MAIN ARTICLES

CONTINUITY OF THE QUENCHING TIME IN A HEAT EQUATION WITH A NONLINEAR BOUNDARY CONDITION

Louis A. Assalé*), Théodore K. Boni*), and Firmin K. N'gohisse)**

*) *Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, (Côte d'Ivoire)*

**) *Université d'Abobo-Adjamé, UFR-SFA, Département de Mathématiques et Informatiques, 02 BP 801 Abidjan 02, (Côte d'Ivoire)*

Abstract: In this paper, we address the following initial-boundary value problem

$$\begin{aligned} &u_t = \Delta u \quad \text{in} \quad \Omega \times (0,T), \\ &\tfrac{\partial u}{\partial \nu} = -b(x)u^{-p} \quad \text{on} \quad \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 0$, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$, $u_0 \in C^2(\overline{\Omega})$, $u_0(x) > 0$ in $\overline{\Omega}$, $b \in C^0(\partial\Omega)$, $b(x) \gneqq 0$ on $\partial\Omega$. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial data u_0. Finally, we give some numerical results to illustrate our analysis.

Key words: Quenching, heat equation, nonlinear boundary condition, numerical quenching time

MSC 2000: 35B40, 35B50, 35K60, 65M06

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial \nu} = -b(x)u^{-p} \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{3}$$

where Δ is the Laplacian, $p > 0$, $u_0 \in C^2(\overline{\Omega})$, $u_0(x) > 0$ in $\overline{\Omega}$, $b \in C^0(\partial\Omega)$, $b(x) \gneqq 0$ on $\partial\Omega$.

Here $(0, T)$ is the maximal time interval on which the solution u of (1)-(3) exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t \to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x \in \overline{\Omega}} u(x, t)$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Thus, in this paper, by virtue of the definition of the time T, we have

$$u(x, t) > 0 \quad \text{in} \quad \overline{\Omega} \times [0, T).$$

Solutions of heat equations with nonlinear boundary conditions which quench in a finite time have been the subject of investigations of many authors (see, [7], [11], [14], [28], and the references cited therein). By standard methods, it is not hard to prove the local in time existence and uniqueness of a classical solution (see, [7], [26]). Let us notice that, in most of the cases, the problem (1)–(3) has been treated in one dimensional space. For instance, when $N = 1$, $\Omega = (0, 1)$, $b(0) = 0$ and $b(1) = 1$, Fila and Levine have shown that the solution u of (1)–(3) quenches in a finite time, and the quenching occurs at the point $x = 1$. However, in [7], Boni has worked in several dimensional spaces, and he has proved that the solution u of (1)-(3) quenches in a finite, and its quenching set is located on the boundary of the domain Ω in the case where $b(x) = 1$ (see, [7]). It is worth noting that in these papers, one may see some estimation of the quenching time when quenching occurs. For quenching results of other problems, one may consult the following references [2]-[4], [10], [13], [24], [25], [27], [29], [32]. In the present paper, we are interested in the dependence of the quenching time with respect to the initial data. In other words, we want to know if the quenching time as a function of the initial data is continuous. More precisely, let us consider the solution v of the initial-boundary value problem below

$$v_t = \Delta v \quad \text{in} \quad \Omega \times (0, T_h), \tag{4}$$

$$\frac{\partial v}{\partial \nu} = -b(x) v^{-p} \quad \text{on} \quad \partial\Omega \times (0, T_h), \tag{5}$$

$$v(x, 0) = u_0^h(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{6}$$

where $u_0^h \in C^2(\overline{\Omega})$, $u_0^h(x) \geq u_0(x)$ in $\overline{\Omega}$, and $\lim_{h \to 0} u_0^h = u_0$. Here $(0, T_h)$ is the maximal time interval of existence of the solution v. Let us notice that, from the maximum principle, we have $v \geq u$ as long as all of them are defined. We deduce that $T_h \geq T$. In the present paper, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time T_h, and the following relation holds

$$\lim_{h\to 0} T_h = T.$$

Similar results have been obtained in [5], [8], [12], [16]-[19], [21], [22], where the authors have considered the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). Our paper is organized as follows. In the next section, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time. In the third section, we prove the continuity of the quenching time, and finally in the last section, we give some computational results.

2 Quenching time

In this section, under some hypotheses, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time.
Using an idea of Friedman and Lacey in [15], we may prove the following result.

