Bulletin of TICMI
Vol. 18, No. 1, 2014, 11–28

On a Theorem of Zhizhiashvili

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(*Received December 9, 2013; Revised March 26, 2014; Accepted May 21, 2014*)

Dedicated to the 80th birthday of Professor Levan Zhizhiashvili

We present a result of Zhizhiashvili concerning the Marcinkiewicz summability and its generalizations for trigonometric and Walsh-Fourier series.

Keywords: Trigonometric, Walsh-Fourier series, Fejér and Cesàro summability, Marcinkiewicz summability, Hardy spaces.

AMS Subject Classification: 42B08, 42A24, 42B30, 42C10.

1. Introduction

In this survey paper we will consider Zhizhiashvili's work in summability theory and its impact up to the present days. We present one of his fundamental theorems and several (recent) extensions and generalizations. We investigate convergence and summations of one- and multi-dimensional trigonometric and Walsh-Fourier series. First we give the corresponding results in the one-dimensional case and then the generalizations for higher dimensions. Two types of summability methods will be investigated, the Fejér and Cesàro or (C, α) methods. The Fejér summation is a special case of the Cesàro method, $(C, 1)$ is exactly the Fejér method.

In the multi-dimensional case the Marcinkiewicz summability and the corresponding maximal operators are considered. Marcinkiewicz [27] proved that the arithmetic means of the cubic partial sums taken on the diagonal (the so called Marcinkiewicz-Fejér means) of a two-dimensional function $f \in L \log L(\mathbb{T}^2)$ converge almost everywhere to f. Later Zhizhiashvili [57, 58] extended this result to all $f \in L_1(\mathbb{T}^2)$ and to Cesàro means, D'yachenko [8] and Weisz [52]) to all $f \in L_1(\mathbb{T}^d)$. The result for Walsh-Fourier series is due to the author [50] and to Goginava [14–17].

We introduce classical and dyadic martingale Hardy spaces $H_p(\mathbb{X})$ (where $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0, 1))$ and prove that the maximal operator of the Marcinkiewicz summability means are bounded from $H_p(\mathbb{X})$ to $L_p(\mathbb{X})$, whenever $p > p_0$ for some $p_0 < 1$. The exact value of p_0, which depends on the type of summability and on the dimension, is given. For $p = 1$ we obtain a weak type inequality by interpolation, which implies the almost everywhere convergence of the summability means just

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ISSN: 1512-0082 print

mentioned.

2. Trigonometric and Walsh system

We consider either the torus $\mathbb{X} = \mathbb{T}$ or the unit interval $\mathbb{X} = [0, 1)$ with the Lebesgue measure λ. We briefly write $L_p(\mathbb{X})$ instead of the real $L_p(\mathbb{X}, \lambda)$ space equipped with the norm (or quasinorm)

$$\|f\|_p := \left(\int_{\mathbb{X}} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \le \infty),$$

where λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. A Banach space B consisting of measurable functions on $\mathbb{X}$ is called a homogeneous Banach space if

(i) for all $f \in B$ and $x \in \mathbb{X}$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,
(ii) the function $x \mapsto T_x f$ from $\mathbb{X}$ to B is continuous for all $f \in B$,
(iii) $\|f\|_1 \le C \|f\|_B$ for all $f \in B$.

Here T_x denotes the usual translation operator for $\mathbb{T}$ and the dyadic translation for $[0, 1)$. For an introduction to homogeneous Banach spaces, see Katznelson [21]. It is easy to see that the spaces $L_p(\mathbb{X})$ $(1 \le p < \infty)$, $C(\mathbb{X})$, the Lorentz spaces $L_{p,q}(\mathbb{X})$ $(1 < p < \infty, 1 \le q < \infty)$ and the Hardy space $H_1(\mathbb{X})$ are homogeneous Banach spaces. The weak $L_p(\mathbb{X})$ space $L_{p,\infty}(\mathbb{X})$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}$ is a quasi-normed space. It is easy to see that

$$L_p(\mathbb{X}) \subset L_{p,\infty}(\mathbb{X}) \qquad \text{and} \qquad \| \cdot \|_{p,\infty} \le \| \cdot \|_p$$

for each $0 < p < \infty$.

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0, 1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (0 \le n_k < 2).$$

In what follows let $\phi_n(x)$ denote the trigonometric system $e^{2\pi \imath n \cdot x}$ $(n \in \mathbb{Z})$ defined on $\mathbb{T}$ or the Walsh system $\phi_n(x) := w_n(x)$ $(n \in \mathbb{N})$ defined on the unit interval. For the Walsh system let $\phi_n = 0$ if $n \in \mathbb{Z} \setminus \mathbb{N}$.

