SOME RESULTS IN THE THEORY OF THE SPECTRAL REPRESENTATION OF THE LINEAR MULTIGROUP TRANSPORT

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Abstract: The spectral representation of the linear multigroup transport problem is applied to the additional example. We obtain the dispersion relations, normalization coefficients and eigenfunctions for any order N of scattering by using the eigenfunctions for isotropic scattering as the basis.

Key words: basis, dispersion, eigenfunctions, normalization coefficients, spectral integral

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In the paper [1] was developed the mathematical reformulation of singular approach to the solution of the one-dimensional equation of multigroup transport theory. A number of simple examples were presented in which the spectral formulation leads to the standard results of singular approach. In this paper we demonstrate that the eigefunctions for isotropic scattering can be used as a basis set for obtaining the dispersion relation, normalization coefficients and eigenfunctions for Nth order scattering.

The phase function for a previously solved transport problem is

$$f_0(\mu \to \mu') = \sum_{s=0}^{N} (2s+1) P_s(\mu) f_s P_s(\mu'),$$

with corresponding characteristic matrix equation

$$(\nu I - \mu \ell)\phi_\nu(\mu) = \frac{c\nu}{2} \int_{-1}^{+1} f_0(\mu' \to \mu)\phi_\nu(\mu') d\mu',$$

and known eigenfunctions $\phi_\nu(\mu)$, eigenvalue spectrum $S_0[\nu]$ and spectral density $d\rho(\nu)$. The phase function for the problem to be solved (N-th order scattering) is

$$f(\mu \to \mu') = \sum_{s=0}^{N+1} (2s+1) P_s(\mu) f_s P_s(\mu'),$$

with corresponding characteristic matrix equation

$$(\omega I-\mu\ell)\psi_\omega(\mu)=\frac{c\omega}{2}\int_{-1}^{+1}f(\mu'\to\mu)\psi_\omega(\mu')d\mu', \qquad (1)$$

and assumed unknown eigenfunctions $\psi_\omega(\mu)$, eigenvalue spectrum $S_N[\omega]$, normalization coefficients $N_N(\omega)$ where $\ell = diag\{l_1,...,l_{i_0}\}, \quad l_i>0$, moreover without loss of generality we can take $\max_i l_i = 1, \quad P_s(\mu)=diag\{p_s(\mu),...,p_s(\mu)\}$, $p_s(\mu)$ is the Legendre polynomial of order s, and f_s is $i_0\times i_0$ matrix, $s = 1,2,...,s_0$. For the sake of simplicity, we have chosen $f_0 = I$ and f_s are symmetric matrics.

Our basic integral equation (see [1]) is

$$(\omega-\nu)K(\nu,\omega)=\frac{\omega\nu c}{2}\int_{S_0[\nu']}(A(\nu,\nu')-A_0(\nu,\nu'))d\rho(\nu')K(\nu',\omega). \qquad (2)$$

where

$$A(\nu,\nu')-A_0(\nu,\nu')$$
$$=\int_{-1}^{+1}d\mu\int_{-1}^{+1}d\mu'\phi_\nu(\mu)(f(\mu'\to\mu)-f_0(\mu'\to\mu))\phi_{\nu'}(\mu').$$

Its solution is equivalent to the solution of Eq.(1). However, where Eq.(1) is an integral equation involving an integration over the μ, Eq.(2) involves the spectral integral over the known eigenvalues of a complete set of solutions to an equation of transport.

The continuum of $S_0[\nu]$ is known to be given by $-1\le\nu\le 1$. After some algebra from Eq.(2) we obtain the dispersion relation giving the discrete values of ω, which lie outside of the continuum of $S_0[\nu]$,

$$det\Big(I-\frac{c\omega}{2}\sum_{s=1}^{N}(2s+1)\Big(L(\omega,S_0)h_s(\omega)+\frac{r_s(\omega)}{\omega}\Big)f_s g_s^0(\omega)\Big)=0.$$

In this equation $L(\omega,S_0)$ is the matrix spectral integral defined by

$$L(\omega,S_0)=\int_{S_0[\nu]}\frac{\nu}{\omega-\nu}h_0(\nu)d\rho(\nu)h_0(\nu)$$

where $h_s(\nu)$ is the matrix defined by

$$h_s(\nu)=\int_{-1}^{+1}P_s(\mu)\phi_\nu(\mu)d\mu,$$

the s-th degree matrix polynomials $r_s(\omega)$ are defined by

$$r_0(\omega)=0, \qquad r_1(\omega)=-2\omega I,$$

and the recursion relation

$$(s+1)r_{s+1}(\omega) + r_s(\omega) = (2s+1)(1-c)\omega r_s(\omega), \qquad s \geq 1,$$

the s-th degree matrix polynomials $g_s^0(\omega)$ are defined by

$$g_0^0(\omega) = I,$$

and the recursion relation

$$(s+1)g_{s+1}^0(\omega) + g_s^0(\omega) = (2s+1)(I - cf_s)\omega g_s^0(\omega), \qquad s \geq 0.$$