Theorem 2.1. *Let v be the solution of (4)–(6), and assume that there exists a constant $A \in (0, 1]$ such that the initial data at (6) satisfies*

$$\Delta u_0^h(x) \leq -A(u_0^h(x))^{-p} \quad in \quad \overline{\Omega}. \tag{7}$$

Then, the solution v quenches in a finite time T_h which obeys the following estimate

$$T_h \leq \frac{(u_{0min}^h)^{p+1}}{A(p+1)},$$

where $u_{0min}^h = \min_{x\in\overline{\Omega}} u_0^h(x)$.

Proof. Since $(0, T_h)$ is the maximal time interval of existence of the solution v, our purpose is to show that T_h is finite and obeys the above inequality. Introduce the function $J(x, t)$ defined as follows

$$J(x, t) = v_t(x, t) + Av^{-p}(x, t) \quad \text{in} \quad \overline{\Omega} \times [0, T_h).$$

A straightforward computation reveals that

$$J_t - \Delta J = (V_t - \Delta v)_t - Apv^{-p-1}v_t - A\Delta v^{-p} \quad \text{in} \quad \Omega \times (0, T_h). \tag{8}$$

Again, by a direct calculation, it is not hard to see that

$$\Delta v^{-p} = p(p+1)v^{-p-2}|\nabla v|^2 - pv^{-p-1}\Delta v \quad \text{in} \quad \Omega \times (0, T_h),$$

which implies that $\Delta v^{-p} \geq -pv^{-p-1}\Delta v$ in $\Omega \times (0,T_h)$. Using this estimate and (8), we arrive at

$$J_t - \Delta J \leq (V_t - \Delta v)_t - Apv^{-p-1}(v_t - \Delta v) \quad \text{in} \quad \Omega \times (0,T_h). \tag{9}$$

It follows from (4) that

$$J_t - \Delta J \leq 0 \quad \text{in} \quad \Omega \times (0,T_h).$$

We also have

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial v}{\partial \nu}\right)_t - Apv^{-p-1}\frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial\Omega \times (0,T_h).$$

We deduce from (5) that

$$\frac{\partial J}{\partial \nu} = pb(x)v^{-p-1}v_t + Apb(x)v^{-2p-1} \quad \text{on} \quad \partial\Omega \times (0,T_h).$$

Due to the expression of J, we find that

$$\frac{\partial J}{\partial \nu} = pb(x)v^{-p-1}J \quad \text{on} \quad \partial\Omega \times (0,T_h).$$

Finally, we get

$$J(x,0) = v_t(x,0) + Av^{-p}(x,0) \leq \Delta v_0(x) + A(u_0^h(x))^{-p} \quad \text{in} \quad \overline{\Omega}.$$

Thanks to (7), we discover that

$$J(x,0) \leq 0 \quad \text{in} \quad \overline{\Omega}.$$

It follows from the maximum principle that

$$J(x,t) \leq 0 \quad \text{in} \quad \overline{\Omega} \times (0,T_h).$$

This estimate may be rewritten in the following manner

$$v^p dv \leq -Adt \quad \text{in} \quad \overline{\Omega} \times (0,T_h). \tag{10}$$

Integrate the above inequality over $(0,T_h)$ to obtain

$$T_h \leq \frac{(v(x,0))^{p+1}}{A(p+1)} \quad \text{in} \quad x \in \overline{\Omega}, \tag{11}$$

which implies that

$$T_h \leq \frac{(u_{0min}^h)^{p+1}}{A(p+1)}. \tag{12}$$

Use the fact that the quantity on the right hand side of (12) is finite to complete the rest of the proof. □

Remark 2.2. *Let $t_0 \in (0, T_h)$. Integrating the inequality (10) over (t_0, T_h), we get*

$$T_h - t_0 \leq \frac{(v(x, t_0))^{p+1}}{A(p+1)} \quad for \quad x \in \overline{\Omega},$$

which implies that

$$T_h - t_0 \leq \frac{(v_{\min}(t_0))^{p+1}}{A(p+1)}. \tag{13}$$

3 Continuity of the quenching time

In this section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time, and its quenching time goes to that of the solution u of (1)–(3) when h goes to zero.
We denote by

$$\|u(\cdot, t)\|_\infty = \max_{x \in \overline{\Omega}} |u(x, t)| \quad \text{and} \quad \|u_0\|_\infty = \max_{x \in \overline{\Omega}} |u_0(x)|.$$

In order to demonstrate our result, we firstly show that the solution v approaches the solution u in $\Omega \times [0, T - \tau]$ with $\tau \in (0, T)$ when h tends to zero. This result is stated in the following theorem.

