluids was studied by Korteweg [5] and by Cahn and Hilliard [6, 7]. The Cosserat theory was extended to coupled stress theories introduced by Toupin [8], Mindlin and Tiersten [9] and others.

The Cosserat theory was extended to full stress theories (including double stress) as follows

- first strain gradient continua was introduced by Mindlin [10] and Mindlin and Eshel [11] (statics and dynamics),

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- second gradient continua was elaborated in Germain [12] and Mindlin [14],
- stress gradient elasticity model was proposed by Eringen [15], who reformulated his earlier studies on nonlocal elasticity,
- considering size effects (cf. Altan and Aifantis [16], Lurie et al. [17], Ma and Gao [18]),
- removing singularities in the stresses and displacements for boundary conditions with discontinues (e.g., Askes et al. [19], Georgiadis and Anagnostou [20], Reiher et al. [21]),
- describing phenomena in the micro- and nanometer range like dislocations [22–24], and
- catching some the phenomena in region with stress concentration [25], and
- including boundary and surface energies [26].

1.1. *Classical elasticity*

Let us assume within the linear elasticity [27] that the strain energy W depends only on the strain tensor $\mathbf{E}_2$

$$W = \frac{1}{2}\mathbf{E}_2 \cdot\cdot \mathbf{C}_4 \cdot\cdot \mathbf{E}_2 \tag{1}$$

with

$$\mathbf{E}_2 = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right], \qquad \mathbf{T}_2 = \mathbf{C}_4 \cdot\cdot \mathbf{E}_2.$$

Here $\mathbf{u}$ is the displacement vector, $\mathbf{T}_2$ is the Cauchy stress tensor and $\mathbf{C}_4$ is the fourth rank elasticity tensor containing the elastic parameters of the considered linear-elastic material. ∇ denotes the Hamilton operator, which acts as follows on the displacement field $\mathbf{u}$

$$\mathbf{u} \otimes \nabla = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i\mathbf{e}_j = u_{i,j}\mathbf{e}_i\mathbf{e}_j$$

and $\cdot$ the scalar (dot) product. In this case the size and the shape of the sample is removed from the material modelling by Hooke's law since the strains are dimensionless.

The classical elasticity has a limited range of application w.r.t.

- size effects and
- discontinuous boundary conditions

among others. The first limitation is due to the fact, that the strains are dimensionless and becomes apparent when small structures are considered. The second limitation is encountered in almost all boundary problems that contain edges, corners and concentrated forces.

1.2. *Strain gradient elasticity*

In this case (first possible extension) the strain energy W depends not only on the strain tensor $\mathbf{E}_2$, but also on the second derivatives of the displacements $\mathbf{E}_3$

$$W(\mathbf{E}_2, \mathbf{E}_3) = \frac{1}{2}\mathbf{E}_2 \cdot\cdot \mathbf{C}_4 \cdot\cdot \mathbf{E}_2 + \frac{1}{2}\mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3, \quad \mathbf{E}_3 = \mathbf{u} \otimes \nabla \otimes \nabla. \tag{2}$$

$\mathbf{C}_6$ is the 6th rank tensor of material parameters. $\otimes$ denotes the dyadic product.

With the increasing miniaturization of components and targeted development of micro-structured materials we need to go beyond the classical theory of elasticity and the limitations of classical elasticity theory can be overcome with gradient expansion of energy ($\mathbf{E}_n isthe (n-1)$-fold gradient of displacement)

$$W(\mathbf{E}_2, \mathbf{E}_3) = \frac{1}{2}\mathbf{E}_2 \cdot\cdot \mathbf{C}_4 \cdot\cdot \mathbf{E}_2 + \frac{1}{2}\mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3 + \ldots + \frac{1}{2}\mathbf{E}_n \underbrace{\cdot\cdot\cdot\cdot}_{n \text{ times}} \mathbf{C}_{2n} \underbrace{\cdot\cdot\cdot\cdot}_{n \text{ times}} \mathbf{E}_n. \tag{3}$$

The last one is a quadratic form without coupled terms.