In this paper the constants C_p depend only on p and may denote different constants in different contexts.

3. Partial sums of one-dimensional Fourier series

For an integrable function $f \in L_1(\mathbb{X})$ ($\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$) its kth trigonometric or Walsh-Fourier coefficient is defined by

$$\widehat{f}(k) := \int_{\mathbb{X}} f \phi_k \, d\lambda \qquad (k \in \mathbb{Z}).$$

The definition of the Fourier coefficients can be extended easily to distributions in case of the trigonometric system and to martingales in case of the Walsh system (see Weisz [51, 54]).

For $f \in L_1(\mathbb{X})$ the nth partial sum $s_n f$ of the Fourier series of f is introduced by

$$s_n f(x) := \sum_{|k| \le n} \widehat{f}(k) \phi_k(x) = \int_{\mathbb{X}} f(x-u) D_n(u) \, du \qquad (n \in \mathbb{N}),$$

where

$$D_n(u) := \sum_{|k| \le n} \phi_k(u)$$

is the nth trigonometric or Walsh-Dirichlet kernel (see Figure 1). In case of the Walsh system we use dyadic addition instead of addition.

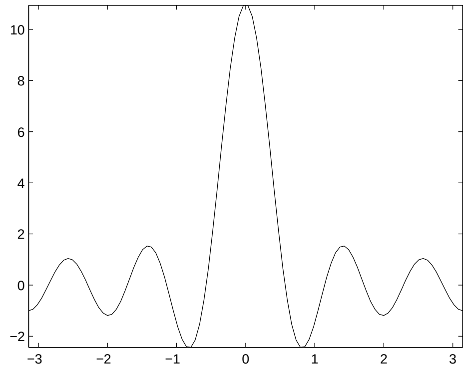
It is a basic question as to whether the function f can be reconstructed from the partial sums of its Fourier series. It can be found in most books about Fourier series (e.g., Zygmund [59], Bary [1], Torchinsky [45], Grafakos [18], Schipp, Wade, Simon and Pál [37]), that the partial sums converge to f in the L_p-norm if $1 < p < \infty$.

Theorem 3.1 : *If $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$, then*

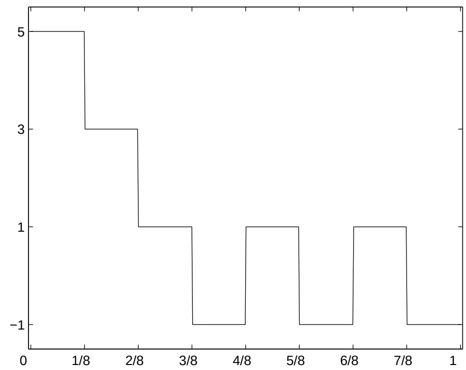
$$\|s_n f\|_p \le C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n \to \infty} s_n f = f \qquad \text{in the } L_p\text{-norm.}$$



(a) The trigonometric Dirichlet kernel



(b) The Walsh-Dirichlet kernel

Figure 1. The Dirichlet kernels D_n with $n = 5$.

One of the deepest results in harmonic analysis is Carleson's result, i.e. the partial sums of the Fourier series converge almost everywhere to $f \in L_p(\mathbb{X})$ $(1 < p < \infty)$ (see Carleson [6], Hunt [20] for the trigonometric series and Billard [4], Sjölin [41], Schipp [35] for Walsh series).

Theorem 3.2: *If $f \in L_p(\mathbb{X})$ for some $1 < p < \infty$, then*

$$\left\| \sup_{n\in\mathbb{N}} |s_n f| \right\|_p \le C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{a.e.}$$

The inequalities of Theorems 3.1 and 3.2 do not hold if $p = 1$ or $p = \infty$, and the almost everywhere convergence does not hold if $p = 1$. du Bois Reymond proved the existence of a continuous function $f \in C(\mathbb{T})$ and a point $x_0 \in \mathbb{T}$ such that the partial sums $s_n f(x_0)$ diverge as $n \to \infty$. Kolmogorov gave an integrable function $f \in L_1(\mathbb{T})$, whose Fourier series diverges almost everywhere or even everywhere (see Kolgomorov [22, 23], Zygmund [59] or Grafakos [18]). The analogous result for

Walsh-Fourier series can be found in Schipp [33] and Simon [38].

4. Hardy spaces

To prove almost everywhere convergence of the summability means introduced in the next section, we will need the concept of Hardy spaces and their atomic decomposition. First we consider the classical Hardy spaces for the trigonometric system and then the dyadic Hardy spaces for the Walsh system.