Theorem 3.1. *Let u be the solution of (1)–(3). Suppose that $u \in C^{2,1}(\overline{\Omega} \times [0, T - \tau])$ and $\min_{t \in [0, T-\tau]} u_{\min}(t) = \alpha > 0$ with $\tau \in (0, T)$. Then, the problem (4)-(6) admits a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T_h))$, and the following relation holds*

$$\sup_{t \in [0, T-\tau]} \|v(\cdot, t) - u(\cdot, t)\|_\infty = 0(\|u_0^h - u_0\|_\infty) \quad as \quad h \to 0.$$

Proof. For each h, the problem (4)–(6) has a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T_h))$. In the introduction of the paper, we have seen that $T_h \geq T$. We observe that

$$\|v(\cdot, 0) - u(\cdot, 0)\|_\infty = \|u_0^h - u_0\|_\infty. \tag{14}$$

Let $t(h) \leq T - \tau$ be the greatest value of $t > 0$ such that

$$\|v(\cdot, t) - u(\cdot, t)\|_\infty < \frac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{15}$$

Making use of the relation (14), we note that $t(h) > 0$ for h sufficiently small. An application of the triangle inequality, gives

$$v_{\min}(t) \geq u_{\min}(t) - \|v(\cdot, t) - u(\cdot, t)\|_\infty \quad \text{for} \quad t \in (0, t(h)),$$

which implies that

$$v_{\min}(t) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)).$$

Introduce the function e defined as follows

$$e(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)).$$

A routine computation reveals that

$$e_t - \Delta e = 0 \quad \text{in} \quad \Omega \times (0, t(h)),$$

$$\frac{\partial e}{\partial \nu} = pb(x)\theta^{-p-1}e \quad \text{on} \quad \partial\Omega \times (0, t(h)),$$

$$e(x,0) = u_0^h(x) - u_0(x) \quad \text{in} \quad \overline{\Omega},$$

where θ is an intermediate value between u and v. Since the domain Ω has a smooth boundary $\partial\Omega$, there exists a function $\rho \in C^2(\overline{\Omega})$ satisfying $\rho(x) \geq 0$ in Ω and $\frac{\partial \rho}{\partial \nu} = 1$ on $\partial\Omega$. Let K be a positive constant such that $K \geq L\Delta\varphi + L^2|\nabla\varphi|^2$ for $x \in \overline{\Omega}$, where $L = p\|b\|_\infty(\frac{\alpha}{2})^{-p-1}$. It is not hard to see that $p(\frac{\alpha}{2})^{-p-1} \geq p\theta^{-p-1}$ in $\partial\Omega \times (0, t(h))$. Introduce the function z defined as follows

$$z(x,t) = e^{Lt + L\varphi(x)}\|u_0^h - u_0\|_\infty \quad \text{in} \quad \overline{\Omega} \times [0, T].$$

A straightforward calculation reveals that

$$z_t - \Delta z = (K - L\Delta\varphi - L^2|\nabla\varphi|^2)z \quad \text{in} \quad \Omega \times (0, t(h)),$$

$$\frac{\partial z}{\partial \nu} = Lz \quad \text{on} \quad \partial\Omega \times (0, t(h)),$$

$$z(x,0) \geq e(x,0) \quad \text{in} \quad \overline{\Omega}.$$

Since $L \geq pb(x)\theta^{-p-1}$ on $\partial\Omega \times (0, t(h))$, and $K \geq L\Delta\varphi + L^2|\Delta\varphi|^2 \quad \text{for} \quad x \in \overline{\Omega}$, we deduce that

$$z_t - \Delta z \geq 0 \quad \text{in} \quad \Omega \times (0, t(h)),$$

$$\frac{\partial z}{\partial \nu} \geq pb(x)\theta^{-p-1}z \quad \text{on} \quad \partial\Omega \times (0, t(h)),$$

$$z(x,0) \geq e(x,0) \quad \text{in} \quad \overline{\Omega}.$$

It follows from the maximum principle that

$$z(x,t) \geq e(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)).$$

In the same way, we also prove that

$$z(x,t) \geq -e(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)),$$

which implies that

$$\|e(\cdot,t)\|_\infty \leq e^{Kt+L\|\varphi\|_\infty} \|u_0^h - u_0\|_\infty \quad \text{for} \quad t \in (0, t(h)).$$

Let us show that $t(h) = T - \tau$. Suppose that $t(h) < T - \tau$. From (15), we obtain

$$\frac{\alpha}{2} = \|v(\cdot, t(h)) - u(\cdot, t(h))\|_\infty \leq e^{KT+L\|\varphi\|_\infty} \|u_0^h - u_0\|_\infty.$$

Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. Consequently $t(h) = T - \tau$, and the proof is complete. □
Now, we are in a position to prove the main result of the paper.