The matrix function $g_s(\omega)$ which defined by

$$g_s(\omega) = \int_{-1}^{+1} P_s(\mu)\psi_\omega(\mu)d\mu$$

is given by the equation

$$g_s(\omega) = g_(\omega)g_s^0(\omega), \qquad s \geq 0.$$

For all ω in the continuum of $S_0[\nu]$, that is $-1 \leq \nu \leq 1$, basic Eq.(2) has the singular solution of the form

$$K(\nu,\omega) = -\omega \sum_{s=1}^{N}(2s+1)\phi_\nu(\omega)h_s(\nu)f_s g_s(\omega)$$

$$+\delta(\nu-\omega)N_0(\omega)\sum_{s=1}^{N}(2s+1)P_s(\omega)f_s g_s(\omega). \qquad (3)$$

For the case of continuum ω, the normalization coefficient $N_N(\omega)$ for the unknown eigenfunctions $\psi_\omega(\mu)$ are given by

$$\int_{S_0[\nu]} K(\nu,\omega)d\rho(\nu)K(\nu,\omega') = \delta(\omega-\omega')N_N(\omega).$$

After the calculation of $N_N(\omega)$ we find that

$$N(\omega) = N_0(\omega)\big([M(\omega,\omega) - \omega N_0^{-1}(\omega)\sum_{s=1}^{N}(2s+1)g_s(\omega)f_s h_s(\omega)\lambda_0(\omega)]$$

$$X[M(\omega,\omega) - \omega N_0^{-1}(\omega)\sum_{s=1}^{N}(2s+1)\lambda_0(\omega)h_s(\omega)f_s g_s(\omega)]$$

$$+[\frac{c\pi\omega^2}{2}N_0^{-1}\sum_{s=1}^{N}(2s+1)g_s(\omega)f_s h_s(\omega)]$$

$$X[\frac{c\pi\omega^2}{2}N_0^{-1}\sum_{s=1}^{N}(2s+1)h_s(\omega)f_s g_s(\omega)])$$

where

$$M(\mu,\omega)=\sum_{s=0}^{N}(2s+1)P_s(\mu)f_s g_s(\omega).$$

The normalization coefficient for $\psi_{\omega_j(\mu)}$ with ω_j discrete follows from

$$N(\omega_j)=\int_{S_0[\nu]}K(\nu,\omega_j)d\rho(\nu)K(\nu,\omega_j).$$

Then employing last formula, we can express $N(\omega_j)$ in the form

$$N(\omega_j)$$

$$=(\frac{c\omega_j}{2})^2\sum_{n=1}^{N}(2n+1)\sum_{s=1}^{N}(2s+1)f_s g_s\frac{d}{d\omega}[\omega h_s(\omega)L(\omega,S_0)h_n(\omega)\mid_{\omega=\omega_j}f_n g_n(\omega_j)$$

$$+(\frac{c\omega_j}{2})^2\sum_{n=1}^{N}(2n+1)\sum_{s=1}^{N}(2s+1)f_s g_s(\omega_j)\frac{d}{d\omega_j}[h_s(\omega_j)r_n(\omega_j)]f_n g_n(\omega_j)$$

$$+(\frac{c\omega_j}{2})^2\sum_{n=1}^{N}(2n+1)\sum_{s=1}^{N}(2s+1)f_s g_n(\omega_j)\frac{d}{d\omega_j}[h_n(\omega_j)r_s(\omega)]f_n g_s(\omega_j)$$

The eigenfunctions $\psi_\omega(\mu)$ for the unsolved problem are given by Eq.(15) from [1]

$$\psi_\omega(\mu)=\int_{S_0[\nu]}\phi_\nu(\mu)d\rho(\nu)K(\nu,\omega). \tag{4}$$

To obtain $\psi_\omega(\mu)$ for continuum ω, substitute the expression for $K(\nu,\omega)$ given by Eq.(3) into Eq.(4), we obtain

$$\psi_\omega(\mu)=-\omega\sum_{s=1}^{N}(2s+1)\int_{S_0[\nu]}\phi_\nu(\mu)d\rho(\nu)\phi_\nu(\omega)h_s(\nu)f_s g_s(\omega)$$

$$+\sum_{s=1}^{N}(2s+1)\phi_\omega(\mu)P_s(\omega)f_s g_s(\omega).$$

The eigenfunction $\psi_{\omega_j}(\mu)$ for discrete ω_j is obtained by substituting $K(\nu,\omega_j)$, as given by Eq. (2), into Eq.(4). We obtain

$$\psi_{\omega_j}(\mu)=\int_{S_0[\nu]}\phi_\nu(\mu)d\rho(\nu)\frac{\omega_j\nu c}{2}(\omega_j-\nu)^{-1}\sum_{s=1}^{N}(2s+1)h_s(\nu)f_s g_s(\omega_j).$$

References

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