4.1. *The $H_p(\mathbb{T})$ classical Hardy spaces*

A distribution f is in the Hardy space $H_p(\mathbb{T})$ and in the weak Hardy space $H_{p,\infty}(\mathbb{T})$ $(0 < p \leq \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{0<t} |f * P_t| \right\|_p < \infty$$

and

$$\|f\|_{H_{p,\infty}} := \left\| \sup_{0<t} |f * P_t| \right\|_{p,\infty} < \infty,$$

respectively, where

$$P_t(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi \imath kx} = \frac{1-r^2}{1+r^2-2r\cos 2\pi x} \qquad (r := e^{-t}, x \in \mathbb{T})$$

is the periodic Poisson kernel. Since $P_t \in L_1(\mathbb{T})$, the convolution in the definition of the norms are well defined.

4.2. *The $H_p[0,1)$ dyadic Hardy spaces*

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \leq k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x. The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in [0,1)\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbb{N})$.

We investigate the class of *martingales* $f = (f_n, n \in \mathbb{N})$ with respect to $(\mathcal{F}_n, n \in \mathbb{N})$. For $0 < p \leq \infty$ the *dyadic Hardy space* $H_p[0,1)$ and the *dyadic weak Hardy space* $H_{p,\infty}[0,1)$ consist of all martingales for which

$$\|f\|_{H_p} := \left\| \sup_{n\in\mathbb{N}} |f_n| \right\|_p < \infty$$

and

$$\|f\|_{H_{p,\infty}} := \left\| \sup_{n\in\mathbb{N}} |f_n| \right\|_{p,\infty} < \infty,$$

respectively.

4.3. *Atomic decomposition*

The results of this subsection hold for both the classical and the dyadic Hardy spaces. It is known (see e.g. Stein [42] or Weisz [51]) that

$$H_p(\mathbb{X}) \sim L_p(\mathbb{X}) \qquad (1 < p \leq \infty)$$

and $H_1(\mathbb{X}) \subset L_1(\mathbb{X}) \subset H^{\square}_{1,\infty}(\mathbb{X})$, where $\sim$ denotes the equivalence of spaces and norms. Moreover,

$$\|f\|_{H_{1,\infty}} \leq C\|f\|_1 \leq C\|f\|_{H_1}.$$

The atomic decomposition provides a useful characterization of Hardy spaces. A function $a \in L_\infty(\mathbb{T})$ is a classical *p-atom* if there exists an interval $I \subset \mathbb{T}$ such that

(i) supp $a \subset I$,
(ii) $\|a\|_\infty \leq |I|^{-1/p}$,
(iii) $\int_I a(x)x^k\, dx = 0$ for all $k \in \mathbb{N}$ with $k \leq \lfloor 1/p - 1 \rfloor$.

While a function $a \in L_\infty[0,1)$ is called a dyadic *p-atom* if there exists a dyadic interval $I \subset [0,1)$ such that (i), (ii) and (iii) with $k = 0$ hold.

The Hardy space $H_p(\mathbb{X})$ has an atomic decomposition. In other words, every function from the Hardy space can be decomposed into the sum of atoms. A first version of the atomic decomposition was introduced by Coifman and Weiss [7] in the classical case and by Herz [19] in the martingale case. The proof of the next theorem can be found in Latter [24], Lu [26], Wilson [55, 56], Stein [42] and Weisz [47, 51].

Theorem 4.1 : *A distribution (resp. martingale) f is in $H_p(\mathbb{X})$ $(0 < p \leq 1)$ if and only if there exists a sequence $(a^k, k \in \mathbb{N})$ of classical (resp. dyadic) p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \qquad \textit{and} \qquad \sum_{k=0}^{\infty} \mu_k a^k = f$$

in the sense of distributions (resp. martingales). Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

The "only if" part of the theorem holds also for $0 < p < \infty$. The following result gives a sufficient condition for an operator to be bounded from $H_p(\mathbb{X})$ to $L_p(\mathbb{X})$ (see Weisz [51, 53]). For $I \subset \mathbb{T}$ let I^r be the interval having the same center as the interval I and length $2^r|I|$. If $I \subset [0,1)$ is a dyadic interval then let I^r be a dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r|I|$ $(r \in \mathbb{N})$.