Theorem 3.2. *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time T such that $u \in C^{2,1}(\overline{\Omega} \times [0,T))$. Under the assumption of Theorem 2.1, the problem (4)-(6) admits a unique solution v which quenches in a finite time T_h, and the following relation holds*

$$\lim_{h \to 0} T_h = T.$$

Proof. Let $0 < \varepsilon \leq T/2$. There exists $\rho > 0$ such that

$$\frac{\rho^{p+1}}{A(p+1)} \leq \frac{\varepsilon}{2}. \tag{16}$$

Since u quenches in a finite time T, there exists $T_0 \in (T - \frac{\varepsilon}{2}, T)$ such that

$$0 < u_{\min}(t) < \frac{\rho}{2} \quad \text{for} \quad t \in [T_0, T).$$

Obviously, we have

$$u_{\min}(t) > 0 \quad \text{for} \quad t \in [0, T_0].$$

Invoking Theorem 3.1, we see that the problem (4)–(6) admits a unique solution v, and the following relation holds

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t \in [0, T_0],$$

which implies that $\|v(\cdot,T_0)-u(\cdot,T_0)\|_\infty \le \frac{\rho}{2}$. An application of the triangle inequality leads us to

$$v_{\min}(T_0) \le \|v(\cdot,T_0)-u(\cdot,T_0)\|_\infty + u_{\min}(T_0) \le \frac{\rho}{2}+\frac{\rho}{2}=\rho.$$

Exploiting Theorem 2.1, we note that the solution v quenches at the time T_h. In the introduction of the paper, we have shown that $T_h \ge T$. We infer from Remark 2.1 and (16) that

$$0 \le T_h - T = T_h - T_0 + T_0 - T \le \frac{(v_{\min}(T_0))^{p+1}}{A(p+1)} + \frac{\varepsilon}{2} \le \varepsilon,$$

and the proof is complete. □

4 Numerical results

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \Delta u \quad \text{in} \quad B \times (0,T),$$

$$\frac{\partial u}{\partial \nu} = -b(x)u^{-p} \quad \text{on} \quad S \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \overline{B},$$

where $B = \{x \in \mathbb{R}^N;\ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N; \|x\| = 1\}$, $b(x) = \beta(|x|)$, and $u_0(x) = \varphi(|x|)$. The above problem may be rewritten in the following form

$$u_t = u_{rr} + \frac{N-1}{r}u_r, \quad r \in (0,1), \quad t \in (0,T), \tag{17}$$

$$u_r(0,t) = 0, \quad u_r(1,t) = -\beta(1)(u(1,t))^{-p}, \quad t \in (0,T), \tag{18}$$

$$u(r,0) = \varphi(r), \quad r \in [0,1]. \tag{19}$$

Here, we take $\beta(1) = 1$, $p = 1$, and $\varphi(r) = \frac{3-r^2}{2} + \varepsilon(\frac{1+\cos(\pi r)}{2})$ with $\varepsilon \in [0,1)$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$, and approximate the solution u of (17)-(19) by the solution $U_h^{(n)} = (U_0^{(n)}, ..., U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h},$$

$$1\leq i\leq I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=\frac{2U_{I-1}^{(n)}-2U_I^{(n)}}{h^2}+(N-1)\frac{U_I^{(n)}-U_{I-1}^{(n)}}{h}-\frac{2}{h}(U_I^{(n)})^{-p},$$

$$U_i^{(0)}=\varphi_i,\quad 0\leq i\leq I,$$

where $n\geq 0$. In order to permit the discrete solution to reproduce the property of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n=\min\{\frac{(1-h^2)h^2}{2N},h^2(U_{hmin}^{(n)})^{p+1}\},$$

with $U_{hmin}^{(n)}=\min_{0\leq i\leq I}U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (17)-(19) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h},\quad 1\leq i\leq I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=\frac{2U_{I-1}^{(n+1)}-2U_I^{(n+1)}}{h^2}+(N-1)\frac{U_I^{(n+1)}-U_{I-1}^{(n+1)}}{h}-\frac{2}{h}(U_I^{(n)})^{-p-1}U_I^{(n+1)}$$

$$U_i^{(0)}=\varphi_i,\quad 0\leq i\leq I,$$

where $n\geq 0$ Here, as in the case of the explicit scheme, we pick

$$\Delta t_n=h^2(U_{hmin}^{(n)})^{p+1}.$$

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [6]). It is not hard to see that $u_{rr}(0,t)=\lim_{r\to 0}\frac{u_r(r,t)}{r}$. Hence, if $r=0$, then, we note that

$$u_t(0,t)=Nu_{rr}(0,t),\quad t\in(0,T).$$

This observation has been taken into account in the construction of our schemes at the first node. We need the following definition.