1.3. *Coupled strain gradient elasticity*

In this case the strain energy W is not only a quadratic form on the strain tensor $\mathbf{E}_2$ and on second derivatives of the displacement $\mathbf{E}_3$, but contains also coupled terms

$$W(\mathbf{E}_2, \mathbf{E}_3) = \frac{1}{2}\mathbf{E}_2 \cdot\cdot \mathbf{C}_4 \cdot\cdot \mathbf{E}_2 + \mathbf{E}_2 \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot \mathbf{E}_3 + \frac{1}{2}\mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3. \tag{4}$$

$\mathbf{C}_5$ is the 5th rank tensor of material parameters.

In classical elasticity there are 8 symmetry classes: triclinic, monoclinic, orthotropic, trigonal, tetragonal, transversely isotropic, cubic and isotropic [28]. In gradient elasticity 17 symmetry classes need to be distinguished, namely triclinic, planar rotations of period 2 to 6 with and without rotations by π around the axes inside the plane, transversely isotropic with and without rotations by π around axes perpendicular to the axis of transverse isotropy, tetra-, octa- and icosahedral symmetries and isotropy.

1.3.1. Basic relations of coupled strain gradient elasticity

Let us introduce the energy density in the simplest case of coupled strain gradient elasticity as Eq. (4). The $\mathbf{C}_4, \mathbf{C}_5$ and $\mathbf{C}_6$ are the relevant stiffness tensors of fourth-, fifth- and sixth-rank. It is obvious that in the case of anisotropic linear elasticity, the number of entries in the tensors becomes very large. Due to a lack of experimental data, concretizations of the tensors for the case of general anisotropy are not known.

With the help of Eq. (4) the stress tensor $\mathbf{T}_2$ and the double stress tensor $\mathbf{T}_3$ can be computed as

$$\mathbf{T}_2 = \frac{\partial W}{\partial \mathbf{E}_2} = \mathbf{C}_4 \cdot\cdot \mathbf{E}_2 + \mathbf{C}_5 \cdot\cdot\cdot \mathbf{E}_3, \quad \mathbf{T}_3 = \frac{\partial W}{\partial \mathbf{E}_3} = \mathbf{C}_5^{\mathrm{T}} \cdot\cdot \mathbf{E}_2 + \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3. \tag{5}$$

The calculation of these tensors is a well-known from the classical elasticity procedure [27]. As shown in [29] the tensors $\mathbf{E}_2, \mathbf{E}_3, \mathbf{C}_4, \mathbf{C}_5$ and $\mathbf{C}_6$ have following general symmetries (not following from the assumption of special cases of anisotropy)

$$E_{ij} = E_{ji}, \quad E_{ijk} = E_{jik},$$

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk},$$

$$C_{ijkln} = C_{jikln} = C_{ijknl},$$

$$C_{ijklmn} = C_{lmnijk} = C_{ikjlmn} = C_{ijklnm}$$

One principle assumption in the theory of elasticity is the positive definiteness of the potential energy (4) [27], which is not easy to prove in the case of coupled strain gradient elasticity. The tensor $\mathbf{C}_5$ is symmetric with respect to the first two and to the last three indices $C^{\mathrm{T}}_{ijklm} = C_{klmij}$, i.e. the first two and the last three entries are exchanged en bloc, such that

$$\mathbf{E}_2 \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot \mathbf{E}_3 = \mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_5^{\mathrm{T}} \cdot\cdot \mathbf{E}_2$$

The positive definiteness can be shown with the help of block diagonalizations. There are two possibilities:

(1) variant
This variant is based on the following modifications of the strain and strain gradient energy density

$$W = \frac{1}{2}\mathbf{E}_2^{\mathrm{m}} \cdot\cdot \mathbf{C}_4 \cdot\cdot \mathbf{E}_2^{\mathrm{m}} + \frac{1}{2}\mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_6^{\mathrm{m}} \cdot\cdot\cdot \mathbf{E}_3 \tag{6}$$

The superscript m denotes the modified strains and the modified stiffness tensor

$$\mathbf{E}_2^{\mathrm{m}} = \mathbf{E}_2 + \mathbf{E}_3 \cdot\cdot\cdot \mathbf{C}_5^{\mathrm{T}} \cdot\cdot \mathbf{C}_4^{-1}, \quad \mathbf{C}_6^{\mathrm{m}} = \mathbf{C}_6 - \mathbf{C}_5^{\mathrm{T}} \cdot\cdot \mathbf{C}_4^{-1} \cdot \mathbf{C}_5$$