Theorem 4.2 : *For each $n \in \mathbb{N}$, let $V_n : L_1(\mathbb{X}) \to L_1(\mathbb{X})$ be a bounded linear* *operator and let*

$$V_*f := \sup_{n\in\mathbb{N}} |V_n f|.$$

Suppose that

$$\int_{\mathbb{X}\setminus I^r} |V_* a|^{p_0}\, d\lambda \le C_{p_0}$$

for all classical (resp. dyadic) p-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{X})$ to $L_{p_1}(\mathbb{X})$ for some $1 < p_1 \le \infty$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X})) \tag{1}$$

for all $p_0 \le p \le p_1$. Moreover, if $p_0 < 1$ then the operator V_* is of weak type $(1,1)$, i.e., if $f \in L_1(\mathbb{X})$ then

$$\sup_{\rho>0} \rho\, \lambda(|V_* f| > \rho) \le C\|f\|_1. \tag{2}$$

Note that (2) can be obtained from (1) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [3] and Bennett and Sharpley [2] or Weisz [47, 51]. The interpolation of martingale Hardy spaces was worked out in [47]. Theorem 4.2 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

5. Summability of one-dimensional Fourier series

Though Theorems 3.1 and 3.2 are not true for $p = 1$ and $p = \infty$, with the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books of Stein and Weiss [44], Butzer and Nessel [5], Trigub and Belinsky [46], Grafakos [18] and Weisz [51, 54] and the references therein.

The best known summability method is the Fejér method. In 1904 Fejér [10] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [25] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus almost everywhere.

Here we consider the *Fejér* and *Cesàro* (or (C,α)) means defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) = \sum_{|j|\le n} \left(1 - \frac{|j|}{n}\right) \widehat{f}(j)\phi_j(x) = \int_{\mathbb{X}} f(x-u)K_n(u)\, du$$

and

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k f(x)$$
$$= \frac{1}{A_{n-1}^\alpha} \sum_{|j|\le n} A_{n-1-|j|}^\alpha \widehat{f}(j)\phi_j(x) = \int_{\mathbb{X}} f(x-u) K_n^\alpha(u)\, du,$$

where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}$$

and the Fejér and Cesàro kernels are given by

$$K_n(u) := \sum_{|j|\le n} \left(1 - \frac{|j|}{n}\right) \phi_j(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u)$$

and

$$K_n^\alpha(u) := \frac{1}{A_{n-1}^\alpha} \sum_{|j|\le n} A_{n-1-|j|}^\alpha \phi_j(u) = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} D_k(u)$$

(see Figure 2). It is known (Zygmund [59]) that

$$A_k^\alpha \sim k^\alpha \qquad (k \in \mathbb{N}).$$

The Riesz means are generalizations of the Fejér means, if $\alpha = 1$, then we get back the Fejér means. We will suppose always that $0 < \alpha \le 1$. The case $\alpha > 1$ can be led back to $\alpha = 1$. The next result extends Theorem 3.1 to the summability means (see Zygmund [59] and Paley [31]).

Theorem 5.1: *If $0 < \alpha \le 1$ and B is a homogeneous Banach space on $\mathbb{X}$, then*

$$\|\sigma_n^\alpha f\|_B \le C \|f\|_B \qquad (f \in B, n \in \mathbb{N}).$$

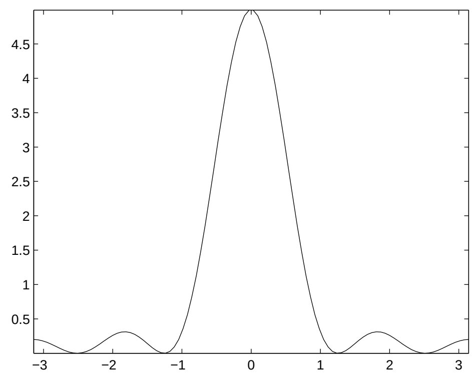
Moreover, for all $f \in B$

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \textit{in the } B\textit{-norm.}$$

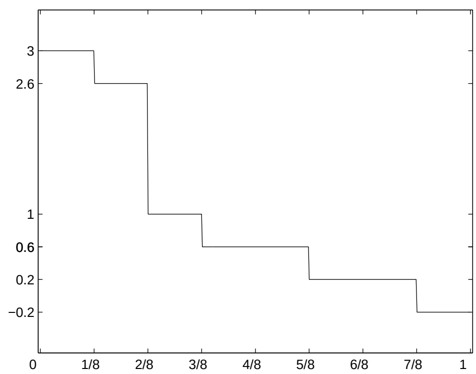
Recall that the $L_p(\mathbb{X})$ $(1 \le p < \infty)$ spaces are all homogeneous Banach spaces, so Theorem 5.1 holds for these spaces, too.

The *maximal operator* of the Cesàro means are defined by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|.$$



(a) The trigonometric Fejér kernel



(b) The Walsh-Fejér kernel

Figure 2. The Fejér kernels K_n with $n = 5$.