Definition 4.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to\infty} U_{hmin}^{(n)} = 0$, *and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the numerical quenching time of the discrete solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times, and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$, which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $p = 1$, $N = 2$
First case: $\varepsilon = 0$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.160184	294	0.5	-
32	0.158065	957	2.6	-
64	0.157405	3387	11.3	1.68
128	0.157205	12697	129	1.72

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.161968	214	1.5	-
32	0.158539	639	2.8	-
64	0.157527	2115	15.8	1.76
128	0.157236	7610	210	1.79

Second case: $\varepsilon = 1$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.312115	451	0.6	-
32	0.309584	1579	2.11	-
64	0.308820	5871	17.2	1.73
128	0.308594	22628	224	1.76

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.312640	251	0.5	-
32	0.309741	786	1.7	-
64	0.308863	2703	18	1.72
128	0.308606	9959	320	1.77

Third case: $\varepsilon = 1/100$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.161288	295	0.7	-
32	0.159164	961	2.2	-
64	0.158503	3405	11	1.68
128	0.158303	12770	185	1.72

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.163064	214	0.5	-
32	0.159636	640	2.4	-
64	0.158625	2121	14.4	1.76
128	0.158334	7635	217	1.80

Third case: $\varepsilon = 1/10000$

Table 7: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.160195	294	0.5	-
32	0.158076	957	1.4	-
64	0.157416	3387	11	1.68
128	0.157216	12698	152	1.72

Table 8 : Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.161979	214	0.5	-
32	0.158506	639	1.4	-
64	0.157538	2115	13.7	1.76
128	0.157247	7610	214	1.80

Remark 4.1. *If we consider the problem (17)-(19) in the case where the initial data* $\varphi(r) = \frac{3-r^2}{2} + \varepsilon(\frac{1+\cos(\pi r)}{2})$, *then we observe from Tables 1 to 6 that, if* $\varepsilon \in (0,1)$ *is small enough, then the numerical quenching time is close to that of the solution of (17)-(19) in the case where* $\varepsilon = 0$. *This computational result confirms the theory established in the previous section.*

In the following, we also give some plots to illustrate our analysis. In the figures below, we can appreciate that the discrete solution quenches and the quenching occurs at the last node.

References

[1] L. M. Abia, J. C. López-Marcos and J. Martínez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, Appl. Numer. Math., 26 (1998), 399-414.

[2] A. Acker and W. Walter, The quenching problem for nonlinear parabolic differential equations, Lecture Notes in Math., Springer-Verlag, New York, 564 (1976), 1-12.

[3] A. Acker and B. Kawohl, Remarks on quenching, Nonl. Anal. TMA, 13 (1989), 53-61.

[4] C. Bandle and C. M. Braumer, Singular perturbation method in a parabolic problem with free boundary, in Proc. BAIL IVth Conference, Boole Press Conf. Ser. 8, Novosibirsk, (1986), 7-14.

[5] P. Baras and L. Cohen, Complete blow-up after T_{max} for the solution of a semilinear heat equation, *J. Funct. Anal.*, 71 (1987), 142-174.

[6] T. K. Boni, Extinction for discretizations of some semilinear parabolic equations, C.R.A.S., Serie I, 333 (2001), 795-800.

[7] T. K. Boni, On quenching of solutions for some semilinear parabolic equations of second order, Bull. Belg. Math. Soc., 7 (2000), 73-95.

[8] C. Cortazar, M. del Pino and M. Elgueta, On the blow-up set for $u_t = \Delta u^m + u^m$, $m > 1$, Indiana Univ. Math. J., 47 (1998), 541-561.

[9] C. Cortazar, M. del Pino and M. Elgueta, Uniqueness and stability of regional blow-up in a porous-medium equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), 927-960.