(2) variant
Now we assume the following modified strain and strain gradient energy density

$$W = \frac{1}{2}\mathbf{E}_2 \cdot\cdot \mathbf{C}_4^{\mathrm{m}} \cdot\cdot \mathbf{E}_2 + \frac{1}{2}\mathbf{E}_3^{\mathrm{m}} \cdot\cdot\cdot \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3^{\mathrm{m}} \tag{7}$$

The superscript m denotes the modified second gradient of displacement and the modified stiffness tensor

$$\mathbf{E}_3^{\mathrm{m}} = \mathbf{E}_3 + \mathbf{E}_2 \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot \mathbf{C}_6^{-1}, \quad \mathbf{C}_4^{\mathrm{m}} = \mathbf{C}_4 - \mathbf{C}_5 \cdot\cdot\cdot \mathbf{C}_6^{-1} \cdot\cdot\cdot \mathbf{C}_5^{\mathrm{T}}$$

The modified relations were obtained for arbitrary material symmetry classes in [30, 31]. It is obvious that (6) and (7) are quadratic forms and the positive definiteness of the potential energy is guaranteed.

1.3.2. Representation of the stiffness tensors for the case of hemitropy

In accordance with dell'Isola et al. [32], Glüge et al. [13], and Mindlin [10] we obtain the following representations for the stiffness tensors $\mathbf{C}_4, \mathbf{C}_5$ and $\mathbf{C}_6$

$$\begin{aligned}
\mathbf{C}_4 &= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})]\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,\\
\mathbf{C}_5 &= [\kappa(\varepsilon_{imk}\delta_{jl} + \varepsilon_{ilk}\delta_{jm} + \varepsilon_{jmk}\delta_{il} + \varepsilon_{jlk}\delta_{im})]\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,\\
\mathbf{C}_6 &= [c_1(\delta_{jk}\delta_{im}\delta_{nl} + \delta_{jk}\delta_{in}\delta_{ml} + \delta_{ij}\delta_{kl}\delta_{nl} + \delta_{jk}\delta_{im}\delta_{mn})\\
&\quad + c_2(\delta_{ij}\delta_{km}\delta_{nl} + \delta_{jm}\delta_{kl}\delta_{nl} + \delta_{ij}\delta_{kn}\delta_{ml} + \delta_{jn}\delta_{ik}\delta_{ml})\\
&\quad + c_3(\delta_{jm}\delta_{kl}\delta_{in} + \delta_{jl}\delta_{in}\delta_{km} + \delta_{jn}\delta_{im}\delta_{kl} + \delta_{jl}\delta_{im}\delta_{nk})\\
&\quad + c_4(\delta_{jn}\delta_{il}\delta_{km} + \delta_{jm}\delta_{kn}\delta_{il})\\
&\quad + c_5\delta_{il}\delta_{jk}\delta_{mn}]\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n.
\end{aligned}$$

λ and μ are Lamé's parameters, and κ and $c_1, \ldots, c_5$ are higher order material parameters.

Assuming hemitropic symmetry the well known constraints from classical mechanics for λ and μ

$$\mu > 0, \quad 3\lambda + 2\mu > 0$$

can be derived. In addition, the constraints for constitutive parameters $c_{1,\ldots,5}$ and κ can be obtained

$$\begin{aligned}
&c_4 > 0, \quad c_4 - 6\kappa^2/\mu > c_3 > c_4/2, \quad c_5 > 2/5(c_3 - c_4),\\
&c_2 > \frac{-10c_1^2 - 12c_1c_3 - 4c_3^2 + 4c_1c_4 + 2c_3c_4 + 2c_4^2 + c_3c_5 + 3c_4c_5}{2(2c_3 - 4c_4 - 5c_5)}
\end{aligned}$$

Note, that

- for hemitropic materials only one of the inequality constraints for constitutive parameters is affected by presence of the coupling tensor and
- due to the coupling, the shear modulus appears when requiring positive definiteness of the sixth-rank stiffness.