Applying Theorem 4.2, we extended the previous result to the $L_p(\mathbb{X})$ spaces $(0 < p < 1)$ and to the maximal operator in [48, 49, 51]. The first inequality was proved by Fujii [12] in the Walsh case for $p = 1$ (see also Schipp, Simon [36]).

Theorem 5.2: *If* $0 < \alpha \leq 1$ *and* $1/(\alpha + 1) < p \leq \infty$*, then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X}))$$

and for $f \in H_{1/(\alpha+1)}(\mathbb{X})$*,*

$$\|\sigma_*^\alpha f\|_{1/(\alpha+1),\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^\alpha f > \rho)^{\alpha+1} \leq C \|f\|_{H_{1/(\alpha+1)}}.$$

The critical index is $p = 1/(\alpha + 1)$, if p is smaller than or equal to this critical index, then σ_*^α is not bounded anymore (see Stein, Taibleson and Weiss [43], Simon and Weisz [40], Simon [39] and Gát and Goginava [13]):

Theorem 5.3: *The operator* σ_*^α $(0 < \alpha \leq 1)$ *is not bounded from* $H_p(\mathbb{X})$ *to* $L_p(\mathbb{X})$ *if* $0 < p \leq 1/(\alpha + 1)$*.*

We get the next weak type $(1, 1)$ inequality from Theorem 5.2 by interpolation

(Weisz [48, 49, 51], Zygmund [59] for the trigonometric system, for $\alpha = 1$ and for the Walsh system Schipp [34]).

Corollary 5.4: *If* $0 < \alpha \leq 1$ *and* $f \in L_1(\mathbb{X})$ *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_1.$$

This weak type $(1,1)$ inequality and the density argument of Marcinkiewicz and Zygmund [28] imply the well known theorem of Fejér [10] and Lebesgue [25] with $\alpha = 1$. Riesz [32] proved it for other α's and Fine [11], Schipp [34] and Weisz [49] for the Walsh system.

Corollary 5.5: *If* $0 < \alpha \leq 1$ *and* $f \in L_1(\mathbb{X})$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad a.e.$$

With the help of the conjugate functions we ([51]) proved also

Theorem 5.6: *If* $0 < \alpha \leq 1$ *and* $1/(\alpha+1) < p \leq \infty$ *then*

$$\|\sigma_n^\alpha f\|_{H_p} \leq C_p\|f\|_{H_p} \qquad (f \in H_p(\mathbb{X})).$$

Corollary 5.7: *If* $0 < \alpha \leq 1$, $1/(\alpha+1) < p < \infty$ *and* $f \in H_p(\mathbb{X})$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{in the } H_p\text{-norm.}$$

6. Multi-dimensional partial sums

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. The $L_p(\mathbb{X}^d)$ spaces are defined in the usual way. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_2 := \left(\sum_{k=1}^d |x_k|^2\right)^{1/2}, \qquad |x| := \|x\|_\infty := \sup_{k=1,\ldots,d} |x_k|.$$

The d-dimensional trigonometric and Walsh system is introduced as a Kronecker product by

$$\phi_k(x) := \phi_{k_1}(x_1) \cdots \phi_{k_d}(x_d),$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{X}^d$. The multi-dimensional *Fourier coefficients* of an integrable function f are defined by

$$\widehat{f}(k) := \int_{\mathbb{X}^d} f\phi_k \, d\lambda \qquad (k \in \mathbb{N}^d).$$

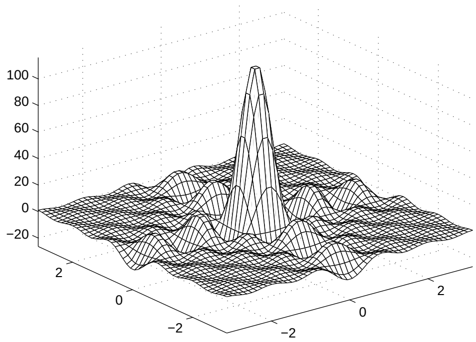
For $f \in L_1(\mathbb{X}^d)$ the nth *cubic partial sum* $s_n f$ $(n \in \mathbb{N})$ is given by

$$s_n f(x) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} \widehat{f}(k) \phi_k(x) = \int_{\mathbb{X}^d} f(x-u) D_n(u)\, du \qquad (n \in \mathbb{N}),$$