[10] K. Deng and H. A Levine, On the blow-up of u_t at quenching, Proc. Amer. Math. Soc., 106 (1989), 1049-1056.

[11] K. Deng and M. Xu, Quenching for a nonlinear diffusion equation with singular boundary condition, Z. Angew. Math. Phys., 50 (1999), 574-584.

[12] C. Fermanian Kammerer, F. Merle and H. Zaag, Stability of the blow-up profile of nonlinear heat equations from the dynamical system point of view, Math. Ann., 317 (2000), 195-237.

[13] M. Fila, B. Kawohl and H. A. Levine, Quenching for quasilinear equations, Comm. Part. Diff. Equat., 17 (1992), 593-614.

[14] M. Fila and H. A. Levine, Quenching on the boundary, Nonl. Anal. TMA, 21 (1993), 795-802.

[15] A. Friedman and B. Mcleod, Blow-up of positive solutions of nonlinear heat equations, Indiana Univ. Math. J., 34 (1985), 425-477.

[16] V. A. Galaktionov, Boundary value problems for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$, Differ. Equat., 17 (1981), 551-555.

[17] V. Galaktionov and J. L. Vazquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math., 50 (1997), 1-67.

[18] V. Galaktionov and J. L. Vazquez, The problem of blow-up in nonlinear parabolic equation, current developments in PDE (Temuco,1999), Discrete Contin. Dyn. Syst., 8 (2002), 399-433.

[19] V. A. Galaktionov, S. P. KMiklhailov and A. A. Samarskii, Unbounded solutions of the Cauchy problem for the parabolic equation $u_t = \nabla(u^{\sigma}\nabla u) + u^{\beta}$, Soviet Phys. Dokl., 25 (1980), 458-459.

[20] J. Guo, On a quenching problem with Robin boundary condition, Nonl. Anal. TMA, 17 (1991), 803-809.

[21] P. Groisman and J. D. Rossi, Dependance of the blow-up time with respect to parameters and numerical approximations for a parabolic problem, Asympt. Anal., 37 (2004), 79-91.

[22] P. Groisman, J. D. Rossi and H. Zaag, On the dependence of the blow-up time with respect to the initial data in a semilinear parabolic problem, Comm. Partial Differential Equations, 28 (2003), 737-744.

[23] M. A. Herreros and J. L. Vazquez, Generic behaviour of one-dimensional blow up patterns, Ann. Scuola. Norm. Sup. Pisa Cl. Sci., 19(1992), 381-450.

[24] H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$, Pul. Res. Inst. Math. Sci., 10 (1975), 729-736.

[25] C. M. Kirk and C. A. Roberts, A review of quenching results in the context of nonlinear volterra equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 10 (2003), 343-356.

[26] A. Ladyzenskaya, V. A Solonnikov and N. N. Ural'ceva, Linear and quasililear equations parabolic type, Trans. Math. Monogr., 23 AMS, Providence, RI, (1968).

[27] H. A. Levine, The phenomenon of quenching: a survey, in Trends In The Theory And Practice Of Nonlinear Analysis, North-Holland, Amsterdam (1985), 275-286.

[28] H. A. Levine, The quenching of solutions of linear parabolic and hyperbolic equations with nonlinear boundary conditions, SIAM J. Math. Anal., 14 (1983), 1139-1152.

[29] H. A. Levine, Quenching, nonquenching and beyond quenching for solution of some parabolic equations, Ann. Math. Pura Appl., 155 (1989), 243-260.

[30] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Comm. Pure Appl. Math., 45 (1992), 293-300.

[31] T. Nakagawa, Blowing up on the finite difference solution to $u_t = u_{xx} + u^2$, Appl. Math. Optim., 2 (1976), 337-350.

[32] D. Nabongo and T. K. Boni, Quenching time for some nonlinear parabolic equations, An. St. Univ. Ovidius Constanta, 16 (2008), 87-102.

[33] D. Phillips, Existence of solution of quenching problems, Appl. Anal., 24 (1987), 253-264.

[34] M. H. Protter and H. F. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, NJ (1967).

[35] P. Quittner, Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems, Houston J. Math., 29 (3): 757-799 (electronic), 2003.

[36] Q. Shang and A. Q. M. Khaliq, A compound adaptive approach to degenerate nonlinear quenching problems, Numer. Methods Partial Differential Equations, 15 (1999), 29-47.

[37] W. Walter, Differential-und Integral-Ungleichungen, Springer, Berlin (1964).

Received May, 10, 2008; accepted October, 15, 2008.