1.3.3. Complementary energy

Applying the Legendre transform to Eq. (4) we get

$$W^*(\mathbf{T}_2, \mathbf{T}_3) = \frac{1}{2}\mathbf{T}_2 \cdot\cdot \mathbf{S}_4 \cdot\cdot \mathbf{T}_2 + \mathbf{T}_2 \cdot\cdot \mathbf{S}_5 \cdot\cdot\cdot \mathbf{T}_3 + \frac{1}{2}\mathbf{T}_3 \cdot\cdot\cdot \mathbf{S}_6 \cdot\cdot\cdot \mathbf{T}_3$$

with the stress tensors (5).

(1) variant
Taking into account (6) modified strain and strain gradient energy density we can write down the complementary energy density

$$W^* = \frac{1}{2}\mathbf{T}_2 \cdot\cdot \mathbf{C}^{-1} \cdot\cdot \mathbf{T}_2 + \frac{1}{2}\mathbf{T}_3^{\mathrm{m}} \cdot\cdot\cdot \mathbf{C}_6^{\mathrm{m}} \cdot\cdot\cdot \mathbf{T}_3^{\mathrm{m}},$$

with

$$\mathbf{T}_2 = \mathbf{C}_4 \cdot\cdot \mathbf{E}_2^{\mathrm{m}}, \quad \mathbf{T}_3 = \mathbf{C}_6^{\mathrm{m}} \cdot\cdot\cdot \mathbf{E}_3.$$

The compliance tensors can be defined as [30, 31]

$$\begin{aligned}
&\mathbf{S}_4 = \mathbf{C}_4^{-1} + \mathbf{C}_4^{-1} \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot (\mathbf{C}_6^{\mathrm{m}})^{-1} \cdot\cdot\cdot \mathbf{C}_5^{\mathrm{T}} \cdot\cdot \mathbf{C}_4^{-1},\\
&\mathbf{S}_5 = \mathbf{C}_4^{-1} + \mathbf{C}_4^{-1} \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot (\mathbf{C}_6^{\mathrm{m}})^{-1},\\
&\mathbf{S}_6 = (\mathbf{C}_6^{\mathrm{m}})^{-1}.
\end{aligned}$$

(2) variant
Taking into account (7) modified strain and strain gradient energy density we can write down the complementary energy density

$$W^* = \frac{1}{2} \mathbf{T}_2^{\mathrm{m}} \cdot\cdot (\mathbf{C}_4^{\mathrm{m}})^{-1} \cdot\cdot \mathbf{T}_2^{\mathrm{m}} + \frac{1}{2} \mathbf{T}_3 \cdot\cdot\cdot \mathbf{C}_6^{-1} \cdot\cdot\cdot \mathbf{T}_3,$$

with

$$\mathbf{T}_2^{\mathrm{m}} = \mathbf{C}_4^{\mathrm{m}} \cdot\cdot \mathbf{E}_2, \quad \mathbf{T}_3 = \mathbf{C}_6 \cdot\cdot\cdot \mathbf{E}_3^{\mathrm{m}}$$

The compliance tensors are in this case [30, 31]

$$\begin{aligned}
&\mathbf{S}_4 = (\mathbf{C}_4^{\mathrm{m}})^{-1},\\
&\mathbf{S}_5 = (\mathbf{C}_4^{\mathrm{m}})^{-1} \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot \mathbf{C}_6^{-1},\\
&\mathbf{S}_6 = \mathbf{C}_6^{-1} + \mathbf{C}_6^{-1} \cdot\cdot\cdot \mathbf{C}_5^{\mathrm{T}} \cdot\cdot (\mathbf{C}_4^{\mathrm{m}})^{-1} \cdot\cdot \mathbf{C}_5 \cdot\cdot\cdot \mathbf{C}_6^{-1}.
\end{aligned}$$

Let us summarize the results. The following changes should be performed to guarantee the equivalence of the potential and the complementary energies:

(1) variant

$$\{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\} \to \{\mathbf{C}_4, \mathbf{C}_6^{\mathrm{m}}\},$$

$$\{\mathbf{C}_4, \mathbf{C}_6^{\mathrm{m}}\} \to \{\mathbf{S}_4, \mathbf{S}_6^{\mathrm{m}}\},$$

$$\{\mathbf{S}_4, \mathbf{S}_5^{\mathrm{m}}\} \to \{\mathbf{S}_4, \mathbf{S}_5, \mathbf{S}_6\}$$

(2) variant

$$\{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\} \to \{\mathbf{C}_4^{\mathrm{m}}, \mathbf{C}_6\},$$

$$\{\mathbf{C}_4^{\mathrm{m}}, \mathbf{C}_6\} \to \{\mathbf{S}_4^{\mathrm{m}}, \mathbf{S}_6\},$$

$$\{\mathbf{S}_4^{\mathrm{m}}, \mathbf{S}_6\} \to \{\mathbf{S}_4, \mathbf{S}_5, \mathbf{S}_6\}$$

In addition, the compliance tensors obtained from the both variants of modified energy density are identical and the tensorial relations for the compliance tensors are obtained for arbitrary material symmetry classes.