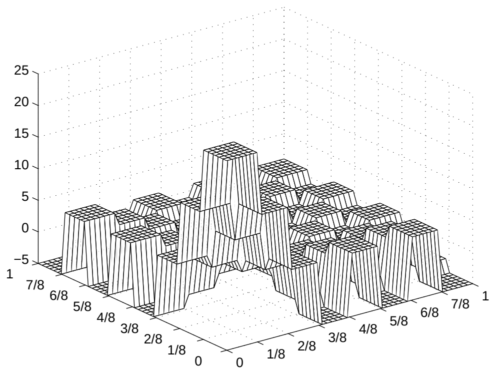
where

$$D_n(u) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} \phi_k(u)$$

is the nth trigonometric or Walsh-Dirichlet kernel (see Figure 3). Other types of



(a) The trigonometric Dirichlet kernel



(b) The Walsh-Dirichlet kernel

Figure 3. The Dirichlet kernels D_n with $n = 5$.

partial sums are considered e.g. in Weisz [51, 54].

By iterating the one-dimensional result, we get easily the next theorem.

Theorem 6.1: *If $f \in L_p(\mathbb{X}^d)$ for some $1 < p < \infty$, then*

$$\|s_n f\|_p \leq C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_p\text{-norm.}$$

The analogue of the Carleson's theorem holds also for higher dimensions and for the trigonometric system (see Fefferman [9] and Grafakos [18]), and it holds for the Walsh system if $p = 2$ (see Móricz [29] or Schipp, Wade, Simon and Pál [37]).

Theorem 6.2 : *If $f \in L_p(\mathbb{X}^d)$ for some $1 < p < \infty$, then for the trigonometric Fourier series*

$$\left\| \sup_{n\in\mathbb{N}} |s_n f| \right\|_p \le C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{a.e.}$$

The same result holds for the Walsh-Fourier series if $p = 2$.

It is an open question, whether this theorem holds for the Walsh system and for $p \neq 2$ (cf. Schipp, Wade, Simon and Pál [37]).

7. Hardy spaces

In this section we introduce the multi-dimensional classical Hardy spaces for the trigonometric system and the multi-dimensional dyadic Hardy spaces for the Walsh system.

7.1. *The $H_p(\mathbb{T}^d)$ multi-dimensional classical Hardy spaces*

A distribution f is in the Hardy space $H_p(\mathbb{T}^d)$ and in the weak Hardy space $H_{p,\infty}^{\square}(\mathbb{T}^d)$ $(0 < p \le \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{0<t} \left| f * P_t^d \right| \right\|_p < \infty$$

and

$$\|f\|_{H_{p,\infty}} := \left\| \sup_{0<t} \left| f * P_t^d \right| \right\|_{p,\infty} < \infty,$$

respectively, where

$$P_t^d(x) := \sum_{k\in\mathbb{Z}^d} e^{-t\|k\|_2} e^{2\pi\imath k\cdot x} \qquad (x \in \mathbb{T}^d, t > 0)$$

is the d-dimensional periodic Poisson kernel. If $d = 1$, then we get back the one-dimensional Poisson kernel.

7.2. *The $H_p[0,1)^d$ multi-dimensional dyadic Hardy spaces*

By a *dyadic rectangle* we mean a Cartesian product of d dyadic intervals. For $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in [0,1)^d$ let $I_n(x) := I_n(x_1) \times \ldots \times I_n(x_d)$ be a dyadic cube. The σ-algebra generated by the dyadic cubes $\{I_n(x) : x \in [0,1)^d\}$ will be denoted again by $\mathcal{F}_n$ $(n \in \mathbb{N})$.

For $0 < p \leq \infty$ the *dyadic Hardy space* $H_p[0,1)^d$ and the *dyadic weak Hardy space* $H_{p,\infty}[0,1)^d$ consist of all d-dimensional dyadic martingales $f = (f_n, n \in \mathbb{N})$ with respect to $(\mathcal{F}_n, n \in \mathbb{N})$, for which

$$\|f\|_{H_p} := \left\| \sup_{n \in \mathbb{N}} |f_n| \right\|_p < \infty$$

and

$$\|f\|_{H_{p,\infty}} := \left\| \sup_{n \in \mathbb{N}} |f_n| \right\|_{p,\infty} < \infty,$$

respectively.