2. Variational equation of the equilibrium and boundary conditions

2.1. *Variational equation*

Let us introduce the variational equation of the equilibrium. The total potential energy with a variation of $\mathbf{u}$ is

$$\delta \int\limits_V W \mathrm{d}V = \int\limits_V (\mathbf{T}_2 \cdot\cdot \delta \mathbf{E}_2 + \mathbf{T}_3 \cdot\cdot\cdot \delta \mathbf{E}_3) \mathrm{d}V$$

or

$$\begin{aligned}\delta \int\limits_V W \mathrm{d}V = & \int\limits_V \delta \mathbf{u} (\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla) \cdot \nabla \mathrm{d}V \\ & + \int\limits_S \delta \mathbf{u} \left\{ (\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla - \mathbf{T}_3 \cdot \nabla_S) \cdot \mathbf{n} \right. \\ & + \left. \mathbf{T}_3 \cdot\cdot [(\mathbf{n} \cdot \nabla_S) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_S] \right\} \mathrm{d}S \\ & + \delta \mathbf{u} \otimes \nabla_{\mathbf{n}} \cdot\cdot \mathbf{T}_3 \cdot \mathbf{n} \mathrm{d}S \\ & + \oint\limits_C \delta \mathbf{u} \cdot (\mathbf{T}_3 \cdot\cdot \mathbf{n} \otimes \mathbf{m})] \mathrm{d}C.\end{aligned}$$

Then the principle of the Lagrangian stationary (principle of virtual power) is

$$\delta \mathcal{L} = \delta A - \delta \int\limits_V W \mathrm{d}V = 0,$$

where A is a work of external forces and double forces and W is a strain and strain gradient energy density. The variation of the strain and strain gradient energy requires an admissible form of the work

$$\begin{aligned}A &= \int\limits_V \mathbf{u} \cdot \mathbf{f} \mathrm{d}V + \int\limits_S (\mathbf{u} \cdot \mathbf{p} + \mathbf{u} \otimes \nabla_{\mathbf{n}} \cdot\cdot \mathbf{R}) \mathrm{d}S + \oint\limits_C \mathbf{u} \cdot \mathbf{c} \mathrm{d}C \\ &= \int\limits_V \mathbf{u} \cdot \mathbf{f} \mathrm{d}V + \int\limits_S (\mathbf{u} \cdot \mathbf{p} + \mathrm{D}\mathbf{u} \cdot \mathbf{r}_{\mathbf{n}}) \mathrm{d}S + \oint\limits_C \mathbf{u} \cdot \mathbf{c} \mathrm{d}C\end{aligned}$$

with

$$\mathbf{u} \otimes \nabla_{\mathbf{n}} = \frac{\partial u_i}{\partial x_j} n_j \mathbf{e}_i \otimes \mathbf{n} = (\mathrm{D}\mathbf{u}) \otimes \mathbf{n}, \quad \mathrm{D}\mathbf{u} = (\otimes \nabla) \cdot \mathbf{n}.$$

On details including the uniqueness is reported in [33].

2.2. *Boundary conditions*

2.2.1. *Natural boundary conditions*

The stress equilibrium equation is

$$(\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}$$

and the natural (static) boundary conditions for the surface tractions and the edge forces, which can be prescribed, are:

- vector field of the tractions on the part of the surface of the body S_d

$$\mathbf{p}_\mathrm{pr} = (\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla - \mathbf{T}_3 \nabla_S) \cdot \mathbf{n} + \mathbf{T}_3 \cdot\cdot [(\mathbf{n} \cdot \nabla_S)\mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_S,$$

- double tractions in normal direction on S_g

$$\mathbf{r}_{\mathbf{n}\ \mathrm{pr}} = \mathbf{T}_3 \cdot\cdot \mathbf{n} \otimes \mathbf{n} \quad \text{on} \quad S_\mathrm{g},$$

- line forces on edge on the part of edge C_g

$$\mathbf{c}_\mathrm{pr} = \mathbf{T}_3 \cdot\cdot \mathbf{n} \otimes \mathbf{m} \quad \text{on} \quad C_\mathrm{g}.$$

Subscript "pr" denotes "prescribed" and S_d is the surface where the natural or traction boundary conditions are prescribed.

2.2.2. *Displacement boundary conditions*

The stress equilibrium equation is as in the previous case. The displacement (kinematic) boundary conditions in terms of the displacement fields $\mathbf{u}$ and its normal gradient, which can be prescribed as:

- displacement field $\mathbf{u}$ on the part of the surface of the body S_g

$$\mathbf{u}_\mathrm{pr} = \mathbf{u} \quad \text{on} \quad S_\mathrm{g},$$

- normal gradient of the displacement field $\mathbf{u}$ on S_g

$$\mathrm{D}\mathbf{u}_\mathrm{pr} = \mathrm{D}\mathbf{u} \quad \text{on} \quad S_\mathrm{g},$$

- displacement $\mathbf{u}$ on the part of the edge C_g

$$\mathbf{u}_\mathrm{pr} = \mathbf{u} \quad \text{on} \quad C_\mathrm{g}.$$

2.2.3. *Mixed boundary conditions*

Now the stress equilibrium equation is as follows

$$(\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}.$$

The mixed boundary conditions, prescribed are:

- body force $\mathbf{f}$ in the interior of the body V,

- vector field of the tractions $\mathbf{p}$ on the part of the surface of the body S_{d},
- displacement fields $\mathbf{u}$ on the surface of the body S_{g},
- double tractions $\mathbf{r_n}$ in normal direction on the surface of the body S_{d},
- normal gradient of the displacement fields $\mathrm{D}\mathbf{u}$ on the surface S_{g},
- line forces $\mathbf{c}$on edge on the edges C_{d} of the surface of the body S,
- displacement fields $\mathbf{u}$ on the edge C_{g} of the surface of the body S,

$$S_{\mathrm{d}} \bigcup S_{\mathrm{g}} = S, \quad C_{\mathrm{g}} \bigcup C_{\mathrm{g}} = C.$$

3. Principles of minimum of the potential and complementary energy

3.1. *Uncoupled strain gradient elasticity*

Let us assume the simplest form (no coupling)

$$W^*(\mathbf{T}_2, \mathbf{T}_3) = \frac{1}{2}\mathbf{T}_2 \cdot\cdot \mathbf{S}_4 \cdot\cdot \mathbf{T}_2 + \frac{1}{2}\mathbf{T}_3 \cdot\cdot\cdot \mathbf{S}_6 \cdot\cdot\cdot \mathbf{T}_3, \quad \mathbf{S}_3 = \mathbf{0}.$$

The generalization of the principle of minimum of potential energy to gradient elasticity is relatively simple, see e.g. Kirchner and Steinmann [34], Polizzotto [35, 36], Gao and Park [37], Georgiadis and Grentzelou [38]. The extension of the principle of minimum of complementary energy is less clear. There are some works, like Polizzotto [36], Georgiadis and Grentzelou [38] examining the stress gradient elasticity. The change of the independent variable from one displacement field to several stress fields ($\mathbf{T}_2$ and $\mathbf{T}_3$) is rather complicated. It is not necessary in Polizzotto [36], since he considers a stress field and its gradient.

3.2. *Coupled strain gradient elasticity*

3.2.1. Principle of minimum of potential energy

Theorem 3.1 : *Let us consider the linear elastic gradient material with the positive definite potential energy density, let A be the set of all (compatible) vector fields $\mathbf{u}(\mathbf{x})$ which fulfil the displacement boundary conditions on the part of the body surface $S_{\mathrm{g}}(S_{\mathrm{d}} \bigcup S_{\mathrm{g}} = S)$ and on the part of the surface edge $C_{\mathrm{g}}(C_{\mathrm{d}} \bigcup C_{\mathrm{g}} = C)$. We define the following functional $\Phi : \mathfrak{A} \rightarrow \mathfrak{R}$*