7.3. *Atomic decomposition of the multi-dimensional Hardy spaces*

It is known again (see e.g. Stein [42] or Weisz [51]) that

$$H_p(\mathbb{X}^d) \sim L_p(\mathbb{X}^d) \qquad (1 < p \leq \infty)$$

and $H_1(\mathbb{X}^d) \subset L_1(\mathbb{X}^d) \subset H_{1,\infty}(\mathbb{X}^d)$ with

$$\|f\|_{H_{1,\infty}} \leq C\|f\|_1 \leq C\|f\|_{H_1}.$$

A function $a \in L_\infty(\mathbb{T}^d)$ is a multi-dimensional classical *p-atom* if there exists a cube $I \subset \mathbb{T}^d$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_\infty \leq |I|^{-1/p}$,
(iii) $\int_I a(x)x^k\,dx = 0$ for all multi-indices $k = (k_1, \ldots, k_d)$ with $\|k\|_2 \leq \lfloor d(1/p - 1) \rfloor$.

A function $a \in L_\infty[0,1)^d$ is called a multi-dimensional dyadic *p-atom* if there exists a dyadic cube $I \subset [0,1)^d$ such that (i), (ii) and (iii) with $k = 0$ hold.

The atomic decomposition holds for the multi-dimensional Hardy spaces, too (see Latter [24], Lu [26], Wilson [55, 56], Stein [42] and Weisz [47, 51]).

Theorem 7.1: *A distribution (resp. martingale) f is in $H_p(\mathbb{X}^d)$ $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of classical (resp. dyadic) p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \qquad \textit{and} \qquad \sum_{k=0}^{\infty} \mu_k a^k = f$$

in the sense of distributions (resp. martingales). Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

For a cube

$$I = I_1 \times \cdots \times I_d \subset \mathbb{X}^d \quad \text{let} \quad I^r = I_1^r \times \cdots \times I_d^r.$$

For the proof of the next theorem see Weisz [51, 53].

Theorem 7.2: *For each $n \in \mathbb{N}$, let $V_n : L_1(\mathbb{X}^d) \to L_1(\mathbb{X}^d)$ be a bounded linear operator and let*

$$V_* f := \sup_{n \in \mathbb{N}} |V_n f|.$$

Suppose that

$$\int_{\mathbb{X}^d \setminus I^r} |V_* a|^{p_0} \, d\lambda \le C_{p_0}$$

for all classical (resp. dyadic) p-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{X}^d)$ to $L_{p_1}(\mathbb{X}^d)$ for some $1 < p_1 \le \infty$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X}^d))$$

for all $p_0 \le p \le p_1$. Moreover, if $p_0 < 1$, then

$$\sup_{\rho > 0} \rho \, \lambda(|V_* f| > \rho) \le C \|f\|_1 \qquad (f \in L_1(\mathbb{X}^d)).$$

8. Marcinkiewicz summability of multi-dimensional Fourier series and Hardy spaces

The summability results can be generalized for higher dimensions in several ways. Here we consider the method, which basically consist of the arithmetic means of the cubic partial sums $s_k f$, and which was first investigated by Marcinkiewicz [27] and Zhizhiashvili [57, 58]. They proved fundamental results in this topic.

The Marcinkiewicz-Fejér and Marcinkiewicz-Cesàro means of the trigonometric Fourier or Walsh-Fourier series of f are defined by

$$\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x) = \sum_{j \in \mathbb{Z}^d, |j| \le n} \left(1 - \frac{|j|}{n}\right) \widehat{f}(j) \phi_j(x) = \int_{\mathbb{X}^d} f(x-u) K_n(u) \, du$$

and

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k f(x)$$
$$= \frac{1}{A_{n-1}^\alpha} \sum_{j \in \mathbb{Z}^d, |j| \le n} A_{n-1-|j|}^\alpha \widehat{f}(j) \phi_j(x) = \int_{\mathbb{X}^d} f(x-u) K_n^\alpha(u)\, du,$$

respectively, where the Marcinkiewicz-Fejér and Marcinkiewicz-Cesàro kernels are given by

$$K_n(u) := \sum_{j \in \mathbb{Z}^d, |j| \le n} \left(1 - \frac{|j|}{n}\right) \phi_j(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u)$$

and

$$K_n^\alpha(u) := \frac{1}{A_{n-1}^\alpha} \sum_{j \in \mathbb{Z}^d, |j| \le n} A_{n-1-|j|}^\alpha \phi_j(u) = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} D_k(u)$$

(see Figure 4).

The next theorem can be found in [50, 52, 54] for multi-dimensional Fourier series and for two-dimensional Walsh-Fourier series and in [14–17] for multi-dimensional Walsh-Fourier series.

Theorem 8.1 : *If $0 < \alpha \le 1$ and B is a homogeneous Banach space on $\mathbb{X}^d$, then*

$$\|\sigma_n^\alpha f\|_B \le C \|f\|_B \qquad (f \in B, n \in \mathbb{N})$$

and

$$\lim_{n \to \infty} \sigma_n^\alpha f = f \qquad \textit{in the } B\textit{-norm for all } f \in B.$$

Of course the theorem holds for the $L_p(\mathbb{X})$ $(1 \le p < \infty)$ spaces as well. The *maximal Marcinkiewicz-Cesàro operator* is defined by

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f|.$$

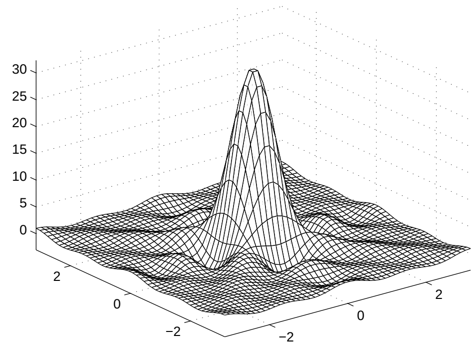
The next theorem follows from Theorem 7.2.

Theorem 8.2 : *If $0 < \alpha \le 1$ and $d/(d+\alpha) < p \le \infty$, then*

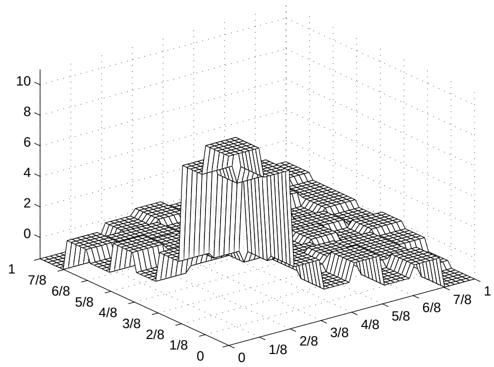
$$\|\sigma_*^\alpha f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X}^d)) \tag{3}$$

and for $f \in H_{d/(d+\alpha)}(\mathbb{X}^d)$,

$$\|\sigma_*^\alpha f\|_{d/(d+\alpha),\infty} = \sup_{\rho > 0} \rho \lambda(\sigma_*^\alpha f > \rho)^{(d+\alpha)/d} \le C \|f\|_{H_{d/(d+\alpha)}}. \tag{4}$$



(a) The trigonometric Marcinkiewicz-Fejér kernel



(b) The Walsh-Marcinkiewicz-Fejér kernel

Figure 4. The Marcinkiewicz-Fejér kernels K_n with $n = 5$.

This theorem was proved by Oswald [30] for Fourier transforms and for Riesz means, by the author for multi-dimensional Fourier series and for two-dimensional Walsh-Fourier series [50–52, 54] and by Goginava [14–17] for multi-dimensional Walsh-Fourier series.

Oswald and Goginava verified also that $d/(d+\alpha)$ is the best possible constant.

Theorem 8.3: *The operator* σ_*^α $(0 < \alpha \leq 1)$ *is not bounded from* $H_p(\mathbb{X}^d)$ *to* $L_p(\mathbb{X}^d)$ *if* $0 < p \leq d/(d+\alpha)$.

Of course (4) is not true for $p < d/(d+\alpha)$, because then (3) would hold for $p < d/(d+\alpha)$ by interpolation.

The weak type $(1,1)$ inequality and the almost everywhere convergence of the Marcinkiewicz-Cesàro means is obtained again by interpolation and by Theorem 8.2.

Corollary 8.4: *If* $0 < \alpha \leq 1$ *and* $f \in L_1(\mathbb{X}^d)$, *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_1.$$

Corollary 8.5: *If* $0 < \alpha \leq 1$ *and* $f \in L_1(\mathbb{X}^d)$, *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad a.e.$$

This corollary was verified first by Marcinkievicz [27] for two-dimensional Fourier series, for $f \in L \log L(\mathbb{T}^2)$ and $\alpha = 1$. Later Zhizhiashvili [57, 58] extended this result to all $f \in L_1(\mathbb{T}^2)$ and $0 < \alpha \leq 1$, D'yachenko [8] and Weisz [52]) to all $f \in L_1(\mathbb{T}^d)$. The result for Walsh-Fourier series is due to the author [50] and to Goginava [14–17]. The next two results can be found in [50, 52].

Theorem 8.6: *If* $0 < \alpha \leq 1$ *and* $d/(d+\alpha) < p \leq \infty$, *then*

$$\|\sigma_n^\alpha f\|_{H_p} \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X}^d)).$$

Corollary 8.7: *If* $0 < \alpha \leq 1$, $d/(d+\alpha) < p < \infty$ *and* $f \in H_p(\mathbb{X}^d)$, *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{in } H_p\text{-norm.}$$

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