$$\Phi(\mathbf{u}) \equiv \delta \int_V W \mathrm{d}V - \int_V \delta\mathbf{u} \cdot \mathbf{f} \mathrm{d}V - \int_{S_{\mathrm{d}}} (\delta\mathbf{u} \cdot \mathbf{p}_{\mathrm{pr}}) + \mathrm{D}\delta\mathbf{u} \cdot r_{\mathrm{n\ pr}}) \mathrm{d}S + \oint_{C_{\mathrm{d}}} \mathbf{u} \cdot \mathbf{c}_{\mathrm{pr}} \mathrm{d}C.$$

where body forces, surface tractions and edge forces are prescribed. Then the functional obtains a minimum for the solution $\mathbf{u}^0$ of the mixed boundary value problem, i.e.,

$$\Phi(\mathbf{u}^0) \leq \Phi(\mathbf{u}) \quad \forall \mathbf{u} \in \mathfrak{A}$$

If the equality holds, then $\mathbf{u}^0$ and $\mathbf{u}$ differ only by a rigid body displacement

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}^0(\mathbf{x}) = \mathbf{u}_{\mathrm{c}} + \mathbf{\Omega} \cdot (\mathbf{x} - \mathbf{x}_0)$$

where $\mathbf{u}_{\mathrm{c}}$ *and* $\mathbf{x}_0$ *are two constant vectors and* $\boldsymbol{\Omega}$ *is a constant antisymmetric tensor.*

The proof of this theorem is given in [39].

3.2.2. Principle of minimum of complementary energy

Theorem 3.2: *Let us consider the linear elastic gradient material with the positive definite complementary energy density, let* $\mathfrak{T}$ *be the set of symmetric stress fields* $\mathbf{T}_2$ *and of couple stress fields* $\mathbf{T}_3$ *that obey the equilibrium conditions* $(\mathbf{T}_2 - \mathbf{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}$ *and the natural boundary conditions for the prescribed vector field of the tractions, the double tractions in normal direction, and the line forces on* S_{d} *and* C_{d}*. Then the functional*

$$\Psi(\mathbf{T}_2, \mathbf{T}_3) = \int\limits_V W^* \mathrm{d}V - \int\limits_{S_{\mathrm{g}}} (\mathbf{u}_{\mathrm{pr}} \cdot \mathbf{p} + \mathrm{D}\mathbf{u}_{\mathrm{pr}} \cdot \mathbf{r}_{\mathbf{n}}) \mathrm{d}S - \oint \mathbf{u}_{\mathrm{pr}} \cdot \mathbf{p} \mathrm{d}C$$

with the complementary elastic energy W^* *obtains for the solution* $\mathbf{T}_2^0, \mathbf{T}_3^0$ *of the mixed boundary value problem a minimum in* $\mathfrak{T}$*, i.e.,*

$\Psi(\mathbf{T}_2^0, \mathbf{T}_3^0) \leq \Psi(\mathbf{T}_2, \mathbf{T}_3) \quad \forall \mathbf{T}_2, \mathbf{T}_3 \in \mathfrak{T}$

$$\Psi(\mathbf{T}_2^0, \mathbf{T}_3^0) \leq \Psi(\mathbf{T}_2, \mathbf{T}_3) \quad \forall \mathbf{T}_2, \mathbf{T}_3 \in \mathfrak{T}$$

If the equality holds, then

$$\mathbf{T}_2^0 = \mathbf{T}_2 \quad \text{and} \quad \mathbf{T}_3^0 = \mathbf{T}_3.$$

The proof of this theorem is given in [39].

4. Concluding remarks

The well-established theoretical foundations of classical elasticity can be expanded to coupled strain gradient elasticity. The positive definiteness conditions for the strain and strain gradient energy within linear coupled gradient elasticity have been obtained for a hemitropic material. In the case of hemitropic materials only one of the inequality constraints for the constitutive parameters $c_{1,2,3,4,5}$ is affected by presence of coupling tensor $\mathbf{C}_5$. The inverse Hooke's law and complementary strain energy density has been examined in the context of the theory of coupled gradient elasticity; results are valid for an arbitrary material symmetry class. The most important theorems of classical elasticity can be proved for coupled strain gradient elasticity (uniqueness theorem, principles of a minimum of potential and complementary energies, reciprocal theorem).

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